ETH zürich

Departement Informatik

Programming and Problem-Solving

Complexity and Primality Testing

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Time Complexity of Algorithms

Primality Testing

Exercise – Primality Testing

Write a function that

- takes an integer x as parameter
- calculates whether x is prime
- uses the % operator
- depending on that either returns `True` or `False`

Primality Test

```python
def primetest(x):
    if x < 2:
        return False
    d = 2
    while d < x:
        if x % d == 0:
            return False
        d += 1
    return True
```

Primality Test

How long does it take the algorithm to produce the output?

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2^{n-1} is $\underbrace{10\ldots00}_{n}$, $2^{n-1}+1$ is $\underbrace{10\ldots01}_{n}$, ..., and $2^n - 1$ is $\underbrace{11\ldots11}_{n}$

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A number that is encoded with n bits has size around 2^n

Time Complexity – Technology Model

Random Access Machine

- **Execution model:** Instructions are executed one after the other (on one processor core)

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- **Memory model:** Constant access time
- **Fundamental operations:** Computations $(+, -, \cdot, \dots)$ comparisons, assignment / copy, flow control (jumps)
- **Unit cost model:** Fundamental operations provide a cost of 1

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We are not completely accurate here

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- Surely an addition is faster than a multiplication

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- We assume that arithmetic operations can be done in constant time
- The time needed to add two n-bit numbers depends on n
- Encoding of a floating point number does not directly correspond to its size
- Surely an addition is faster than a multiplication
- Logarithmic cost model takes this into account, but we also won't use it here

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- Algorithm executes five operations per iteration
- In total roughly $5 \cdot 2^n$ operations

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- In total roughly $5 \cdot 2^n$ operations
- We would like to know how time complexity behaves when n grows
- Ignore constant 5

Time Complexity of Algorithms

Asymptotic Upper Bounds

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The exact time complexity can usually not be predicted even for small inputs

- We are interested in **upper bounds**
- We consider the asymptotic behavior of the algorithm
- And ignore all constant factors

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- And ignore all constant factors

Example

- Linear growth with gradient 5 is as good as linear growth with gradient 1
- Quadratic growth with coefficient 10 is as good as quadratic growth with coefficient 1

Asymptotic Upper Bounds

Big-$\mathcal{O}$ Notation

The set $\mathcal{O}(2^n)$ contains all functions that do not grow faster than $c \cdot 2^n$ for some constant c

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- Use asymptotic notation to specify the time complexity of algorithms
- We write $\mathcal{O}(n^2)$ and mean that the algorithm behaves for large n like n^2: when the input length is doubled, the time taken multiplies by four (at most)

Asymptotic Upper Bounds – Formal Definition

$\mathcal{O}$ Notation

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$$f(n) \in \mathcal{O}(g(n))$$

$$\Longleftrightarrow$$

$$\exists c > 0, n_0 \in \mathbb{N} \text{ such that } \forall n \geq n_0 \colon f(n) \leq c \cdot g(n)$$

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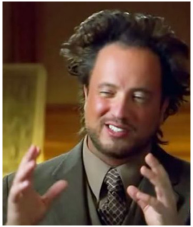
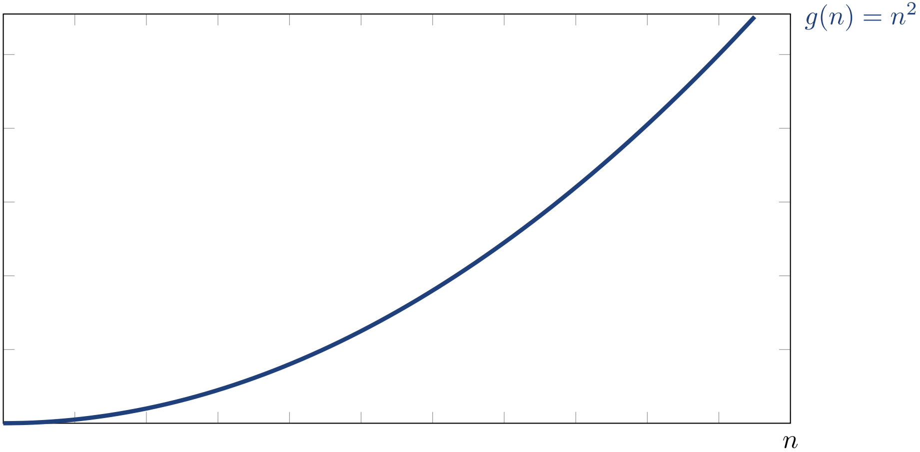


Asymptotic Upper Bounds – Illustration



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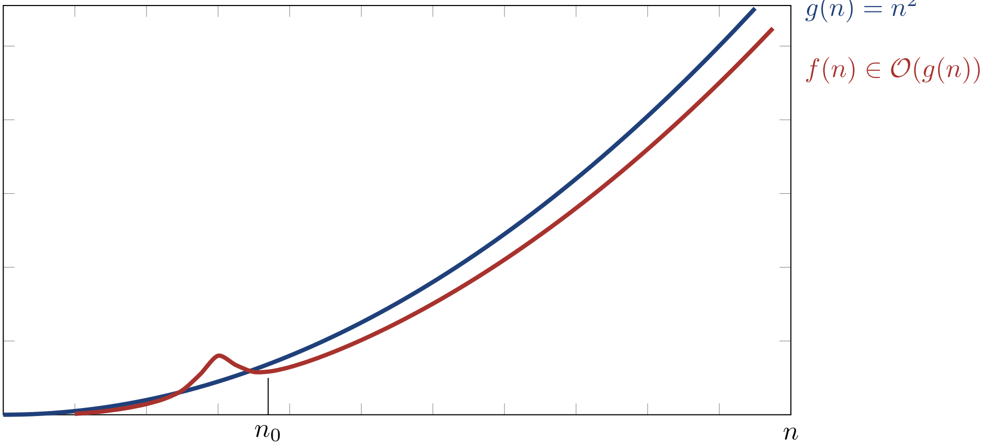

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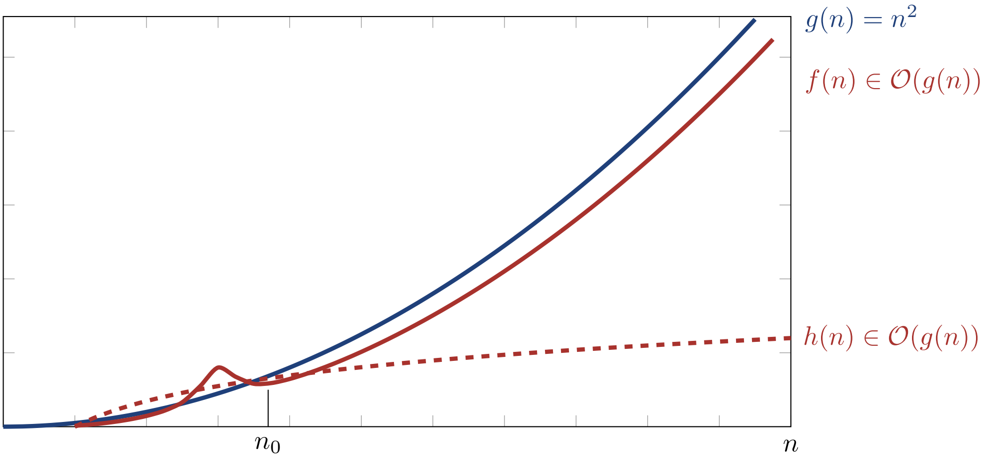

Asymptotic Upper Bounds – Examples

$$\mathcal{O}(g(n)) = \{f\colon \mathbb{N} \to \mathbb{R}^+ \mid \exists c > 0, n_0 \in \mathbb{N}\colon \forall n \geq n_0\colon f(n) \leq c \cdot g(n)\}$$

$f(n)$	$f \in \mathcal{O}(?)$	**Example**
$3n+4$		
$2n$		
n^2+100n		
$n+\sqrt{n}$		

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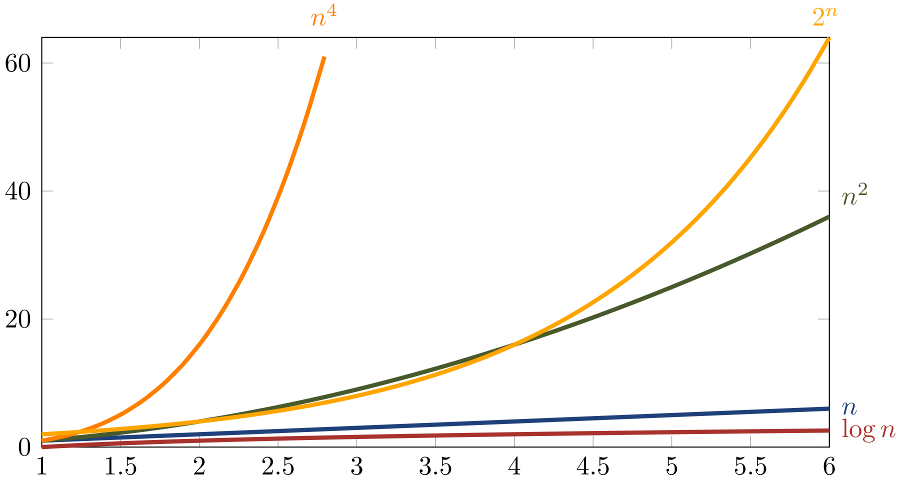
$n^2 + 100n$	$\mathcal{O}(n^2)$	$c = 2, n_0 = 100$
$n + \sqrt{n}$	$\mathcal{O}(n)$	$c = 2, n_0 = 1$

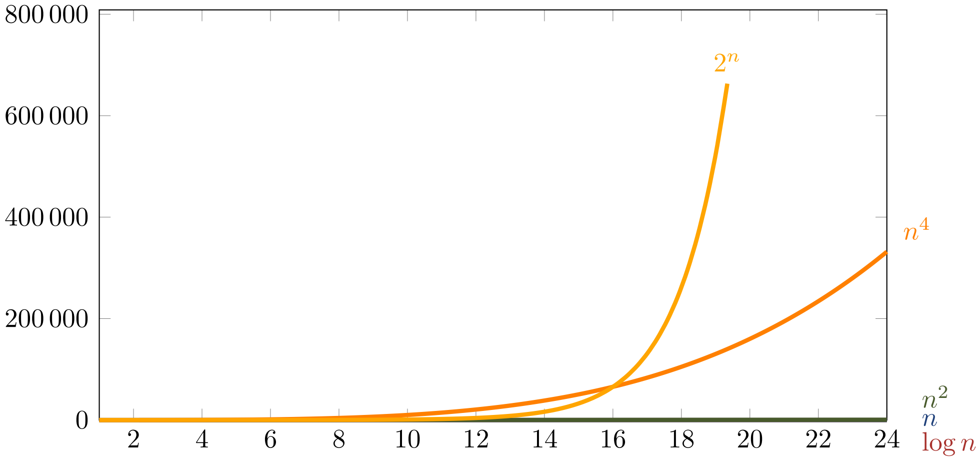
Time Complexity of Algorithms

Time Complexity Analysis

Small n

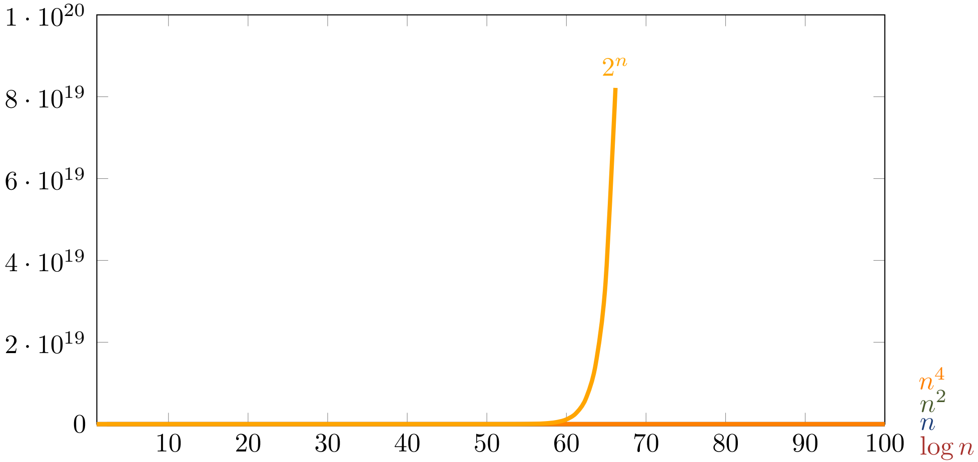


Larger n



800 000
600 000
400 000
200 000
0
2
4
6
8
10
12
14
16
18
20
22
24
2^n
n^4
n^2
n
$\log n$

“Large” n



$1 \cdot 10^{20}$
$8 \cdot 10^{19}$
$6 \cdot 10^{19}$
$4 \cdot 10^{19}$
$2 \cdot 10^{19}$
0
10
20
30
40
50
60
70
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Faster Primality Testing

First Attempt

Faster Primality Testing

Goal

Time complexity better than $\Omega(2^n)$

Faster Primality Testing

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Time complexity better than $\Omega(2^n)$

Observation

- If `x` is not divisible by 2, then it also is not divisible by 4, 6, 8, etc.

Faster Primality Testing

Goal

Time complexity better than $\Omega(2^n)$

Observation

- If $\mathtt{x}$ is not divisible by 2, then it also is not divisible by 4, 6, 8, etc.
- We then only have to check odd numbers
- Algorithm only has to test half the numbers
- Loop is only iterated around $\mathtt{x}/2$ times

Faster Primality Testing

```python
def primetest2(x):
    if x < 2 or (x > 2 and x % 2 == 0):
        return False

    d = 3
    while d < x:
        if x % d == 0:
            return False
        d += 2

    return True
```

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- Loop is iterated roughly `x`$/2$ times instead of `x` times
- Time complexity improves by a factor of 2

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What is the gain?

- Loop is iterated roughly $\texttt{x}/2$ times instead of $\texttt{x}$ times
- Time complexity improves by a factor of 2
- Again assume $\texttt{x}$ is encoded using n bits
- Around $5 \cdot 2^n/2 = 2.5 \cdot 2^n$ fundamental operations in total

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⇨ No asymptotic improvement

Faster Primality Testing

Second Attempt

Faster Primality Testing

Observation

- If $\mathtt{x}$ with $\mathtt{x} > 2$ is not a prime number, then $\mathtt{x}$ is divisible by a number a with

$$1 < a < \mathtt{x}$$

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$$a \cdot b = \mathtt{x} \quad \text{and} \quad 1 < b < \mathtt{x}$$

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- Then $\mathtt{x}$ is also divisible by a number b with

$$a \cdot b = \mathtt{x} \quad \text{and} \quad 1 < b < \mathtt{x}$$

- It cannot be the case that

$$a > \sqrt{\mathtt{x}} \quad \text{and} \quad b > \sqrt{\mathtt{x}},$$

  since then

$$a \cdot b > \mathtt{x}$$

Faster Primality Testing

Including Modules

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So far all functions have been defined in a single file

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Modules

- Distribute functions over multiple files
- Files cannot "see" each other
- Functions can be **imported**
- Structured code

Including Modules

File `functions.py`

```
def square_root(n):
    i = 1
    while i * i < n: # Computer root of next larger square number
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File `applications.py`

```


print(square_root(81))
```

Including Modules

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```

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```
from functions import square_root

print(square_root(81))
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def square_root(n):
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Including Modules

- A large number of modules already exists
- For instance, there is a module `math` which includes a function **`sqrt()`** to compute square roots

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```
NameError: name 'sqrt' is not defined
```

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- For instance, there is a module `math` which includes a function `sqrt()` to compute square roots

```
from math import sqrt

print(sqrt(9))
```

Output: 3

Faster Primality Testing

```
def primetest3(x):
    if x < 2 or (x > 2 and x % 2 == 0):
        return False

    d = 3
    while d < x:
        if x % d == 0:
            return False
        d += 2
    return True
```

Faster Primality Testing

```python
from math import sqrt

def primetest3(x):
    if x < 2 or (x > 2 and x % 2 == 0):
        return False

    d = 3
    while d <= sqrt(x):
        if x % d == 0:
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    return True
```

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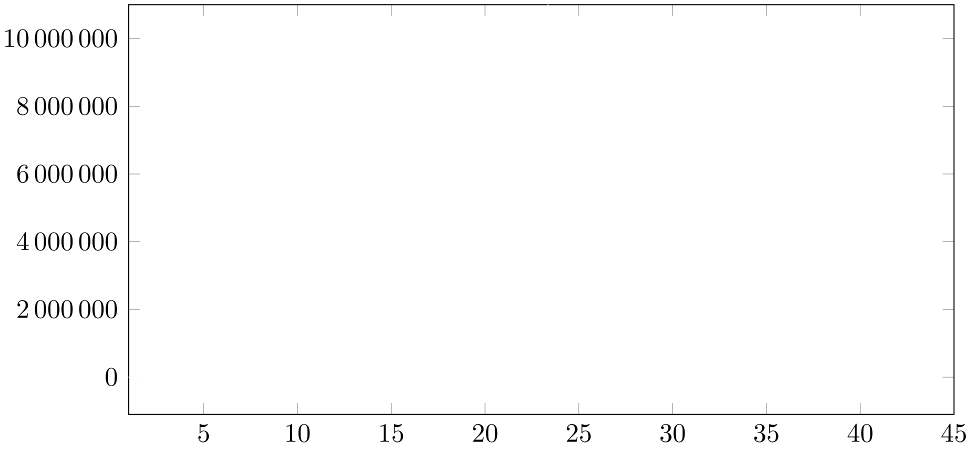
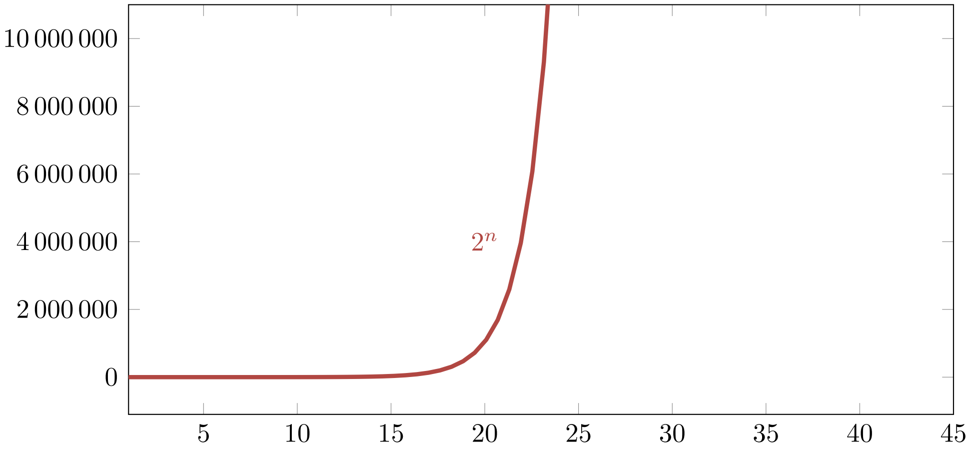
What is the gain this time?

- What is the time complexity of this algorithm?
- Loop is iterated $\sqrt{\mathtt{x}}/2$ times
- Time complexity “grows” with $\sqrt{\mathtt{x}}$
- Time complexity is in $\mathcal{O}(\sqrt{2^n}) = \mathcal{O}(2^{n/2}) = \mathcal{O}(1.415^n)$

Faster Primality Testing



Faster Primality Testing



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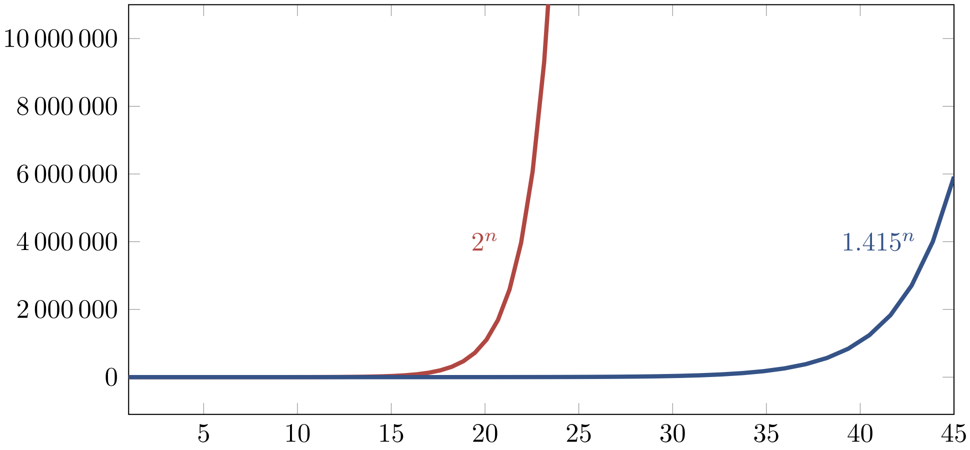

Faster Primality Testing

Suppose our computer can do 1000 iterations of the loop per second

Faster Primality Testing

Suppose our computer can do 1000 iterations of the loop per second; for $\texttt{x} = 100\,000\,000\,000\,031$ this means:

```
... d < x ...
```

$$\frac{100\,000\,000\,000\,031 \text{ iterations}}{1000\ \frac{\text{iterations}}{\text{second}}}$$

$$> 100\,000\,000\,000 \text{ seconds}$$

$$> 3100 \text{ years}$$

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Suppose our computer can do 1000 iterations of the loop per second; for $\mathtt{x} = 100\,000\,000\,000\,031$ this means:

`... d < x ...`	`... d <= sqrt(x) ...`

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$$> 100\,000\,000\,000 \text{ seconds}$$

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$$\frac{\sqrt{100\,000\,000\,000\,031} \text{ iterations}}{1000\ \frac{\text{iterations}}{\text{second}}}$$

$$< \frac{10\,000\,000 \text{ iterations}}{1000\ \frac{\text{iterations}}{\text{second}}}$$

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$$< 3 \text{ hours}$$

Even if the computer that runs the slower program is 100 time faster, it still needs 31 years

Faster Primality Testing

Or the other way around...

Suppose we want to spend 10 minutes

Faster Primality Testing

Or the other way around…

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Then there are at most “testable” primes in the magnitude of:

Faster Primality Testing

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Then there are at most "testable" primes in the magnitude of:

```
... d < x ...
```

$$\frac{\texttt{x}\ \text{iterations}}{1000\ \frac{\text{iterations}}{\text{second}}} = 600\ \text{seconds}$$

$$\Longleftrightarrow \texttt{x} = 600\,000$$

Faster Primality Testing

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Then there are at most "testable" primes in the magnitude of:

```
... d < x ...
```

$$\frac{\mathtt{x}\text{ iterations}}{1000\,\frac{\text{iterations}}{\text{second}}} = 600\text{ seconds}$$

$$\Longleftrightarrow \mathtt{x} = 600\,000$$

```
... d <= sqrt(x) ...
```

$$\frac{\sqrt{\mathtt{x}}\text{ iterations}}{1000\,\frac{\text{iterations}}{\text{second}}} = 600\text{ seconds}$$

$$\Longleftrightarrow \mathtt{x} = 600\,000^2$$

$$\Longleftrightarrow \mathtt{x} = 360\,000\,000\,000$$

Faster Primality Testing

Best and Worst Case Analysis

Best and Worst Case Analysis

Which algorithm is faster?

```
def primetest3(x):
    if x < 2 or (x > 2 and x % 2 == 0):
        return False

    d = 3
    while d <= sqrt(x):
        if x % d == 0:
            return False
        d += 2
    return True
```

```
def primetest4(x):
    if x < 2 or (x > 2 and x % 2 == 0):
        return False

    d = 3
    isprime = True
    while d <= sqrt(x):
        if x % d == 0:
            isprime = False
        d += 2
    return isprime
```

Best and Worst Case Analysis

Suppose `x` is a multiple of 3

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- Then the left algorithm is faster

⇨ Loop is left after first iteration

- "“Early Exit"”
- Right algorithm makes roughly $1.415^n/2$ comparisons

Best and Worst Case Analysis

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- "“Early Exit"”
- Right algorithm makes roughly $1.415^n/2$ comparisons

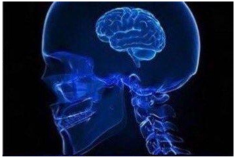
Suppose `x` is prime

- Then both algorithms make $1.415^n/2$ comparisons
- (Of course, still the left one should be implemented)

What else can we do?

Primality test

Test every number between 1 and $\texttt{x}$



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Test every second number
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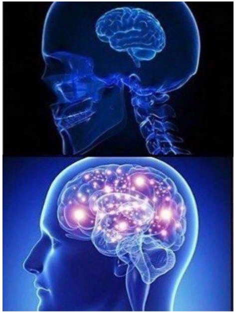

Primality test

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Test every second number
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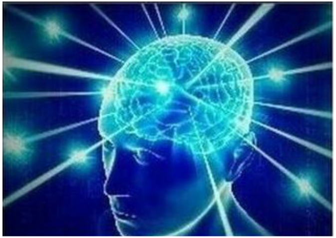
Test every second number
between 1 and $\sqrt{\mathtt{x}}$

Primality test

Randomized Monte Carlo algorithm



Primality test

Randomized Monte Carlo algorithm

Polynomial AKS algorithm

Monte-Carlo Algorithm

Monte-Carlo Algorithm – Basic Idea

Randomized Algorithms make random decisions

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- Input x does not "determine" output anymore
- The same x may result in different outputs
- **Monte-Carlo Algorithm** (MC Algorithm) has bounded error probability
- For `True`/`False` problems (primality test etc.) there are MC algorithms with one-sided error (1MC algorithms)

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- **Las Vegas Algorithm** has error probability 0

Monte-Carlo Algorithm (1MC) – Example

Consider urn with 10^{100} balls colored white (and possibly red)

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Red balls are **witnesses** for claim

Simplified Solovay-Strassen Algorithm

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- Test whether `x` is a prime

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For $\mathtt{x} = p \cdot q$ with p and q being primes, probability to find a witness is

$$\frac{2}{\mathtt{x} - 2}$$

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- (Not exactly trivial number theory)
- **Fermat’s little theorem**

If $\mathtt{x}$ is prime $\Rightarrow a^{\mathtt{x}-1} \equiv 1 \pmod{\mathtt{x}} \quad \forall a \in \{2, \ldots, \mathtt{x}-1\}$

Pierre de Fermat (1607–1665)

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- If $\mathtt{x}$ is prime $\quad\Rightarrow\quad a^{\mathtt{x}-1} \mod \mathtt{x} = 1 \quad \forall a \in \{2, \ldots, \mathtt{x}-1\}$

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  - a is witness that $\mathtt{x}$ is no prime
  - It can be proven that there are $> (\mathtt{x}-2)/2$ witnesses

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- Can be computed in polynomial time
- Time complexity $\mathcal{O}(n^3)$ instead of $\mathcal{O}(1.415^n)$
- Efficient algorithm

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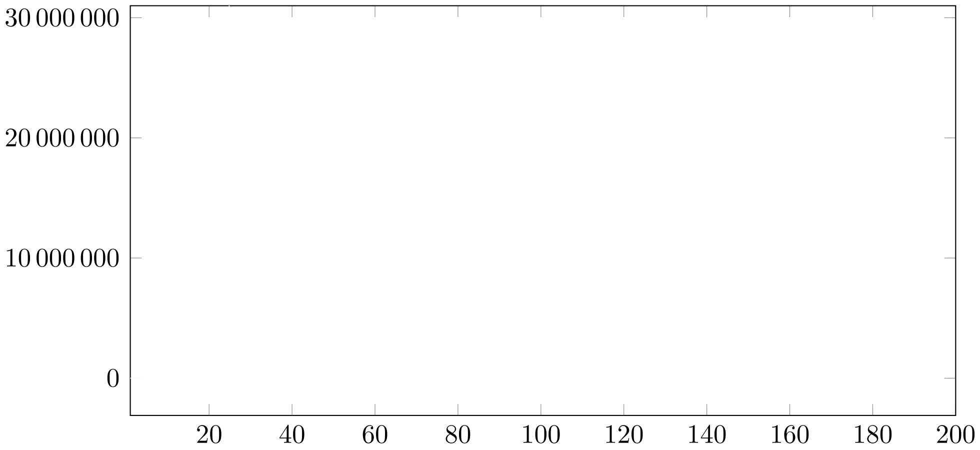

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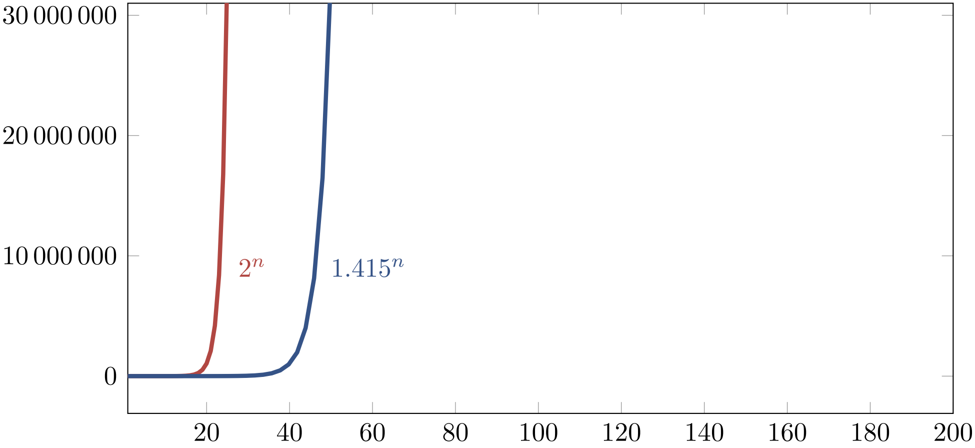

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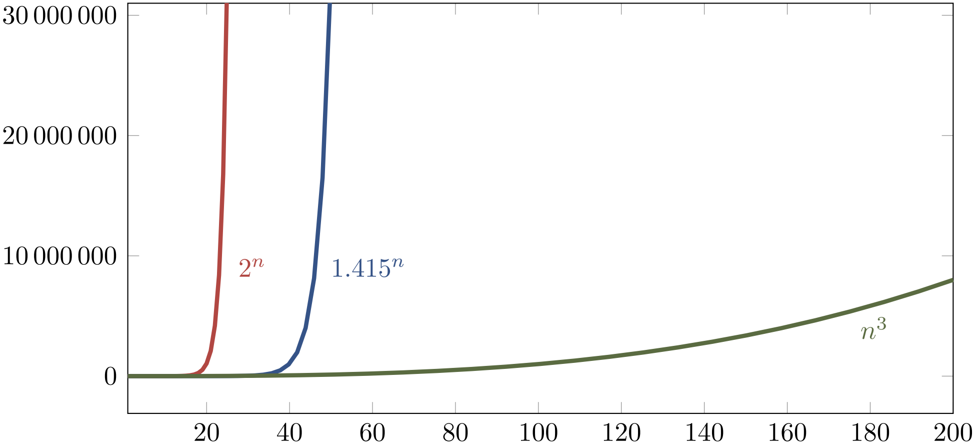

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- Correct output with probability 1
- Suppose $\mathtt{x}$ is no prime
- At least half of $\{2, \ldots, \mathtt{x} - 1\}$ are witnesses
- Correct output with probability $1/2$

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Thanks for your attention

ETH zürich