ETH zürich

Departement Informatik

Programming and Problem-Solving

Graphs and Graph Algorithms

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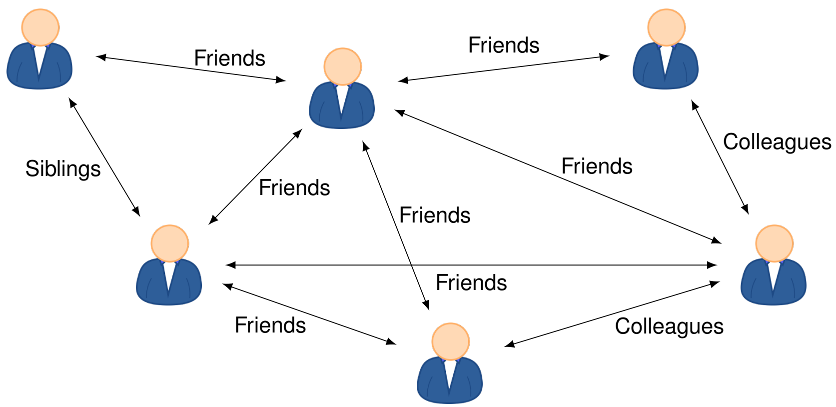
Spring 2021 – May 27, 2021

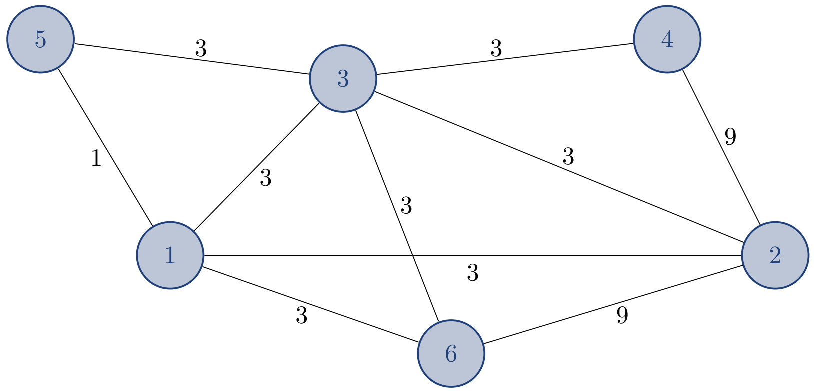
Graphs

Searching in Networks

Social Network



Abstract Modelling



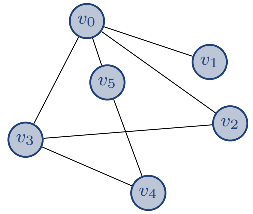
Abstract Modelling

A **graph** $G = (V, E, w)$ consists of

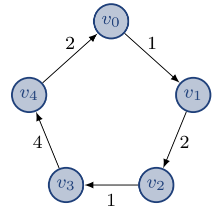
1. a set V of vertices
2. a set E of edges between some of the vertices
3. (a weight function w)

- Vertices are called $v_0, v_1, v_2, \ldots$
- Graphs are either **weighted** or **unweighted**
- Graphs are either **directed** or **undirected**
- Graphs are either **connected** or **unconnected**

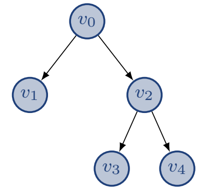
Abstract Modelling



Undirected unweighted graph



Undirected weighted graph



Directed unweighted graph

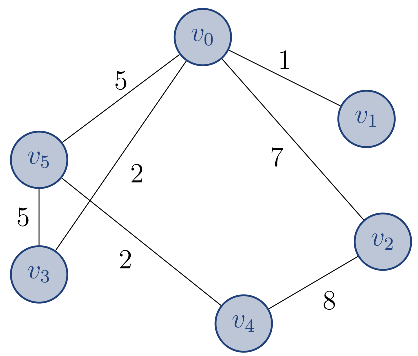
Which type of graph is used depends on what we want to model

We mostly consider undirected, unweighted, connected graphs

Graphs

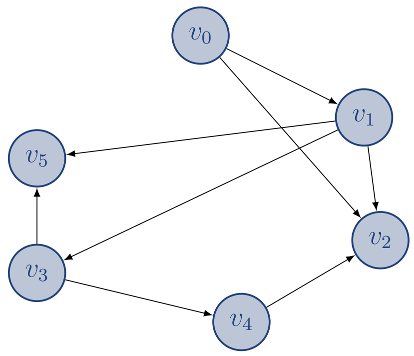
On the Computer

Adjacency Matrices – Undirected Weighted Graphs



$$\begin{pmatrix} 0 & 1 & 7 & 2 & 0 & 5 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 7 & 0 & 0 & 0 & 8 & 0 \\ 2 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 8 & 0 & 0 & 2 \\ 5 & 0 & 0 & 5 & 2 & 0 \end{pmatrix}$$

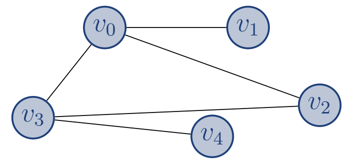
Adjacency Matrices – Directed Unweighted Graphs



$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

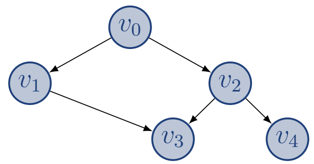
Adjacency Matrices – Directed / Undirected Graphs

Matrices of undirected graphs are symmetric



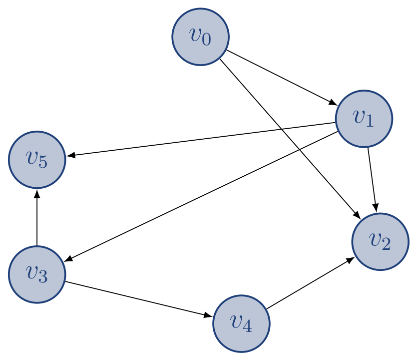
$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Matrices of directed graphs are not (always) symmetric



$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Adjacency Lists – Directed Unweighted Graphs



$$\begin{array}{l} ((1,2), \\ (2,3,5), \\ (), \\ (4,5), \\ (2), \\ ()) \end{array}$$

Adjacency Matrices and Lists in Python

Use 2-dimensional lists

Matrix: Weighted

```
G = [ [ 0, 1, 7, 2, 0, 5 ],
      [ 1, 0, 0, 0, 0, 0 ],
      [ 7, 0, 0, 0, 8, 0 ],
      [ 2, 0, 0, 0, 0, 5 ],
      [ 0, 0, 8, 0, 0, 2 ],
      [ 5, 0, 0, 5, 2, 0 ] ]
```

Matrix: Unweighted

```
G = [ [ 0, 1, 1, 0, 0, 0 ],
      [ 1, 0, 1, 1, 0, 1 ],
      [ 1, 1, 0, 0, 1, 0 ],
      [ 0, 1, 0, 0, 1, 1 ],
      [ 0, 0, 1, 1, 0, 0 ],
      [ 0, 1, 0, 1, 0, 0 ] ]
```

List: Unweighted

```
G = [ [1,2], [0,2,3,5], [0,1,4], [1,4,5], [2,3], [1,3] ]
```

Graph Algorithms

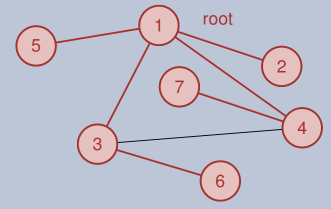
Breadth-First and Depth-First Search

Breadth-First (BFS) and Depth-First Search (DFS)

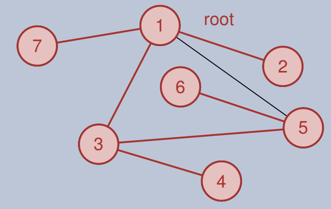
Many applications need the systematic exploration of a given graph

- Start and an arbitrary vertex
- Follow edges through graph
- Store vertices in the respective order

BFS: First go broadly and than deeply, just as with the Heap; break ties in favor of smaller indices



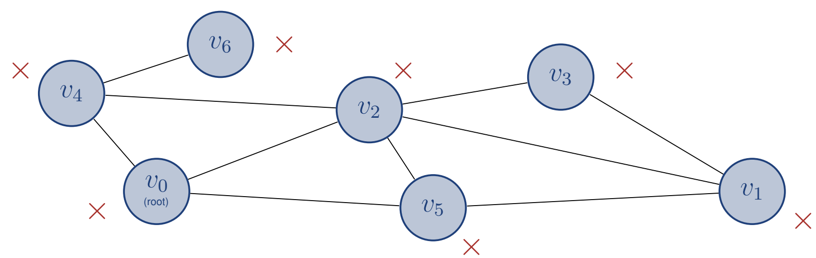
DFS: Go into the graph as deep as possible, then broadly; again break ties in favor of smaller indices

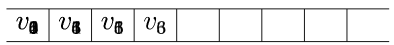


Breadth-First Search

Iteratively with a Queue

BFS with a Queue



Queue: 

Output: v_0 v_2 v_4 v_5 v_1 v_3 v_6

BFS with Queue and Adjacency Matrix

```
G = [ [0,0,1,0,1,1,0], [0,0,1,1,0,1,0], [1,1,0,1,1,1,0], [0,1,1,0,0,0,0],
      [1,0,1,0,0,0,1], [1,1,1,0,0,0,0], [0,0,0,0,1,0,0] ]
queue = []
visited = [ 0 for i in range(len(G)) ]
```

- Consider first vertex in queue and print it
- Add unvisited neighbors to queue
- `visited` stores which vertices have been visited
- Repeat as long as queue is not empty

Exercise – BFS with Queue and Adjacency Matrix

Implement BFS

- as a Python function
- with a 2-dimensional list as parameter
- using a queue
- and an adjacency matrix

```
G = [ [0,0,1,0,1,1,0], [0,0,1,1,0,1,0], [1,1,0,1,1,1,0], [0,1,1,0,0,0,0],
      [1,0,1,0,0,0,1], [1,1,1,0,0,0,0], [0,0,0,0,1,0,0] ]
```


BFS with Queue and a Adjacency Matrix

```
def BFS(G):
    queue = []
    visited = [ 0 for i in range(len(G)) ]
    queue.append(0)
    visited[0] = 1
    while len(queue) > 0:
        current = queue.pop(0)
        print(current, end=" ")
        for j in range(len(G)):
            if G[current][j] == 1 and visited[j] == 0:
                visited[j] = 1
                queue.append(j)
BFS([ [0,0,1,0,1,1,0], [0,0,1,1,0,1,0], [1,1,0,1,1,1,0], [0,1,1,0,0,0,0],
     [1,0,1,0,0,0,1], [1,1,1,0,0,0,0], [0,0,0,0,1,0,0] ])
```


Exercise – BFS with Queue and Adjacency List

Implement BFS

- as a Python function
- with a 2-dimensional list as parameter
- using a queue
- and an adjacency list

```
G = [ [2,4,5], [2,3,5], [0,1,3,4,5], [1,2],
      [0,2,6], [0,1,2], [4] ]
```

BFS with Queue and a Adjacency List

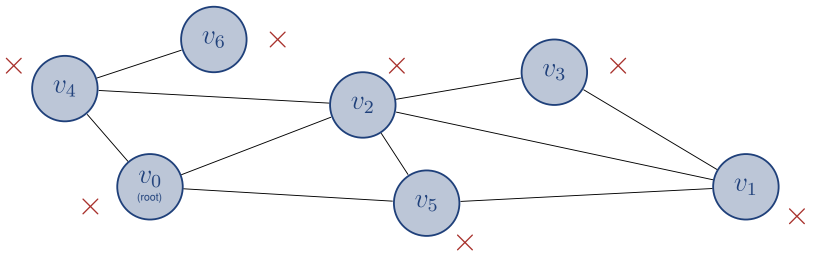
```python
def BFS(G):
    queue = []
    visited = [ 0 for i in range(len(G)) ]
    queue.append(0)
    visited[0] = 1
    while len(queue) > 0:
        current = queue.pop(0)
        print(current, end=" ")
        for j in G[current]:
            if visited[j] == 0:
                visited[j] = 1
                queue.append(j)

BFS([ [2,4,5], [2,3,5], [0,1,3,4,5], [1,2], [0,2,6], [0,1,2], [4] ])
```


Depth-First Search

Iteratively with a Stack

DFS with a Stack



Stack:

v_0/v_5	v_4	v_2/v_3	v_1/v_6	v_3	v_1/v_5	v_3		

Output: v_0 v_2 v_1 v_3 v_5 v_4 v_6

DFS with Stack and Adjacency Matrix

```python
G = [ [0, 0, 1, 0, 1, 1, 0],
      [0, 0, 1, 1, 0, 1, 0],
      [1, 1, 0, 1, 1, 1, 0],
      [0, 1, 1, 0, 0, 0, 0],
      [1, 0, 1, 0, 0, 0, 1],
      [1, 1, 1, 0, 0, 0, 0],
      [0, 0, 0, 0, 1, 0, 0] ]
stack = []
visited = [ 0 for i in range(len(G)) ]
```

- Consider first vertex in stack and print it
- Add unvisited neighbors to stack
- `visited` stores which vertices have been visited
- Repeat as long as stack is not empty

Exercise – DFS with Stack and Adjacency Matrix

Implement DFS

- as a Python function
- with a 2-dimensional list as parameter
- using a stack
- and an adjacency matrix

```
G = [ [0,0,1,0,1,1,0], [0,0,1,1,0,1,0],
      [1,1,0,1,1,1,0], [0,1,1,0,0,0,0],
      [1,0,1,0,0,0,1], [1,1,1,0,0,0,0],
      [0,0,0,0,1,0,0] ]
```

DFS with Stack and Adjacency Matrix

```
def DFS(G):
    stack = []
    visited = [ 0 for i in range(len(G)) ]
    stack.append(0)
    while len(stack) > 0:
        current = stack.pop()
          if visited[current] == 0:
               visited[current] = 1
               print(current, end=" ")
               for j in reversed(range(len(G))):
                   if G[current][j] == 1 and visited[j] == 0:
                       stack.append(j)

DFS( [ [0,0,1,0,1,1,0], [0,0,1,1,0,1,0], [1,1,0,1,1,1,0], [0,1,1,0,0,0,0],
      [1,0,1,0,0,0,1], [1,1,1,0,0,0,0], [0,0,0,0,1,0,0] ] )
```


Exercise – DFS with Stack and Adjacency List

Implement DFS

- as a Python function
- with a 2-dimensional list as parameter
- using a stack
- and an adjacency matrix

```
G = [ [2,4,5], [2,3,5], [0,1,3,4,5], [1,2], [0,2,6], [0,1,2], [4] ]
```


DFS with Stack and Adjacency List

```
def DFS(G):
    stack = []
    visited = [ 0 for i in range(len(G)) ]
    stack.append(0)
    while len(stack) > 0:
        current = stack.pop()
        if visited[current] == 0:
             visited[current] = 1
             print(current, end=" ")
             for j in reversed(G[current]):
                 if visited[j] == 0:
                     stack.append(j)
DFS([ [2,4,5], [2,3,5], [0,1,3,4,5], [1,2], [0,2,6], [0,1,2], [4] ])
```

Depth-First Search

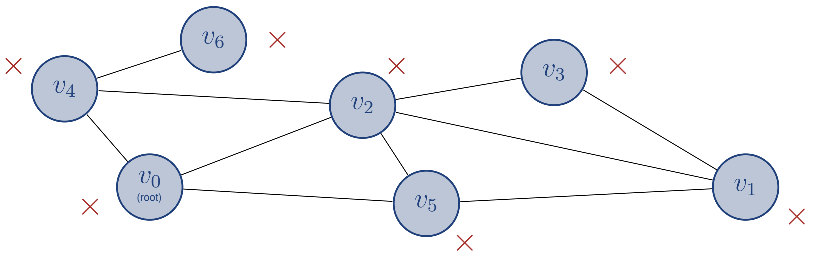
Recursively

Recursive DFS

- Global list `visited`
- Function `DFS`, which is called recursively
- Two parameters
  1. graph `G`
  2. Start vertex `current`

```
visited = [ 0 for i in range(len(G)) ]
def DFS(G, current):
    visited[current] = 1
    print(current, end=" ")
    for i in range(len(G)):
        if G[current][i] == 1 and visited[i] == 0:
            DFS(G, i)
```


Recursive DFS



`DFS(G,0)` → `DFS(G,2)` → `DFS(G,4)` → `DFS(G,6)`

Output: v_0 v_2 v_1 v_3 v_5 v_4 v_6

Recursive DFS

Applications

Applications

Is graph connected?

⇨ DFS from arbitrary vertex; are all vertices visited when done?

Is vertex w reachable from vertex v?

⇨ DFS from v; is w visited when done?

Is a graph 2-colorable?

⇨ DFS from arbitrary vertex and color levels differently

Does a graph contain a cycle?

⇨ DFS from arbitrary vertex; is there a back edge?

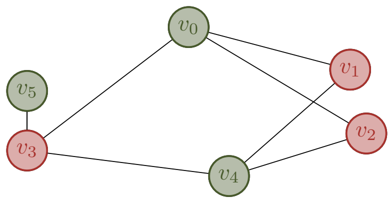
Recursive DFS

Graph Coloring

Graph Coloring

Consider arbitrary graph

- Can it be colored with two colors?
- Connected vertices (“neighbors”) have different color
- Compute recursively



- List `color` instead of `visited`
- 0: not yet visited
- 1: colored green
- 2: colored red

Graph Coloring

We use recursive DFS

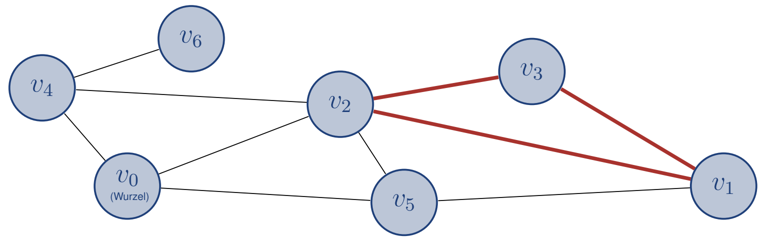
- All neighbors of `current` get a color different from that of `current`
- If neighbor already has same color as `current`, coloring is invalid

```
def coloring(G, current):
    for i in range(len(G)):
        if G[current][i] == 1 and color[i] == 0:
            color[i] = 3 - color[current]
            coloring(G, i)
        elif G[current][i] == 1 and color[i] == color[current]:
            print("Coloring impossible.")
            return
```

Recursive DFS

Finding Cycles

Finding Cycles



- DFS computes this
- Traverse graph as before
- Is there an edge to a vertex we already visited?
- Back-Edge
- **Attention:** Single edge is not a cycle

Finding Cycles

Compute whether graph contains a cycle

Extend DFS such that parent is considered

```
def find_cycle(G, current, parent):
    visited[current] = 1
    print(current, end=" ")
    for i in range(len(G)):
        if G[current][i] == 1 and visited[i] == 0:
            find_cycle(G, i, current)
        elif G[current][i] == 1 and visited[i] == 1 and i != parent:
            print("Found cycle.")
            return
```