# 8. Floating-point Numbers II

Floating-point Number Systems; IEEE Standard; Limits of Floating-point Arithmetics; Floating-point Guidelines; Harmonic Numbers



## Floating-point Number Systems

A Floating-point number system is defined by the four natural numbers:

- $\beta \geq 2$, the base,
- $p \geq 1$, the precision (number of places),
- $e_{\min}$, the smallest possible exponent,
- $e_{\max}$, the largest possible exponent.

Notation:

$$F(\beta, p, e_{\min}, e_{\max})$$



## Floating-point number Systems

$F(\beta, p, e_{\min}, e_{\max})$ contains the numbers

$$\pm \sum_{i=0}^{p-1} d_i \beta^{-i} \cdot \beta^e,$$

$$d_i \in \{0, \ldots, \beta - 1\}, \quad e \in \{e_{\min}, \ldots, e_{\max}\}.$$

represented in base $\beta$:

$$\pm\, d_{0\bullet} d_1 \ldots d_{p-1} \times \beta^e,$$



## Floating-point Number Systems

Example

- $\beta = 10$

Representations of the decimal number 0.1

$$1.0 \cdot 10^{-1}, \quad 0.1 \cdot 10^{0}, \quad 0.01 \cdot 10^{1}, \quad \ldots$$

# Normalized representation

Normalized number:

$$\pm\, d_{0\bullet}d_1 \ldots d_{p-1} \times \beta^e, \qquad d_0 \neq 0$$

**Remark 1**

The normalized representation is unique and therefore prefered.

**Remark 2**

The number 0 (and all numbers smaller than $\beta^{e_{\min}}$) have no normalized representation (we will deal with this later)!



# Set of Normalized Numbers

$$F^*(\beta, p, e_{\min}, e_{\max})$$



# Normalized Representation

**Example $F^*(2, 3, -2, 2)$ (only positive numbers)**

| $d_{0\bullet}d_1d_2$ | $e=-2$ | $e=-1$ | $e=0$ | $e=1$ | $e=2$ |
|---|---|---|---|---|---|
| $1.00_2$ | 0.25 | 0.5 | 1 | 2 | 4 |
| $1.01_2$ | 0.3125 | 0.625 | 1.25 | 2.5 | 5 |
| $1.10_2$ | 0.375 | 0.75 | 1.5 | 3 | 6 |
| $1.11_2$ | 0.4375 | 0.875 | 1.75 | 3.5 | 7 |




# Binary and Decimal Systems

- Internally the computer computes with $\beta = 2$ (binary system)
- Literals and inputs have $\beta = 10$ (decimal system)
- Inputs have to be converted!

## Conversion Decimal → Binary

Assume, $0 < x < 2$.

Binary representation:

$$x = \sum_{i=-\infty}^{0} b_i 2^i = b_0 \bullet b_{-1} b_{-2} b_{-3} \dots$$

$$= b_0 + \sum_{i=-\infty}^{-1} b_i 2^i = b_0 + \sum_{i=-\infty}^{0} b_{i-1} 2^{i-1}$$

$$= b_0 + \underbrace{\left( \sum_{i=-\infty}^{0} b_{i-1} 2^i \right)}_{x' = b_{-1} \bullet b_{-2} b_{-3} b_{-4}} / 2$$



## Conversion Decimal → Binary

Assume $0 < x < 2$.

- Hence: $x' = b_{-1} \bullet b_{-2} b_{-3} b_{-4} \dots = 2 \cdot (x - b_0)$
- Step 1 (for $x$): Compute $b_0$:

$$b_0 = \begin{cases} 1, & \text{if } x \geq 1 \\ 0, & \text{otherwise} \end{cases}$$

- Step 2 (for $x$): Compute $b_{-1}, b_{-2}, \dots$:

  Go to step 1 (for $x' = 2 \cdot (x - b_0)$)



## Binary representation of $1.1$

| $x$ | $b_i$ | $x - b_i$ | $2(x - b_i)$ |
|---|---|---|---|
| 1.1 | $b_0 = \mathbf{1}$ | 0.1 | 0.2 |
| 0.2 | $b_{-1} = \mathbf{0}$ | 0.2 | 0.4 |
| 0.4 | $b_{-2} = \mathbf{0}$ | 0.4 | 0.8 |
| 0.8 | $b_{-3} = \mathbf{0}$ | 0.8 | 1.6 |
| 1.6 | $b_{-4} = \mathbf{1}$ | 0.6 | 1.2 |
| 1.2 | $b_{-5} = \mathbf{1}$ | 0.2 | 0.4 |

$\Rightarrow 1.0\overline{0011}$, periodic, *not* finite



## Binary Number Representations of $1.1$ and $0.1$

- are not finite, hence there are errors when converting into a (finite) binary floating-point system.
- `1.1f` and `0.1f` do not equal $1.1$ and $0.1$, but are slightly inaccurate approximation of these numbers.
- In `diff.cpp`: $1.1 - 1.0 \neq 0.1$

## Binary Number Representations of $1.1$ and $0.1$

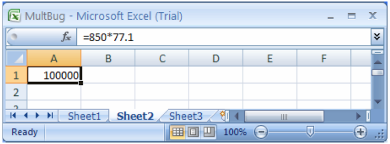
on my computer:

$$
\begin{aligned}
\texttt{1.1} &= \underline{1.1000000000000000}888178\ldots \\
\texttt{1.1f} &= \underline{1.1000000}238418\ldots
\end{aligned}
$$



## The Excel-2007-Bug

```
std::cout << 850 * 77.1; // 65535
```




- $77.1$ does not have a finite binary representation, we obtain $65534.9999999999927\ldots$
- For this and exactly 11 other "rare" numbers the output (and only the output) was wrong.

http://www.lomont.org/Math/Papers/2007/Excel2007Bug.pdf



## Computing with Floating-point Numbers

Example ($\beta = 2, p = 4$):

$$
\begin{array}{rr}
 & 1.111 \cdot 2^{-2} \\
+ & 1.011 \cdot 2^{-1} \\
\hline
 & = 1.001 \cdot 2^{0}
\end{array}
$$

1. adjust exponents by denormalizing one number 2. binary addition of the significands 3. renormalize 4. round to $p$ significant places, if necessary



## The IEEE Standard 754

- defines floating-point number systems and their rounding behavior
- is used nearly everywhere
- Single precision (**`float`**) numbers:

  $F^*(2, 24, -126, 127)$ plus $0, \infty, \ldots$
- Double precision (**`double`**) numbers:

  $F^*(2, 53, -1022, 1023)$ plus $0, \infty, \ldots$
- All arithmetic operations round the *exact* result to the next representable number

## The IEEE Standard 754

Why

$$F^*(2, 24, -126, 127)?$$

- 1 sign bit
- 23 bit for the significand (leading bit is 1 and is not stored)
- 8 bit for the exponent (256 possible values)(254 possible exponents, 2 special values: $0$, $\infty$,...)

$\Rightarrow$ 32 bit in total.



## The IEEE Standard 754

Why

$$F^*(2, 53, -1022, 1023)?$$

- 1 sign bit
- 52 bit for the significand (leading bit is 1 and is not stored)
- 11 bit for the exponent (2046 possible exponents, 2 special values: $0$, $\infty$,...)

$\Rightarrow$ 64 bit in total.



## Floating-point Rules — Rule 1

**Rule 1**
Do not test rounded floating-point numbers for equality.

```
for (float i = 0.1; i != 1.0; i += 0.1)
    std::cout << i << "\n";
```

endless loop because i never becomes exactly 1



## Floating-point Rules — Rule 2

**Rule 2**
Do not add two numbers of very different orders of magnitude!

$$\begin{aligned} & 1.000 \cdot 2^5 \\ & +1.000 \cdot 2^0 \\ & = 1.00001 \cdot 2^5 \\ & \text{“=”}\ 1.000 \cdot 2^5 \end{aligned}$$ (Rounding on 4 places)

Addition of 1 does not have any effect!

- The $n$-the harmonic number is

$$H_n = \sum_{i=1}^{n} \frac{1}{i} \approx \ln n.$$

- This sum can be computed in forward or backward direction, which is mathematically clearly equivalent





```
// Program: harmonic.cpp
// Compute the n-th harmonic number in two ways.

#include <iostream>

int main()
{
  // Input
  std::cout << "Compute H_n for n =? ";
  unsigned int n;
  std::cin >> n;

  // Forward sum
  float fs = 0;
  for (unsigned int i = 1; i <= n; ++i)
    fs += 1.0f / i;

  // Backward sum
  float bs = 0;
  for (unsigned int i = n; i >= 1; --i)
    bs += 1.0f / i;

  // Output
  std::cout << "Forward sum  = " << fs << "\n"
            << "Backward sum = " << bs << "\n";
  return 0;
}
```





Results:

- ```
  Compute H_n for n =?  10000000
  Forward sum = 15.4037
  Backward sum = 16.686
  ```
- ```
  Compute H_n for n =?  100000000
  Forward sum = 15.4037
  Backward sum = 18.8079
  ```





Observation:

- The forward sum stops growing at some point and is “really” wrong.
- The backward sum approximates $H_n$ well.

Explanation:

- For $1 + 1/2 + 1/3 + \cdots$ , later terms are too small to actually contribute
- Problem similar to $2^5 + 1$ “=” $2^5$

**Rule 4**

Do not subtract two numbers with a very similar value.

Cancellation problems, cf. lecture notes.



## Literature

David Goldberg: What Every Computer Scientist Should Know About Floating-Point Arithmetic (1991)


Randy Glasbergen, 1996



# 9. Functions I

Defining and Calling Functions, Evaluation of Function Calls, the Type `void`, Pre- and Post-Conditions



## Functions

- encapsulate functionality that is frequently used (e.g. computing powers) and make it easily accessible
- structure a program: partitioning into small sub-tasks, each of which is implemented as a function

$\Rightarrow$ Procedural programming; procedure: a different word for function.

## Example: Computing Powers

```
double a;
int n;
std::cin >> a; // Eingabe a
std::cin >> n; // Eingabe n

double result = 1.0;
if (n < 0) { // a^n = (1/a)^(−n)
  a = 1.0/a;
  n = −n;
}
for (int i = 0; i < n; ++i)
  result *= a;
```

"Funktion pow"

```
std::cout << a << "^" << n << " = " << resultpow(a,n) << ".\n";
```



## Function to Compute Powers

```
// PRE: e >= 0 || b != 0.0
// POST: return value is b^e
double pow(double b, int e)
{
    double result = 1.0;
    if (e < 0) { // b^e = (1/b)^(−e)
        b = 1.0/b;
        e = −e;
    }
    for (int i = 0; i < e; ++i)
        result *= b;
    return result;
}
```



## Function to Compute Powers

```
// Prog: callpow.cpp
// Define and call a function for computing powers.
#include <iostream>

double pow(double b, int e){...}

int main()
{
  std::cout << pow( 2.0, −2) << "\n"; // outputs 0.25
  std::cout << pow( 1.5, 2) << "\n"; // outputs 2.25
  std::cout << pow(−2.0, 9) << "\n"; // outputs −512

  return 0;
}
```



## Function Definitions

*return type* *argument types*

$T$ fname ($T_1$ pname$_1$, $T_2$ pname$_2$, $\ldots$, $T_N$ pname$_N$)
block

*body*

*function name* *formal arguments*

## Defining Functions

- may not occur *locally*, i.e. not in blocks, not in other functions and not within control statements
- can be written consecutively without separator in a program

```
double pow (double b, int e)
{
    ...
}

int main ()
{
    ...
}
```



## Example: Xor

```
// post: returns l XOR r
bool Xor(bool l, bool r)
{
    return l && !r || !l && r;
}
```



## Example: Harmonic

```
// PRE: n >= 0
// POST: returns nth harmonic number
//       computed with backward sum
float Harmonic(int n)
{
    float res = 0;
    for (unsigned int i = n; i >= 1; --i)
        res += 1.0f / i;
    return res;
}
```



## Example: min

```
// POST: returns the minimum of a and b
int min(int a, int b)
{
    if (a<b)
        return a;
    else
        return b;
}
```

## Function Calls

fname ( *expression*$_1$, *expression*$_2$, ..., *expression*$_N$)

- All call arguments must be convertible to the respective formal argument types.
- The function call is an expression of the return type of the function. Value and effect as given in the postcondition of the function *fname*.

Example: `pow(a,n)`: Expression of type `double`



## Function Calls

For the types we know up to this point it holds that:

- Call arguments are R-values
- The function call is an R-value.

*fname:* R-value $\times$ R-value $\times \cdots \times$ R-value $\longrightarrow$ R-value



## Evaluation of a Function Call

- Evaluation of the call arguments
- Initialization of the formal arguments with the resulting values
- Execution of the function body: formal arguments behave laike local variables
- Execution ends with
  `return` *expression*;

Return value yiels the value of the function call.



## Example: Evaluation Function Call

```
double pow(double b, int e){
    assert (e >= 0 || b != 0);
    double result = 1.0;
    if (e<0) {
        // b^e = (1/b)^(-e)
        b = 1.0/b;
        e = -e;
    }
    for (int i = 0; i < e ; ++i)
        result * = b;
    return result;
}

...
pow (2.0, -2)
```

Call of pow

Return

## Formal arguments

- Declarative region: function definition
- are *invisible* outside the function definition
- are allocated for each call of the function (automatic storage duration)
- modifications of their value do not have an effect to the values of the call arguments (call arguments are R-values)



## Scope of Formal Arguments

```
double pow(double b, int e){
    double r = 1.0;
    if (e<0) {
        b = 1.0/b;
        e = -e;
    }
    for (int i = 0; i < e ; ++i)
        r * = b;
    return r;
}
```

```
int main(){
    double b = 2.0;
    int e = -2;
    double z = pow(b, e);

    std::cout << z; // 0.25
    std::cout << b; // 2
    std::cout << e; // -2
    return 0;
}
```

Not the formal arguments `b` and `e` of pow but the variables defined here locally in the body of `main`



## The type `void`

- Fundamental type with empty value range
- Usage as a return type for functions that do *only* provide an effect

```
// POST: "(i, j)" has been written to
//       standard output
void print_pair (int i, int j)
{
   std::cout << "(" << i << ", " << j << ")\n";
}

int main()
{
   print_pair(3,4); // outputs (3, 4)
   return 0;
}
```



## `void`-Functions

- do not require **`return`**.
- execution ends when the end of the function body is reached or if
- **`return;`** is reached
  or
- **`return`** *expression*; is reached.

Expression with type `void` (e.g. a call of a function with return type `void`

## Pre- and Postconditions

- characterize (as complete as possible) what a function does
- document the function for users and programmers (we or other people)
- make programs more readable: we do not have to understand *how* the function works
- are ignored by the compiler
- Pre and postconditions render statements about the correctness of a program possible – provided they are correct.



## Preconditions

precondition:

- what is required to hold when the function is called?
- defines the *domain* of the function

$0^e$ is undefined for $e < 0$

```
// PRE: e >= 0 || b != 0.0
```



## Postconditions

postcondition:

- What is guaranteed to hold after the function call?
- Specifies *value* and *effect* of the function call.

Here only value, no effect.

```
// POST: return value is b^e
```



## Pre- and Postconditions

- should be correct:
- *if* the precondition holds when the function is called *then* also the postcondition holds after the call.

Funktion `pow`: works for all numbers $b \neq 0$

## Pre- and Postconditions

- We do not make a statement about what happens if the precondition does not hold.
- C++-standard-slang: „Undefined behavior".

Function `pow`: division by 0



## Pre- and Postconditions

- pre-condition should be as *weak* as possible (largest possible domain)
- post-condition should be as *strong* as possible (most detailed information)



## White Lies. . .

```
// PRE: e >= 0 || b != 0.0
// POST: return value is b^e
```

is formally incorrect:

- Overflow if e or b are too large
- $b^e$ potentially not representable as a double (holes in the value range!)



## White Lies are Allowed

```
// PRE: e >= 0 || b != 0.0
// POST: return value is b^e
```

The exact pre- and postconditions are platform-dependent and often complicated. We abstract away and provide the mathematical conditions. ⇒ compromise between formal correctness and lax practice.

## Checking Preconditions...

- Preconditions are only comments.
- How can we ensure that they hold when the function is called?



## ...with assertions

```
#include <cassert>
...
// PRE: e >= 0 || b != 0.0
// POST: return value is b^e
double pow(double b, int e) {
    assert (e >= 0 || b != 0);
    double result = 1.0;
    ...
}
```



## Postconditions with Asserts

- The result of “complex” computations is often easy to check.
- Then the use of asserts for the postcondition is worthwhile.

```
// PRE: the discriminant p*p/4 - q is nonnegative
// POST: returns larger root of the polynomial x^2 + p x + q
double root(double p, double q)
{
    assert(p*p/4 >= q); // precondition
    double x1 = - p/2 + sqrt(p*p/4 - q);
    assert(equals(x1*x1+p*x1+q,0)); // postcondition
    return x1;
}
```



## Exceptions

- Assertions are a rough tool; if an assertions fails, the program is halted in a unrecoverable way.
- C++provides more elegant means (exceptions) in order to deal with such failures depending on the situation and potentially without halting the program
- Failsafe programs should only halt in emergency situations and therefore should work with exceptions. For this course, however, this goes too far.