8. Floating-point Numbers II

Floating-point Number Systems; IEEE Standard; Limits of Floating-point Arithmetics; Floating-point Guidelines; Harmonic Numbers

Floating-point Number Systems

A Floating-point number system is defined by the four natural numbers:

- $\beta \geq 2$, the base,
- $p \geq 1$, the precision (number of places),
- $e_{\min}$, the smallest possible exponent,
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Notation:

$$F(\beta, p, e_{\min}, e_{\max})$$

Floating-point number Systems

$F(\beta, p, e_{\min}, e_{\max})$ contains the numbers

$$\pm \sum_{i=0}^{p-1} d_i \beta^{-i} \cdot \beta^e,$$

$$d_i \in \{0, \ldots, \beta - 1\}, \quad e \in \{e_{\min}, \ldots, e_{\max}\}.$$

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represented in base β:

$$\pm\, d_{0\bullet} d_1 \ldots d_{p-1} \times \beta^e,$$

Floating-point Number Systems

Representations of the decimal number 0.1 (with $\beta = 10$):

$$1.0 \cdot 10^{-1}, \quad 0.1 \cdot 10^{0}, \quad 0.01 \cdot 10^{1}, \quad \ldots$$

Different representations due to choice of exponent

Normalized representation

Normalized number:

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Remark 1

The normalized representation is unique and therefore prefered.

Normalized representation

Normalized number:

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Remark 2

The number 0, as well as all numbers smaller than $\beta^{e_{\min}}$, have no normalized representation (we will come back to this later)

Set of Normalized Numbers

$$F^*(\beta, p, e_{\min}, e_{\max})$$

Normalized Representation

Example $F^*(2, 3, -2, 2)$ (only positive numbers)

$d_{0\bullet}d_1d_2$	$e = -2$	$e = -1$	$e = 0$	$e = 1$	$e = 2$
1.00_2	0.25	0.5	1	2	4
1.01_2	0.3125	0.625	1.25	2.5	5
1.10_2	0.375	0.75	1.5	3	6
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$1.00 \cdot 2^{-2} = \frac{1}{4}$ $\qquad$ $1.11 \cdot 2^{2} = 7$

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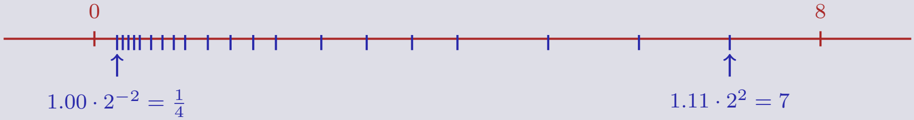

Binary and Decimal Systems

- Internally the computer computes with $\beta = 2$ (binary system)
- Literals and inputs have $\beta = 10$ (decimal system)

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Conversion $(0 < x < 2)$

Computation of the *binary representation*:

$$x = \sum_{i=0}^{\infty} b_i 2^{-i}$$

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$$x = b_0 \bullet b_1 b_2 b_3 \ldots$$

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Computation of the *binary representation*:

$$\begin{aligned} x &= b_0{\bullet}b_1b_2b_3\ldots \\ &= b_0 + 0{\bullet}b_1b_2b_3\ldots \end{aligned}$$

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$$\begin{aligned} x &= b_0{\bullet}b_1b_2b_3\ldots \\ &= b_0 + 0{\bullet}b_1b_2b_3\ldots \\ &\Longrightarrow \\ (x - b_0) &= 0{\bullet}b_1b_2b_3b_4\ldots \end{aligned}$$

Conversion $(0 < x < 2)$

Computation of the *binary representation*:

$$
\begin{aligned}
x &= b_0{\bullet}b_1b_2b_3\ldots \\
&= b_0 + 0{\bullet}b_1b_2b_3\ldots \\
&\Longrightarrow \\
2\cdot(x-b_0) &= b_1{\bullet}b_2b_3b_4\ldots
\end{aligned}
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```
for (int b_0; x != 0; x = 2 * (x - b_0)) {
  b_0 = (x >= 1);
  std::cout << b_0;
}
```

Example (binary)

$$\begin{aligned} x &= \mathbf{1}_{\bullet}01011 \\ &= \mathbf{1} + 0_{\bullet}01011 \\ &\Longrightarrow \\ 2 \cdot (x - \mathbf{1}) &= 0_{\bullet}1011 \end{aligned}$$

Example (binary)

$$
\begin{aligned}
x &= 1_\bullet \mathbf{01011} \\
&= 1 + 0_\bullet 01011 \\
&\Longrightarrow \\
2 \cdot (x - 1) &= \mathbf{0_\bullet 1011}
\end{aligned}
$$

Example (binary)

$$
\begin{aligned}
x &= \mathbf{0}\bullet 1011 \\
&= \mathbf{0} + 0\bullet 1011 \\
&\Longrightarrow \\
2 \cdot (x - \mathbf{0}) &= 1\bullet 011
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Example (binary)

$$
\begin{aligned}
x &= 0.\mathbf{1011} \\
&= 0 + 0.1011 \\
&\implies \\
2 \cdot (x - 0) &= \mathbf{1.011}
\end{aligned}
$$

Example (binary)

$$
\begin{aligned}
x &= \mathbf{1}_\bullet 011 \\
&= \mathbf{1} + 0_\bullet 011 \\
&\Longrightarrow \\
2 \cdot (x - \mathbf{1}) &= 0_\bullet 11
\end{aligned}
$$

Example (binary)

$$
\begin{aligned}
x &= 1.\mathbf{011} \\
&= 1 + 0.011 \\
&\implies \\
2 \cdot (x - 1) &= \mathbf{0.11}
\end{aligned}
$$

Example (binary)

$$
\begin{aligned}
x &= \mathbf{0}\bullet 11 \\
&= \mathbf{0} + 0\bullet 11 \\
&\implies \\
2 \cdot (x - \mathbf{0}) &= 1\bullet 1
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Example (binary)

$$
\begin{aligned}
x &= 0{\bullet}11 \\
&= 0 + 0{\bullet}11 \\
&\Longrightarrow \\
2 \cdot (x - 0) &= 1{\bullet}1
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Example (binary)

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x &= \mathbf{1} \\
&= \mathbf{1} + 0 \\
&\Longrightarrow \\
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\end{aligned}
$$

Example (binary)

$$\begin{aligned} x &= 1 \\ &= 1 + 0 \\ &\implies \\ 2 \cdot (x - 1) &= \mathbf{0} \end{aligned}$$

Binary representation of 1.1_{10}

x	b_i	$x - b_i$	$2(x - b_i)$
1.1	$b_0 = \mathbf{1}$		

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1.6	$b_4 = 1$	0.6	1.2
1.2	$b_5 = 1$	0.2	0.4

$\Rightarrow$ $1.0\overline{0011}$, periodic, *not* finite

Binary Number Representations of 1.1 and 0.1

- are not finite $\Rightarrow$ conversion errors
- `1.1f` und `0.1f`: *Approximations* of 1.1 and 0.1
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Binary Number Representations of 1.1 and 0.1

on my computer:

$$\begin{aligned} \texttt{1.1} &= \underline{1.1000000000000000}888178\ldots \\ \texttt{1.1f} &= \underline{1.1000000}238418\ldots \end{aligned}$$

Computing with Floating-point Numbers

is nearly as simple as with integers.

Computing with Floating-point Numbers

Example ($\beta = 2$, $p = 4$):

$$
\begin{array}{rr}
 & 1.111 \cdot 2^{-2} \\
+ & 1.011 \cdot 2^{-1}
\end{array}
$$

1. adjust exponents by denormalizing one number

Computing with Floating-point Numbers

Example ($\beta = 2$, $p = 4$):

$$
\begin{array}{rr}
 & 1.111 \cdot 2^{-2} \\
+ & 10.110 \cdot 2^{-2} \checkmark
\end{array}
$$

1. adjust exponents by denormalizing one number

Computing with Floating-point Numbers

Example ($\beta = 2, p = 4$):

$$\begin{array}{rr} & 1.111 \cdot 2^{-2} \\ + & 10.110 \cdot 2^{-2} \\ \hline \end{array}$$

2. binary addition of the significands

Computing with Floating-point Numbers

Example ($\beta = 2$, $p = 4$):

$$
\begin{array}{rr}
 & 1.111 \cdot 2^{-2} \\
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\hline
= & 100.101 \cdot 2^{-2} \checkmark
\end{array}
$$

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3. renormalize

Computing with Floating-point Numbers

Example ($\beta = 2$, $p = 4$):

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4. round to p significant places, if necessary

Computing with Floating-point Numbers

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- Single precision (`float`) numbers:

  $F^*(2, 24, -126, 127)$ (32 bit) plus $0, \infty, \ldots$

- Double precision (`double`) numbers:

  $F^*(2, 53, -1022, 1023)$ (64 bit) plus $0, \infty, \ldots$

- All arithmetic operations round the *exact* result to the next representable number

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Example: 32-bit Representation of a Floating Point Number

31	30	29	28	27	26	25	24	23	22	21	20	19	18	17	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	0

$\pm$ Exponent Mantisse

$$\pm \begin{array}{c} 2^{-126}, \ldots, 2^{127} \\ 0, \infty, \ldots \end{array}$$

$$\begin{array}{c} 1.00000000000000000000000 \\ \cdots \\ 1.11111111111111111111111 \end{array}$$

Floating-point Rules Rule 1

Rule 1

Do not test rounded floating-point numbers for equality.

```
for (float i = 0.1; i != 1.0; i += 0.1)
    std::cout << i << "\n";
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for (float i = 0.1; i != 1.0; i += 0.1)
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```

endless loop because i never becomes exactly 1

Floating-point Rules

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Do not add two numbers of very different orders of magnitude!

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$$\begin{array}{r} 1.000 \cdot 2^5 \\ +1.000 \cdot 2^0 \end{array}$$

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$$
\begin{aligned}
&1.000 \cdot 2^5 \\
&+1.000 \cdot 2^0 \\
&= 1.00001 \cdot 2^5
\end{aligned}
$$

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Do not add two numbers of very different orders of magnitude!

$$
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& \text{“}{=}\text{”}\ 1.000 \cdot 2^5 \quad \text{(Rounding on 4 places)}
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$$

Addition of 1 does not have any effect!

Harmonic Numbers

- The n-the harmonic number is

$$H_n = \sum_{i=1}^{n} \frac{1}{i}$$

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Harmonic Numbers

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$$H_n = \sum_{i=1}^{n} \frac{1}{i} \approx \ln n.$$

- This sum can be computed in forward or backward direction, which is mathematically clearly equivalent

Harmonic Numbers — Rule 2

```
std::cout << "Compute H_n for n =? ";
unsigned int n;
std::cin >> n;

float fs = 0;
for (unsigned int i = 1; i <= n; ++i)
    fs += 1.0f / i;
std::cout << "Forward sum = " << fs << "\n";

float bs = 0;
for (unsigned int i = n; i >= 1; --i)
    bs += 1.0f / i;
std::cout << "Backward sum = " << bs << "\n";
```

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std::cout << "Backward sum = " << bs << "\n";
```

Input: **10000000**

forwards: **15.4037**

backwards: **16.686**

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unsigned int n;
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    fs += 1.0f / i;
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float bs = 0;
for (unsigned int i = n; i >= 1; --i)
    bs += 1.0f / i;
std::cout << "Backward sum = " << bs << "\n";
```

Input: **100000000**

forwards: **15.4037**

backwards: **18.8079**

Harmonic Numbers

Observation:

- The forward sum stops growing at some point and is “really” wrong.
- The backward sum approximates H_n well.

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Explanation:

- For $1 + 1/2 + 1/3 + \cdots$, later terms are too small to actually contribute
- Problem similar to $2^5 + 1$ "$=$" 2^5

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- The forward sum stops growing at some point and is “really” wrong.
- The backward sum approximates H_n well.

Explanation:

- For $1 + 1/2 + 1/3 + \cdots$, later terms are too small to actually contribute
- Problem similar to $2^5 + 1$ “$=$” 2^5

Rule 4

Do not subtract two numbers with a very similar value.

Cancellation problems, cf. lecture notes.

Literature

David Goldberg: What Every Computer Scientist Should Know About Floating-Point Arithmetic (1991)

Randy Glasbergen, 1996

9. Functions I

Defining and Calling Functions, Evaluation of Function Calls, the Type `void`

Computing Powers

```
double a;
int n;
std::cin >> a; // Eingabe a
std::cin >> n; // Eingabe n

double result = 1.0;
if (n < 0) { // a^n = (1/a)^(-n)
  a = 1.0/a;
  n = -n;
}
for (int i = 0; i < n; ++i)
  result *= a;

std::cout << a << "^" << n << " = " << result << ".\n";
```

Computing Powers

```
double a;
int n;
std::cin >> a; // Eingabe a
std::cin >> n; // Eingabe n

double result = 1.0;
if (n < 0) { // a^n = (1/a)^(-n)
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  n = -n;
}
for (int i = 0; i < n; ++i)
  result *= a;

std::cout << a << "^" << n << " = " << result << ".\n";
```

Computing Powers

```
double a;
int n;
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std::cin >> n; // Eingabe n

double result = 1.0;
if (n < 0) { // a^n = (1/a)^(-n)
  a = 1.0/a;
  n = -n;
}
for (int i = 0; i < n; ++i)
  result *= a;
```

"Funktion `pow`"

```
std::cout << a << "^" << n << " = " << pow(a,n) << ".\n";
```

Function to Compute Powers

```
// PRE: e >= 0 || b != 0.0
// POST: return value is b^e
double pow(double b, int e)
{
    double result = 1.0;
    if (e < 0) { // b^e = (1/b)^(-e)
        b = 1.0/b;
        e = -e;
    }
    for (int i = 0; i < e; ++i)
        result *= b;
    return result;
}
```

Function to Compute Powers

```
double pow(double b, int e){...}
```

Function to Compute Powers

```
// Prog: callpow.cpp
// Define and call a function for computing powers.
#include <iostream>

double pow(double b, int e){...}

int main()
{
  std::cout << pow( 2.0, -2) << "\n"; // outputs 0.25
  std::cout << pow( 1.5, 2) << "\n"; // outputs 2.25
  std::cout << pow(-2.0, 9) << "\n"; // outputs -512

  return 0;
}
```

Function Definitions

return type *argument types*

T fname (T_1 pname$_1$, T_2 pname$_2$, $\ldots$, T_N pname$_N$)
block

body

function name *formal arguments*

Function Definitions

return type   argument types

T fname (T_1 pname$_1$, T_2 pname$_2$, $\ldots$, T_N pname$_N$)
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Function Definitions

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body

function name *formal arguments*

Function Definitions

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T fname $(T_1$ pname$_1$, T_2 pname$_2, \ldots, T_N$ pname$_N)$
block

body

function name *formal arguments*

Function Definitions

return type *argument types*

T fname (T_1 pname$_1$, T_2 pname$_2$, $\ldots$, T_N pname$_N$)
    block

body

function name *formal arguments*

Xor

```
// post: returns l XOR r
bool Xor(bool l, bool r)
{
    return l != r;
}
```

Harmonic

```
// PRE: n >= 0
// POST: returns nth harmonic number
//       computed with backward sum
float Harmonic(int n)
{
    float res = 0;
    for (unsigned int i = n; i >= 1; --i)
        res += 1.0f / i;
    return res;
}
```

min

```
// POST: returns the minimum of a and b
int min(int a, int b)
{
    if (a<b)
        return a;
    else
        return b;
}
```

Function Calls

fname (*expression*$_1$, *expression*$_2$, $\ldots$, *expression*$_N$)

- All call arguments must be convertible to the respective formal argument types.
- The function call is an expression of the return type of the function.

Example: `pow(a,n)`: Expression of type `double`

Function Calls

fname (*expression*$_1$, *expression*$_2$, . . . , *expression*$_N$)

- All call arguments must be convertible to the respective formal argument types.
- The function call is an expression of the return type of the function.

Example: `pow(a,n)`: Expression of type `double`

Function Calls

fname (*expression*$_1$, *expression*$_2$, ..., *expression*$_N$)

- All call arguments must be convertible to the respective formal argument types.
- The function call is an expression of the return type of the function.

Example: `pow(a,n)`: Expression of type `double`

Function Calls

For the types we know up to this point it holds that:

- Call arguments are R-values
  $\hookrightarrow$ *call-by-value* (also *pass-by-value*), more on this soon
- The function call is an R-value.

Function Calls

For the types we know up to this point it holds that:

- Call arguments are R-values
  $\hookrightarrow$ *call-by-value* (also *pass-by-value*), more on this soon
- The function call is an R-value.

fname: R-value $\times$ R-value $\times \cdots \times$ R-value $\longrightarrow$ R-value

Evaluation Function Call

```
double pow(double b, int e){
    assert (e >= 0 || b != 0);
    double result = 1.0;
    if (e<0) {
        // b^e = (1/b)^(-e)
        b = 1.0/b;
        e = -e;
    }
    for (int i = 0; i < e ; ++i)
        result * = b;
    return result;
}

...
pow (2.0, -2)
```

Evaluation Function Call

```
double pow(double b, int e){
    assert (e >= 0 || b != 0);
    double result = 1.0;
    if (e<0) {
        // b^e = (1/b)^(-e)
        b = 1.0/b;
        e = -e;
    }
    for (int i = 0; i < e ; ++i)
        result * = b;
    return result;
}

...
pow (2.0, -2)
```

Call of pow

Evaluation Function Call

```
double pow(double b, int e){
    assert (e >= 0 || b != 0);
    double result = 1.0;
    if (e<0) {
        // b^e = (1/b)^(-e)
        b = 1.0/b;
        e = -e;
    }
    for (int i = 0; i < e ; ++i)
        result *= b;
    return result;
}

...
pow (2.0, -2)
```

b=2.0,e=-2

Evaluation Function Call

```
double pow(double b, int e){
    assert (e >= 0 || b != 0);
    double result = 1.0;
    if (e<0) {
        // b^e = (1/b)^(-e)
        b = 1.0/b;
        e = -e;
    }
    for (int i = 0; i < e ; ++i)
        result * = b;
    return result;
}

...
pow (2.0, -2)
```

```
b=2.0,e=-2
// ok
```

Evaluation Function Call

```
double pow(double b, int e){
    assert (e >= 0 || b != 0);
    double result = 1.0;
    if (e<0) {
        // b^e = (1/b)^(-e)
        b = 1.0/b;
        e = -e;
    }
    for (int i = 0; i < e ; ++i)
        result * = b;
    return result;
}

...
pow (2.0, -2)
```

result=1.0

Evaluation Function Call

```
double pow(double b, int e){
    assert (e >= 0 || b != 0);
    double result = 1.0;
    if (e<0) {
        // b^e = (1/b)^(-e)
        b = 1.0/b;
        e = -e;
    }
    for (int i = 0; i < e ; ++i)
        result * = b;
    return result;
}

...
pow (2.0, -2)
```

e == -2

Evaluation Function Call

```
double pow(double b, int e){
    assert (e >= 0 || b != 0);
    double result = 1.0;
    if (e<0) {
        // b^e = (1/b)^(-e)
        b = 1.0/b;
        e = -e;
    }
    for (int i = 0; i < e ; ++i)
        result * = b;
    return result;
}

...
pow (2.0, -2)
```

b=0.5

Evaluation Function Call

```
double pow(double b, int e){
    assert (e >= 0 || b != 0);
    double result = 1.0;
    if (e<0) {
        // b^e = (1/b)^(-e)
        b = 1.0/b;
        e = -e;
    }
    for (int i = 0; i < e ; ++i)
        result * = b;
    return result;
}

...
pow (2.0, -2)
```

e=2

Evaluation Function Call

```
double pow(double b, int e){
    assert (e >= 0 || b != 0);
    double result = 1.0;
    if (e<0) {
        // b^e = (1/b)^(-e)
        b = 1.0/b;
        e = -e;
    }
    for (int i = 0; i < e ; ++i)
        result * = b;
    return result;
}

...
pow (2.0, -2)
```

i=0

Evaluation Function Call

```
double pow(double b, int e){
    assert (e >= 0 || b != 0);
    double result = 1.0;
    if (e<0) {
        // b^e = (1/b)^(-e)
        b = 1.0/b;
        e = -e;
    }
    for (int i = 0; i < e ; ++i)
        result * = b;
    return result;
}

...
pow (2.0, -2)
```

i=0

result=0.5

Evaluation Function Call

```
double pow(double b, int e){
    assert (e >= 0 || b != 0);
    double result = 1.0;
    if (e<0) {
        // b^e = (1/b)^(-e)
        b = 1.0/b;
        e = -e;
    }
    for (int i = 0; i < e ; ++i)
        result * = b;
    return result;
}

...
pow (2.0, -2)
```

i=1

Evaluation Function Call

```
double pow(double b, int e){
    assert (e >= 0 || b != 0);
    double result = 1.0;
    if (e<0) {
        // b^e = (1/b)^(-e)
        b = 1.0/b;
        e = -e;
    }
    for (int i = 0; i < e ; ++i)        i=1
        result * = b;                    result=0.25
    return result;
}

...
pow (2.0, -2)
```

Evaluation Function Call

```
double pow(double b, int e){
    assert (e >= 0 || b != 0);
    double result = 1.0;
    if (e<0) {
        // b^e = (1/b)^(-e)
        b = 1.0/b;
        e = -e;
    }
    for (int i = 0; i < e ; ++i)
        result * = b;
    return result;
}

...
pow (2.0, -2)
```

i=2

Evaluation Function Call

```
double pow(double b, int e){
    assert (e >= 0 || b != 0);
    double result = 1.0;
    if (e<0) {
        // b^e = (1/b)^(-e)
        b = 1.0/b;
        e = -e;
    }
    for (int i = 0; i < e ; ++i)
        result * = b;
    return result;
}

...
pow (2.0, -2)
```

result=0.25

Evaluation Function Call

```
double pow(double b, int e){
    assert (e >= 0 || b != 0);
    double result = 1.0;
    if (e<0) {
        // b^e = (1/b)^(-e)
        b = 1.0/b;
        e = -e;
    }
    for (int i = 0; i < e ; ++i)
        result * = b;
    return result;
}

...
pow (2.0, -2)
```

result=0.25

Return

Evaluation Function Call

```
double pow(double b, int e){
    assert (e >= 0 || b != 0);
    double result = 1.0;
    if (e<0) {
        // b^e = (1/b)^(-e)
        b = 1.0/b;
        e = -e;
    }
    for (int i = 0; i < e ; ++i)
        result * = b;
    return result;
}

...
pow (2.0, -2)
```

Return

value: 0.25

Evaluation Function Call

```
double pow(double b, int e){
    assert (e >= 0 || b != 0);
    double result = 1.0;
    if (e<0) {
        // b^e = (1/b)^(-e)
        b = 1.0/b;
        e = -e;
    }
    for (int i = 0; i < e ; ++i)
        result * = b;
    return result;
}

...
pow (2.0, -2)
```

value: 0.25

Scope of Formal Arguments

```
int main(){
    double b = 2.0;
    int e = -2;
    double z = pow(b, e);

    std::cout << z; // 0.25
    std::cout << b; // 2
    std::cout << e; // -2
    return 0;
}
```

Scope of Formal Arguments

```
double pow(double b, int e){
    double r = 1.0;
    if (e<0) {
        b = 1.0/b;
        e = -e;
    }
    for (int i = 0; i < e ; ++i)
        r *= b;
    return r;
}
```

```
int main(){
    double b = 2.0;
    int e = -2;
    double z = pow(b, e);

    std::cout << z; // 0.25
    std::cout << b; // 2
    std::cout << e; // -2
    return 0;
}
```

Scope of Formal Arguments

```
double pow(double b, int e){
    double r = 1.0;
    if (e<0) {
        b = 1.0/b;
        e = -e;
    }
    for (int i = 0; i < e ; ++i)
        r * = b;
    return r;
}
```

```
int main(){
    double b = 2.0;
    int e = -2;
    double z = pow(b, e);

    std::cout << z; // 0.25
    std::cout << b; // 2
    std::cout << e; // -2
    return 0;
}
```

Not the formal arguments `b` and `e` of pow but the variables defined here locally in the body of `main`

The type `void`

```
// POST: "(i, j)" has been written to standard output
???? print_pair(int i, int j) {
    std::cout << "(" << i << ", " << j << ")\n";
}

int main() {
    print_pair(3,4); // outputs (3, 4)
    return 0;
}
```

The type `void`

```
// POST: "(i, j)" has been written to standard output
void print_pair(int i, int j) {
    std::cout << "(" << i << ", " << j << ")\n";
}

int main() {
    print_pair(3,4); // outputs (3, 4)
    return 0;
}
```

The type `void`

- Fundamental type with empty value range

The type `void`

- Fundamental type with empty value range
- Usage as a return type for functions that do *only* provide an effect

`void`-Functions

- do not require `return`.
- execution ends when the end of the function body is reached or if
- `return;` is reached