# 2. Integers

Evaluation of Arithmetic Expressions, Associativity and Precedence, Arithmetic Operators, Domain of Types `int`, `unsigned int`



## Celsius to Fahrenheit

```
// Program: fahrenheit.cpp
// Convert temperatures from Celsius to Fahrenheit.
#include <iostream>

int main() {
  // Input
  std::cout << "Temperature in degrees Celsius =? ";
  int celsius;
  std::cin >> celsius;

  // Computation and output
  std::cout << celsius << " degrees Celsius are "
            << 9 * celsius / 5 + 32  << " degrees Fahrenheit.\n";
  return 0;
}
```



## `9 * celsius / 5 + 32`

- Arithmetic expression,
- contains three literals, a variable, three operator symbols

How to put the expression in parentheses?



## Precedence

**Multiplication/Division before Addition/Subtraction**

`9 * celsius / 5 + 32`

bedeutet

`(9 * celsius / 5) + 32`

**Rule 1: precedence**

Multiplicative operators (`*`, `/`, `%`) have a higher precedence ("bind more strongly") than additive operators (`+`, `-`)

## Associativity

### From left to right

`9 * celsius / 5 + 32`

bedeutet

`((9 * celsius) / 5) + 32`

### Rule 2: Associativity

Arithmetic operators (*, /, %, +, −) are left associative: operators of same precedence evaluate from left to right



## Arity

### Rule 3: Arity

Unary operators +, − first, then binary operators +, −.

`-3 - 4`

means

`(-3) - 4`



## Parentheses

Any expression can be put in parentheses by means of

- associativities
- precedences
- arities (number of operands)

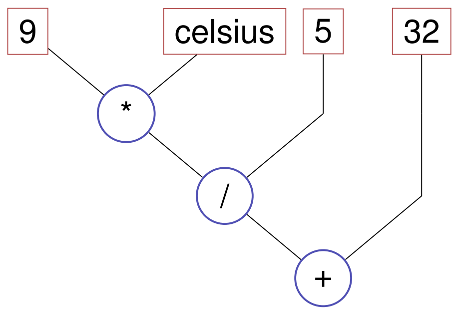
of the operands in an unambiguous way (Details in the lecture notes).



## Expression Trees

Parentheses yield the expression tree

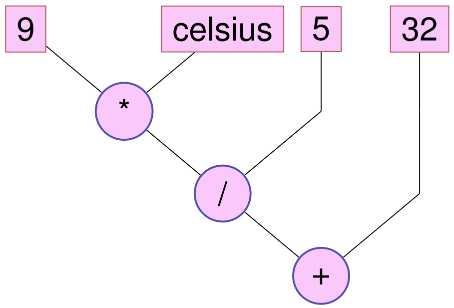
`(((9 * celsius) / 5) + 32)`



## Evaluation Order

"From top to bottom" in the expression tree

`9 * celsius / 5 + 32`

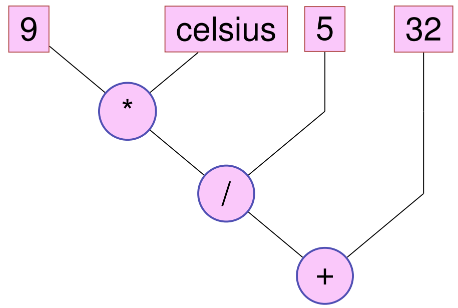





## Evaluation Order

Order is not determined uniquely:

`9 * celsius / 5 + 32`

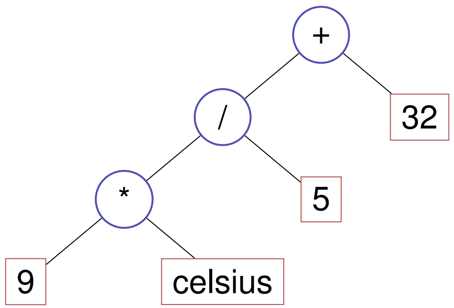





## Expression Trees – Notation

Common notation: root on top

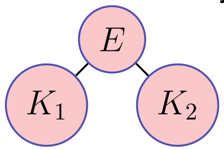
`9 * celsius / 5 + 32`






## Evaluation Order – more formally

- Valid order: any node is evaluated *after* its children




In C++, the valid order to be used is not defined.

- "Good expression": any valid evaluation order leads to the same result.
- Example for a “bad expression”: `a*(a=2)`

## Evaluation order

**Guideline**

**Avoid** modifying variables that are used in the same expression more than once.



## Arithmetic operations

| | Symbol | Arity | Precedence | Associativity |
|---|---|---|---|---|
| Unary + | + | 1 | 16 | right |
| Negation | - | 1 | 16 | right |
| Multiplication | * | 2 | 14 | left |
| Division | / | 2 | 14 | left |
| Modulo | % | 2 | 14 | links |
| Addition | + | 2 | 13 | left |
| Subtraction | - | 2 | 13 | left |

All operators: [R-value $\times$] R-value $\rightarrow$ R-value



## Interlude: Assignment expression – in more detail

- Already known: `a = b` means
  Assignment of `b` (R-value) to `a` (L-value).
  Returns: L-value
- What does `a = b = c` mean?
- Answer: assignment is right-associative

`a = b = c` $\Longleftrightarrow$ `a = (b = c)`

Example multiple assignment:
`a = b = 0` $\Longrightarrow$ `b=0; a=0`



## Division

- Operator / implements integer division

`5 / 2` has value 2

- In `fahrenheit.cpp`

```
9 * celsius / 5 + 32
```

```
15 degrees Celsius are 59 degrees Fahrenheit
```

- Mathematically equivalent...but not in C++!

```
9 / 5 * celsius + 32
```

```
15 degrees Celsius are 47 degrees Fahrenheit
```

## Loss of Precision

**Guideline**

- Watch out for potential loss of precision
- Postpone operations with potential loss of precision to avoid “error escalation”



## Division and Modulo

- Modulo-operator computes the rest of the integer division

`5 / 2` has value 2, `5 % 2` has value 1.

- It holds that:

`(a / b) * b + a % b` has the value of `a`.



## Increment and decrement

- Increment / Decrement a number by one is a frequent operation
- works like this for an L-value:

```
expr = expr + 1.
```

Disadvantages

- relatively long
- `expr` is evaluated twice
  - Later: L-valued expressions whose evaluation is “expensive”
  - `expr` could have an effect (but should not, cf. guideline)



## In-/Decrement Operators

**Post-Increment**

```
expr++
```

Value of `expr` is increased by one, the *old* value of `expr` is returned (as R-value)

**Pre-increment**

```
++expr
```

Value of `expr` is increased by one, the *new* value of `expr` is returned (as L-value)

**Post-Dekrement**

```
expr--
```

Value of `expr` is decreased by one, the *old* value of `expr` is returned (as R-value)

**Prä-Dekrement**

```
--expr
```

Value of `expr` is increased by one, the *new* value of `expr` is returned (as L-value)

## In-/decrement Operators

| | use | arity | prec | assoz | L-/R-value |
|---|---|---|---|---|---|
| **Post-increment** | `expr++` | 1 | 17 | left | L-value → R-value |
| **Pre-increment** | `++expr` | 1 | 16 | right | L-value → L-value |
| **Post-decrement** | `expr--` | 1 | 17 | left | L-value → R-value |
| **Pre-decrement** | `--expr` | 1 | 16 | right | L-value → L-value |



## In-/Decrement Operators

**Example**

```
int a = 7;
std::cout << ++a << "\n"; // 8
std::cout << a++ << "\n"; // 8
std::cout << a << "\n"; // 9
```



## In-/Decrement Operators

Is the expression

`++expr;` ← we favour this

equivalent to

`expr++;`?

Yes, but

- Pre-increment can be more efficient (old value does not need to be saved)
- Post In-/Decrement are the only left-associative unary operators (not very intuitive)



## `C++` vs. `++C`

Strictly speaking our language should be named `++C` because

- it is an advancement of the language `C`
- while `C++` returns the old `C`.

## Arithmetic Assignments

```
a += b
```

$\Leftrightarrow$

```
a = a + b
```

analogously for `-`, `*`, `/` and `%`



## Arithmetic Assignments

| | Gebrauch | Bedeutung |
|---|---|---|
| `+=` | `expr1 += expr2` | `expr1 = expr1 + expr2` |
| `-=` | `expr1 -= expr2` | `expr1 = expr1 - expr2` |
| `*=` | `expr1 *= expr2` | `expr1 = expr1 * expr2` |
| `/=` | `expr1 /= expr2` | `expr1 = expr1 / expr2` |
| `%=` | `expr1 %= expr2` | `expr1 = expr1 % expr2` |

Arithmetic expressions evaluate expr1 only once.
Assignments have precedence 4 and are right-associative.



## Binary Number Representations

Binary representation (Bits from $\{0, 1\}$)

$$b_n b_{n-1} \dots b_1 b_0$$

corresponds to the number $b_n \cdot 2^n + \cdots + b_1 \cdot 2 + b_0$

Example: 101011 corresponds to 43.

*Least Significant Bit (LSB)*

*Most Significant Bit (MSB)*



## Binary Numbers: Numbers of the Computer?

Truth: Computers calculate using binary numbers.


NEUE ZÜRCHER ZEITUNG

TECHNIK

Das programmgesteuerte Rechengerät an der Eidgenössischen Technischen Hochschule in Zürich

## Binary Numbers: Numbers of the Computer?

Stereotype: computers are talking 0/1 gibberish


01001110 01011010 01011010

01000010 01100101 01110010 01101001

01100011 01101000 01110100 01100101

00100000 01110110

01100110 01100101



## Computing Tricks

- Estimate the orders of magnitude of powers of two.[2]:

$2^{10} = 1024 = 1\text{Ki} \approx 10^3.$
$2^{20} = 1\text{Mi} \approx 10^6,$
$2^{30} = 1\text{Gi} \approx 10^9,$
$2^{32} = 4 \cdot (1024)^3 = 4\text{Gi}.$
$2^{64} = 16\text{Ei} \approx 16 \cdot 10^{18}.$

[2] Decimal vs. binary units: MB - Megabyte vs. MiB - Megabibyte (etc.)
kilo (K, Ki) – mega (M, Mi) – giga (G, Gi) – tera(T, Ti) – peta(P, Pi) – exa (E, Ei)



## Hexadecimal Numbers

Numbers with base 16

$$h_n h_{n-1} \dots h_1 h_0$$

corresponds to the number

$$h_n \cdot 16^n + \cdots + h_1 \cdot 16 + h_0.$$

notation in C++: prefix `0x`

Example: `0xff` corresponds to 255.

**Hex Nibbles**

| hex | bin | dec |
|---|---|---|
| 0 | 0000 | 0 |
| **1** | **0001** | **1** |
| **2** | **0010** | **2** |
| 3 | 0011 | 3 |
| **4** | **0100** | **4** |
| 5 | 0101 | 5 |
| 6 | 0110 | 6 |
| 7 | 0111 | 7 |
| **8** | **1000** | **8** |
| 9 | 1001 | 9 |
| a | 1010 | 10 |
| b | 1011 | 11 |
| c | 1100 | 12 |
| d | 1101 | 13 |
| e | 1110 | 14 |
| f | 1111 | 15 |



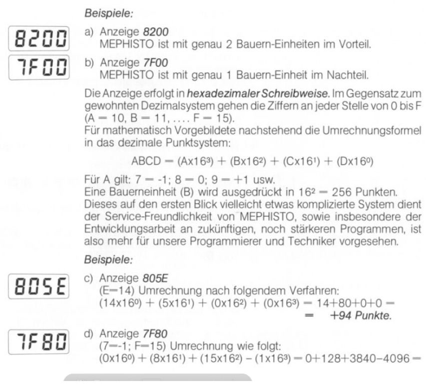
## Why Hexadecimal Numbers?

- A Hex-Nibble requires exactly 4 bits. Numbers 1, 2, 4 and 8 represent bits 0, 1, 2 and 3.
- “compact representation of binary numbers”

32-bit numbers consist of eight hex-nibbles: `0x00000000 -- 0xffffffff` .
`0x400` $= 1Ki = 1'024.$
`0x100000` $= 1Mi = 1'048'576.$
`0x40000000` $= 1Gi = 1'073.741,824.$
`0x80000000`: highest bit of a 32-bit number is set
`0xffffffff`: all bits of a 32-bit number are set
“`0x8a20aaf0` is an address in the upper 2G of the 32-bit address space”

## Why Hexadecimal Numbers?

“For programmers and technicians” (Excerpt of a user manual of the chess computers *Mephisto II*, 1981)



*Beispiele:*

8200 a) Anzeige *8200*
MEPHISTO ist mit genau 2 Bauern-Einheiten im Vorteil.

7F00 b) Anzeige *7F00*
MEPHISTO ist mit genau 1 Bauern-Einheit im Nachteil.

Die Anzeige erfolgt in ***hexadezimaler Schreibweise.*** Im Gegensatz zum gewohnten Dezimalsystem gehen die Ziffern an jeder Stelle von 0 bis F (A = 10, B = 11, .... F = 15).
Für mathematisch Vorgebildete nachstehend die Umrechnungsformel in das dezimale Punktsystem:

$$ABCD = (Ax16^3) + (Bx16^2) + (Cx16^1) + (Dx16^0)$$

Für A gilt: 7 = -1; 8 = 0; 9 = +1 usw.
Eine Bauerneinheit (B) wird ausgedrückt in $16^2$ = 256 Punkten.
Dieses auf den ersten Blick vielleicht etwas komplizierte System dient der Service-Freundlichkeit von MEPHISTO, sowie insbesondere der Entwicklungsarbeit an zukünftigen, noch stärkeren Programmen, ist also mehr für unsere Programmierer und Techniker vorgesehen.

*Beispiele:*

805E c) Anzeige *805E*
(E=14) Umrechnung nach folgendem Verfahren:
$(14x16^0) + (5x16^1) + (0x16^2) + (0x16^3) = 14+80+0+0 =$
= *+94 Punkte.*

7F80 d) Anzeige *7F80*
(7=-1; F=15) Umrechnung wie folgt:
$(0x16^0) + (8x16^1) + (15x16^2) - (1x16^3) = 0+128+3840-4096 =$

http://www.zanchetta.net/default.aspx?Categorie=ECHIQUIERS&Page=documentations



## Example: Hex-Colors

$$\#\underbrace{00}_{r}\underbrace{FF}_{g}\underbrace{00}_{b}$$



## Why Hexadecimal Numbers?

The NZZ could have saved a lot of space ...




## Domain of Type `int`

```
// Output the smallest and the largest value of type int.
#include <iostream>
#include <limits>

int main() {
  std::cout << "Minimum int value is "
            << std::numeric_limits<int>::min() << ".\n"
            << "Maximum int value is "
            << std::numeric_limits<int>::max() << ".\n";
  return 0;
}
```

```
Minimum int value is -2147483648.
Maximum int value is 2147483647.
```

Where do these numbers come from?

## Domain of the Type `int`

- Representation with $B$ bits. Domain comprises the $2^B$ integers:

$$\{-2^{B-1}, -2^{B-1}+1, \ldots, -1, 0, 1, \ldots, 2^{B-1}-2, 2^{B-1}-1\}$$

  Where does this partitioning come from?

- On most platforms $B = 32$
- For the type `int` $\mathrm{C}{++}$ guarantees $B \geq 16$
- Background: Section 2.2.8 (Binary Representation) in the lecture notes.



## Over- and Underflow

- Arithmetic operations (`+,-,*`) can lead to numbers outside the valid domain.
- Results can be incorrect!

`power8.cpp`: $15^8 = -1732076671$

`power20.cpp`: $3^{20} = -808182895$

- There is *no error message!*



## The Type `unsigned int`

- Domain

$$\{0, 1, \ldots, 2^B - 1\}$$

- All arithmetic operations exist also for `unsigned int`.
- Literals: `1u, 17u` ...



## Mixed Expressions

- Operators can have operands of different type (e.g. `int` and `unsigned int`).

```
17 + 17u
```

- Such mixed expressions are of the “more general” type `unsigned int`.
- `int`-operands are *converted* to `unsigned int`.

## Conversion

| `int` Value | Sign | `unsigned int` Value |
|---|---|---|
| $x$ | $\geq 0$ | $x$ |
| $x$ | $< 0$ | $x + 2^B$ |



## Conversion "reversed"

The declaration

```
int a = 3u;
```

converts `3u` to `int`.

The value is preserved because it is in the domain of `int`; otherwise the result depends on the implementation.



## Signed Number Representation

- (Hopefully) clear by now: binary number representation without sign, e.g.

$$[b_{31}b_{30}\dots b_0]_u \quad \widehat{=} \quad b_{31}\cdot 2^{31} + b_{30}\cdot 2^{30} + \cdots + b_0$$

- Obviously required: use a bit for the sign.
- Looking for a consistent solution

> The representation with sign should coincide with the unsigned solution as much as possible. Positive numbers should arithmetically be treated equal in both systems.



## Computing with Binary Numbers (4 digits)

Simple Addition

$$\begin{array}{r} 2 \\ +3 \\ \hline 5 \end{array} \qquad \begin{array}{r} 0010 \\ +0011 \\ \hline 0101 \end{array}$$

Simple Subtraction

$$\begin{array}{r} 5 \\ -3 \\ \hline 2 \end{array} \qquad \begin{array}{r} 0101 \\ -0011 \\ \hline 0010 \end{array}$$

## Computing with Binary Numbers (4 digits)

Addition with Overflow

$$\begin{array}{r} 7 \\ +9 \\ \hline 16 \end{array} \qquad \begin{array}{r} 0111 \\ +1001 \\ \hline (1)0000 \end{array}$$

Negative Numbers?

$$\begin{array}{r} 5 \\ +(-5) \\ \hline 0 \end{array} \qquad \begin{array}{r} 0101 \\ ???? \\ \hline (1)0000 \end{array}$$



## Computing with Binary Numbers (4 digits)

Simpler -1

$$\begin{array}{r} 1 \\ +(-1) \\ \hline 0 \end{array} \qquad \begin{array}{r} 0001 \\ 1111 \\ \hline (1)0000 \end{array}$$

Utilize this:

$$\begin{array}{r} 3 \\ +? \\ \hline -1 \end{array} \qquad \begin{array}{r} 0011 \\ +???? \\ \hline 1111 \end{array}$$



## Computing with Binary Numbers (4 digits)

Invert!

$$\begin{array}{r} 3 \\ +(-4) \\ \hline -1 \end{array} \qquad \begin{array}{l} 0011 \\ +1100 \\ \hline 1111 \widehat{=} 2^B - 1 \end{array}$$

$$\begin{array}{r} a \\ +(-a-1) \\ \hline -1 \end{array} \qquad \begin{array}{l} a \\ \bar{a} \\ \hline 1111 \widehat{=} 2^B - 1 \end{array}$$



## Computing with Binary Numbers (4 digits)

- Negation: inversion and addition of $1$

$$-a \quad \widehat{=} \quad \bar{a} + 1$$

- Wrap around semantics (calculating modulo $2^B$

$$-a \quad \widehat{=} \quad 2^B - a$$

## Why this works

Modulo arithmetics: Compute on a circle[3]

$11 \equiv 23 \equiv -1 \equiv \ldots \mod 12$ $\quad + \quad$ $4 \equiv 16 \equiv \ldots \mod 12$ $\quad = \quad$ $3 \equiv 15 \equiv \ldots \mod 12$

[3] The arithmetics also work with decimal numbers (and for multiplication).



## Negative Numbers (3 Digits)

| | $a$ | $-a$ | |
|---|---|---|---|
| 0 | 000 | 000 | 0 |
| 1 | 001 | **111** | -1 |
| 2 | 010 | **110** | -2 |
| 3 | 011 | **101** | -3 |
| 4 | 100 | **100** | -4 |
| 5 | 101 | | |
| 6 | 110 | | |
| 7 | 111 | | |

The most significant bit decides about the sign *and* it contributes to the value.



## Two's Complement

- Negation by bitwise negation and addition of $1$

$$-2 = -[0010] = [1101] + [0001] = [1110]$$

  - Arithmetics of addition and subtraction *identical* to unsigned arithmetics

$$3 - 2 = 3 + (-2) = [0011] + [1110] = [0001]$$

  - Intuitive "wrap-around" conversion of negative numbers.

$$-n \to 2^B - n$$

  - Domain: $-2^{B-1} \ldots 2^{B-1} - 1$