3. Logical Values

Boolean Functions; the Type `bool`; logical and relational operators; shortcut evaluation

Our Goal

```
int a;
std::cin >> a;
if (a % 2 == 0)
    std::cout << "even";
else
    std::cout << "odd";
```

Behavior depends on the value of a Boolean expression

Boolean Values in Mathematics

Boolean expressions can take on one of two values:

0 or *1*

- *0* corresponds to *“wrong”*
- *1* corresponds to *“true”*

The Type `bool` in C++

- represents *logical values*
- Literals `false` and `true`
- Domain $\{$*false*, *true*$\}$

```
bool b = true; // Variable with value true
```

Relational Operators

`a < b` (smaller than)
`a >= b` (greater than)
`a == b` (equals)
`a != b` (unequal)

number type $\times$ number type $\rightarrow$ `bool`

R-value $\times$ R-value $\rightarrow$ R-value

Table of Relational Operators

	Symbol	Arity	Precedence	Associativity
smaller	<	2	11	left
greater	>	2	11	left
smaller equal	<=	2	11	left
greater equal	>=	2	11	left
equal	==	2	10	left
unequal	!=	2	10	left

number type $\times$ number type $\rightarrow$ `bool`

R-value $\times$ R-value $\rightarrow$ R-value

Boolean Functions in Mathematics

- Boolean function

$$f : \{0,1\}^2 \rightarrow \{0,1\}$$

- 0 corresponds to "false".
- 1 corresponds to "true".

$\mathrm{AND}(x, y)$ $\qquad x \wedge y$

- "logical and"

$$f : \{0,1\}^2 \rightarrow \{0,1\}$$

- 0 corresponds to "false".
- corresponds to "true".

x	y	$\mathrm{AND}(x,y)$
0	0	0
0	1	0
1	0	0
1	1	1

Logical Operator &&

`a && b` (logical and)

`bool` $\times$ `bool` $\to$ `bool`

R-value $\times$ R-value $\to$ R-value

```
int n = −1;
int p = 3;
bool b = (n < 0) && (0 < p); // b = true
```


$\mathrm{OR}(x, y)$ $x \vee y$

- “logical or”

$$f : \{0, 1\}^2 \to \{0, 1\}$$

- 0 corresponds to “false”.
- corresponds to “true”.

x	y	$\mathrm{OR}(x, y)$
0	0	0
0	1	1
1	0	1
1	1	1

Logical Operator ||

`a || b` (logical or)

`bool` $\times$ `bool` $\to$ `bool`

R-value $\times$ R-value $\to$ R-value

```
int n = 1;
int p = 0;
bool b = (n < 0) || (0 < p); // b = false
```


$\mathrm{NOT}(x)$ $\neg x$

- “logical not”

$$f : \{0, 1\} \to \{0, 1\}$$

- 0 corresponds to “false”.
- corresponds to “true”.

x	$\mathrm{NOT}(x)$
0	1
1	0

Logical Operator !

`!b` (logical not)

`bool` → `bool`

R-value → R-value

```
int n = 1;
bool b = !(n < 0); // b = true
```


Precedences

```
!b && a
⇕
(!b) && a

a && b || c && d
⇕
(a && b) || (c && d)

a || b && c || d
⇕
a || (b && c) || d
```


Table of Logical Operators

	Symbol	Arity	Precedence	Associativity
Logical and (AND)	&&	2	6	left
Logical or (OR)	||	2	5	left
Logical not (NOT)	!	1	16	right

Precedences

The unary logical operator !

provides a stronger binding than

binary arithmetic operators. These

bind stronger than

relational operators,

and these bind stronger than

binary logical operators.

```
7 + x < y && y != 3 * z || ! b
7 + x < y && y != 3 * z || (!b)
```

Completeness

- AND, OR and NOT are the boolean functions available in $\mathrm{C{+}{+}}$.
- Any other *binary* boolean function can be generated from them.

x	y	$\mathrm{XOR}(x,y)$
0	0	0
0	1	1
1	0	1
1	1	0

Completeness: $\mathrm{XOR}(x, y)$ $x \oplus y$

$$\mathrm{XOR}(x,y) = \mathrm{AND}(\mathrm{OR}(x,y), \mathrm{NOT}(\mathrm{AND}(x,y))).$$

$$x \oplus y = (x \vee y) \wedge \neg(x \wedge y).$$

```
(x || y) && !(x && y)
```


Completenss Proof

- Identify binary boolean functions with their characteristic vector.

x	y	$\mathrm{XOR}(x,y)$
0	0	0
0	1	1
1	0	1
1	1	0

characteristic vector: 0110

$$\mathrm{XOR} = f_{0110}$$

Completenss Proof

- Step 1: generate the *fundamental* functions f_{0001}, f_{0010}, f_{0100}, f_{1000}

$$\begin{aligned} f_{0001} &= \mathrm{AND}(x,y) \\ f_{0010} &= \mathrm{AND}(x, \mathrm{NOT}(y)) \\ f_{0100} &= \mathrm{AND}(y, \mathrm{NOT}(x)) \\ f_{1000} &= \mathrm{NOT}(\mathrm{OR}(x,y)) \end{aligned}$$

Completenss Proof

- Step 2: generate all functions by applying logical or

$$f_{1101} = \mathrm{OR}(f_{1000}, \mathrm{OR}(f_{0100}, f_{0001}))$$

- Step 3: generate f_{0000}

$$f_{0000} = 0.$$

`bool` vs `int`: Conversion

- `bool` can be used whenever `int` is expected – and vice versa.
- Many existing programs use `int` instead of `bool`
  This is bad style originating from the language C .

`bool`	→ `int`
true	→ 1
false	→ 0

`int`	→ `bool`
≠0	→ *true*
0	→ *false*

```
bool b = 3; // b=true
```


DeMorgan Rules

- `!(a && b) == (!a || !b)`
- `!(a || b) == (!a && !b)`

! (rich *and* beautiful) == (poor *or* ugly)

Application: either ... or (XOR)

`(x || y)`	`&& !(x && y)`	x or y, and not both
`(x || y)`	`&& (!x || !y)`	x or y, and one of them not
`!(!x && !y)`	`&& !(x && y)`	not none and not both
`!(!x && !y || x && y)`		not: both or none

Shortcut Evaluation

- Logical operators `&&` and `||` evaluate the *left operand first*.
- If the result is then known, the right operand will *not be* evaluated.

```
x != 0 && z / x > y
```

$\Rightarrow$ No division by 0

Sources of Errors

- Errors that the compiler can find: syntactical and some semantical errors
- Errors that the compiler cannot find: runtime errors (always semantical)

Avoid Sources of Bugs

1. Exact knowledge of the wanted program behavior

   ≫ It's not a bug, it's a feature !!≪

2. Check at many places in the code if the program is still on track!
3. Question the (seemingly) obvious, there could be a typo in the code.

Against Runtime Errors: *Assertions*

```
assert(expr)
```

- halts the program if the boolean expression `expr` is false
- requires `#include <cassert>`
- can be switched off

DeMorgan's Rules

Question the obvious Question the seemingly obvious!

```
// Prog: assertion.cpp
// use assertions to check De Morgan's laws

#include<cassert>

int main()
{
  bool x; // whatever x and y actually are,
  bool y; // De Morgan's laws will hold:
  assert ( !(x && y) == (!x || !y) );
  assert ( !(x || y) == (!x && !y) );
  return 0;
}
```


Switch off Assertions

```
// Prog: assertion2.cpp
// use assertions to check De Morgan's laws. To tell the
// compiler to ignore them, #define NDEBUG ("no debugging")
// at the beginning of the program, before the #includes

#define NDEBUG
#include<cassert>

int main()
{
  bool x; // whatever x and y actually are,
  bool y; // De Morgan's laws will hold:
  assert ( !(x && y) == (!x || !y) );  // ignored by NDEBUG
  assert ( !(x || y) == (!x && !y) );  // ignored by NDEBUG
  return 0;
}
```


Div-Mod Identity   a/b * b + a%b == a

Check if the program is on track...

```
std::cout << "Dividend a =? ";
int a;
std::cin >> a;

std::cout << "Divisor b =? ";
int b;
std::cin >> b;

// check input
assert (b != 0);
```

Input arguments for calculation

← Precondition for the ongoing computation

Div-Mod identity   a/b * b + a%b == a

...and question the obvious!

```
// check input
assert (b != 0);

// compute result
int div = a / b;
int mod = a % b;

// check result
assert (div * b + mod == a);
...
```

← Precondition for the ongoing computation (for `assert (b != 0);`)

← Div-Mod identity (for `assert (div * b + mod == a);`)

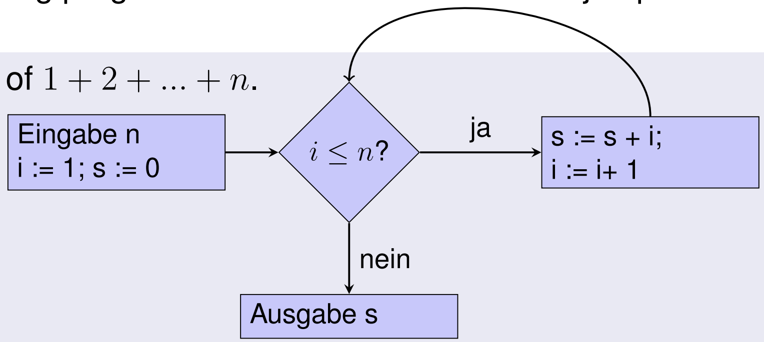
4. Control Structures I

Selection Statements, Iteration Statements, Termination, Blocks

Control Flow

- up to now *linear* (from top to bottom)
- For interesting programs we need “branches” and “jumps”

Computation of $1 + 2 + ... + n$.



Selection Statements

implement branches

- `if` statement
- `if-else` statement

`if`-Statement

`if` (*condition*)
  statement

```
int a;
std::cin >> a;
if (a % 2 == 0)
    std::cout << "even";
```

If *condition* is true then *statement* is executed

- *statement*: arbitrary statement (*body* of the `if`-Statement)
- *condition*: convertible to `bool`

`if-else`-statement

if (*condition*)
    statement1
else
    statement2

```
int a;
std::cin >> a;
if (a % 2 == 0)
    std::cout << "even";
else
    std::cout << "odd";
```

If *condition* is true then *statement1* is executed, otherwise *statement2* is executed.

- *condition*: convertible to `bool`.
- *statement1*: *body* of the `if`-branch
- *statement2*: *body* of the `else`-branch

Layout!

```
int a;
std::cin >> a;
if (a % 2 == 0)
    std::cout << "even";    ← Indentation
else
    std::cout << "odd";     ← Indentation
```


Iteration Statements

implement "loops"

- `for`-statement
- `while`-statement
- `do`-statement

Compute $1 + 2 + ... + n$

```
// Program: sum_n.cpp
// Compute the sum of the first n natural numbers.

#include <iostream>

int main()
{
  // input
  std::cout << "Compute the sum 1+...+n for n =? ";
  unsigned int n;
  std::cin >> n;

  // computation of sum_{i=1}^n i
  unsigned int s = 0;
  for (unsigned int i = 1; i <= n; ++i) s += i;

  // output
  std::cout << "1+...+" << n << " = " << s << ".\n";
  return 0;
}
```

for-Statement Example

```
for ( unsigned int i=1;  i <= n ; ++i )
    s += i;
```

Assumptions: `n == 2`, `s == 0`

i		s
i==1	wahr	s == 1
i==2	wahr	s == 3
i==3	falsch	
		s == 3

for-Statement: Syntax

`for` (*init statement* *condition* ; *expression*)
statement

- *init-statement*: expression statement, declaration statement, null statement
- *condition*: convertible to bool
- *expression*: any expression
- *statement* : any statement (*body* of the for-statement)

for-Statement: semantics

`for` (*init statement* *condition* ; *expression*)
statement

- *init-statement* is executed
- *condition* is evaluated
    - `true`: Iteration starts
      statement is executed
      expression is executed
    - false: `for`-statement is ended.

Gauß as a Child (1777 - 1855)

- Math-teacher wanted to keep the pupils busy with the following task:

Compute the sum of numbers from 1 to 100 !

- Gauß finished after one minute.

The Solution of Gauß

- The requested number is

$$1 + 2 + 3 + \cdots + 98 + 99 + 100.$$

- being half of

$$\begin{array}{rrrrrrrrrr} & 1 & + & 2 & + & \cdots & + & 99 & + & 100 \\ + & 100 & + & 99 & + & \cdots & + & 2 & + & 1 \\ \hline = & 101 & + & 101 & + & \cdots & + & 101 & + & 101 \end{array}$$

- Answer: $100 \cdot 101/2 = 5050$

`for`-Statement: Termination

```
for (unsigned int i = 1; i <= n; ++i)
    s += i;
```

Hier und meistens:

- *expression* changes its value that appears in *condition* .
- After a finite number of iterations *condition* becomes false: *Termination*

Endless Loops

- Endless loops are easy to generate:

```
for ( ; ; ) ;
```

  - Die *empty condition* is true.
  - Die *empty expression* has no effect.
  - Die *null statement* has no effect.

- ... but can in general not be automatically detected.

```
for ( e; v; e) r;
```


Halting Problem

Undecidability of the Halting Problem

There is no $\mathrm{C++}$ program that can determine for each $\mathrm{C++}$-Program P and each input I if the program P terminates with the input I.

This means that the correctness of programs can in general *not* be automatically checked.[5]

[5]Alan Turing, 1936. Theoretical quesitons of this kind were the main motivation for Alan Turing to construct a computing machine.

Example: Prime Number Test

Def.: a natural number $n \geq 2$ is a prime number, if no $d \in \{2, \ldots, n-1\}$ divides n .

A loop that can test this:

```
unsigned int d;
for (d=2; n%d != 0; ++d);
```

- Observation 1:
  After the `for`-statement it holds that $d \leq n$.
- Observation 2:
  n is a prime number if and only if finally $d = n$.

Blocks

- Blocks group a number of statements to a new statement
  `{statement1 statement2 ... statementN}`
- Example: body of the main function

```
int main() {
    ...
}
```

- Example: loop body

```
for (unsigned int i = 1; i <= n; ++i) {
    s += i;
    std::cout << "partial sum is " << s << "\n";
}
```