2. Integers

Evaluation of Arithmetic Expressions, Associativity and Precedence, Arithmetic Operators, Domain of Types **int**, **unsigned int**

Example: power8.cpp

```
int a; // Input
int r; // Result

std::cout << "Compute a^8 for a = ?";
std::cin >> a;

r = a * a; // r = a^2
r = r * r; // r = a^4

std::cout << "a^8 = " << r*r << '\n';
```


Terminology: L-Values and R-Values

L-Wert ("**L**eft of the assignment operator")

- Expression identifying a **memory location**
- For example a variable
  (we'll see other L-values later in the course)
- **Value** is the content at the memory location according to the type of the expression.
- L-Value can change its value (e.g. via assignment)

Terminology: L-Values and R-Values

R-Wert ("**R**ight of the assignment operator")

- Expression that is no L-value
- Example: integer literal `0`
- Any L-Value can be used as R-Value (but not the other way round) ...
- ... by using the *value* of the L-value
  (e.g. the L-value **a** could have the value **2**, which is then used as an R-value)
- An R-Value *cannot change* its value

L-Values and R-Values

R-Value

```
std::cout << "Compute a^8 for a = ? ";
int a;
std::cin >> a;
```

L-value (expression + address)

L-value (expression + address)

```
int r = a * a;  // r = a^2
r = r * r; // r = a^4
```

R-Value

```
std::cout << a<< "^8 = " << r * r << ".\ n";

return 0;
```

R-Value (expression that is not an L-value)

Celsius to Fahrenheit

```
// Program: fahrenheit.cpp
// Convert temperatures from Celsius to Fahrenheit.
#include <iostream>

int main() {
  // Input
  std::cout << "Temperature in degrees Celsius =? ";
  int celsius;
  std::cin >> celsius;

  // Computation and output
  std::cout << celsius << " degrees Celsius are "
            << 9 * celsius / 5 + 32  << " degrees Fahrenheit.\n";
  return 0;
}
```


`9 * celsius / 5 + 32`

```
9 * celsius / 5 + 32
```

- Arithmetic expression,
- contains three literals, a variable, three operator symbols

How to put the expression in parentheses?

Precedence

Multiplication/Division before Addition/Subtraction

`9 * celsius / 5 + 32`

bedeutet

`(9 * celsius / 5) + 32`

Rule 1: precedence

Multiplicative operators (`*`, `/`, `%`) have a higher precedence ("bind more strongly") than additive operators (`+`, `-`)

Associativity

From left to right

`9 * celsius / 5 + 32`

bedeutet

`((9 * celsius) / 5) + 32`

Rule 2: Associativity

Arithmetic operators (*, /, %, +, −) are left associative: operators of same precedence evaluate from left to right

Arity

Sign

`-3 - 4`

means

`(-3) - 4`

Rule 3: Arity

Unary operators +, − first, then binary operators +, −.

Parentheses

Any expression can be put in parentheses by means of

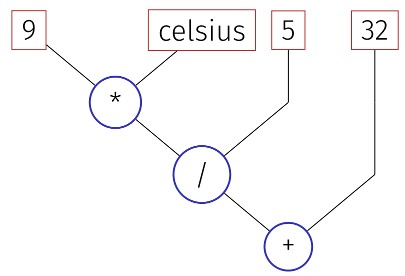
- associativities
- precedences
- arities (number of operands)

of the operands in an unambiguous way (Details in the lecture notes).

Expression Trees

Parentheses yield the expression tree

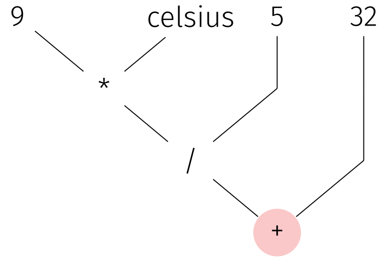
`(((9 * celsius) / 5) + 32)`



Evaluation Order

"From top to bottom" in the expression tree

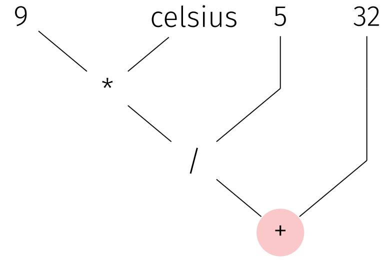
`9 * celsius / 5 + 32`



Evaluation Order

Order is not determined uniquely:

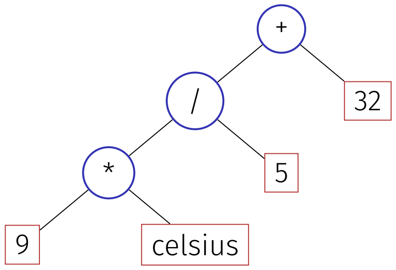
`9 * celsius / 5 + 32`



Expression Trees – Notation

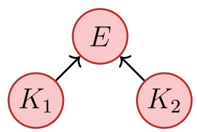
Common notation: root on top

`9 * celsius / 5 + 32`



Evaluation Order – more formally

Valid order: any node is evaluated **after** its children



C++: the valid order to be used is not defined.

- "Good expression": any valid evaluation order leads to the same result.
- Example for a “bad expression”: `a*(a=2)`

Evaluation order

Guideline

Avoid modifying variables that are used in the same expression more than once.

Arithmetic operations

	Symbol	Arity	Precedence	Associativity
Unary +	+	1	16	right
Negation	-	1	16	right
Multiplication	*	2	14	left
Division	/	2	14	left
Modulo	%	2	14	links
Addition	+	2	13	left
Subtraction	-	2	13	left

All operators: [R-value $\times$] R-value $\rightarrow$ R-value

Interlude: Assignment expression – in more detail

- Already known: `a = b` means Assignment of `b` (R-value) to `a` (L-value). Returns: L-value.
- What does `a = b = c` mean?
- Answer: assignment is right-associative

`a = b = c` $\iff$ `a = (b = c)`

Multiple assignment: `a = b = 0` $\Longrightarrow$ `b=0; a=0`

Division

- Operator **/** implements integer division

  `5 / 2` has value 2

- In **`fahrenheit.cpp`**

```
9 * celsius / 5 + 32

15 degrees Celsius are 59 degrees Fahrenheit
```

- Mathematically equivalent... but not in C++!

```
9 / 5 * celsius + 32

15 degrees Celsius are 47 degrees Fahrenheit
```

Loss of Precision

Guideline

- Watch out for potential loss of precision
- Postpone operations with potential loss of precision to avoid “error escalation”

Division and Modulo

- Modulo-operator computes the rest of the integer division

`5 / 2` has value 2, `5 % 2` has value 1.

- It holds that

```
(-a)/b == -(a/b)
```

- It also holds:

`(a / b) * b + a % b` has the value of `a`.

- From the above one can conclude the results of division and modulo with negative numbers

Increment and decrement

- Increment / Decrement a number by one is a frequent operation
- works like this for an L-value:

```
expr = expr + 1.
```

Disadvantages

- relatively long
- `expr` is evaluated twice
  - Later: L-valued expressions whose evaluation is “expensive”
  - `expr` could have an effect (but should not, cf. guideline)

In-/Decrement Operators

Post-Increment

```
expr++
```

Value of `expr` is increased by one, the **old** value of `expr` is returned (as R-value)

Pre-increment

```
++expr
```

Value of `expr` is increased by one, the **new** value of `expr` is returned (as L-value)

Post-Dekrement

```
expr--
```

Value of `expr` is decreased by one, the **old** value of `expr` is returned (as R-value)

Prä-Dekrement

```
--expr
```

Value of `expr` is increased by one, the **new** value of `expr` is returned (as L-value)

In-/decrement Operators

	use	arity	prec	assoz	L-/R-value
Post-increment	`expr++`	1	17	left	L-value → R-value
Pre-increment	`++expr`	1	16	right	L-value → L-value
Post-decrement	`expr--`	1	17	left	L-value → R-value
Pre-decrement	`--expr`	1	16	right	L-value → L-value

In-/Decrement Operators

```
int a = 7;
std::cout << ++a << "\n"; // 8
std::cout << a++ << "\n"; // 8
std::cout << a << "\n"; // 9
```


In-/Decrement Operators

Is the expression

`++expr;` ← we favour this

equivalent to

`expr++;`?

Yes, but

- Pre-increment can be more efficient (old value does not need to be saved)
- Post In-/Decrement are the only left-associative unary operators (not very intuitive)

C++ vs. ++C

Strictly speaking our language should be named `++C` because

- it is an advancement of the language `C`
- while `C++` returns the old `C`.

Arithmetic Assignments

$$\texttt{a += b}$$
$$\Leftrightarrow$$
$$\texttt{a = a + b}$$

analogously for `-`, `*`, `/` and `%`

Arithmetic Assignments

	Gebrauch	Bedeutung
`+=`	`expr1 += expr2`	`expr1 = expr1 + expr2`
`-=`	`expr1 -= expr2`	`expr1 = expr1 - expr2`
`*=`	`expr1 *= expr2`	`expr1 = expr1 * expr2`
`/=`	`expr1 /= expr2`	`expr1 = expr1 / expr2`
`%=`	`expr1 %= expr2`	`expr1 = expr1 % expr2`

Arithmetic expressions evaluate expr1 only once.
Assignments have precedence 4 and are right-associative.

Binary Number Representations

Binary representation (Bits from $\{0, 1\}$)

$$b_n b_{n-1} \dots b_1 b_0$$

corresponds to the number $b_n \cdot 2^n + \cdots + b_1 \cdot 2 + b_0$

`101011` corresponds to 43.

Least Significant Bit (LSB)

Most Significant Bit (MSB)

Computing Tricks

- Estimate the orders of magnitude of powers of two.[2]:

$2^{10} = 1024 = 1\mathrm{Ki} \approx 10^3$,
$2^{20} = 1\mathrm{Mi} \approx 10^6$,
$2^{30} = 1\mathrm{Gi} \approx 10^9$,
$2^{32} = 4 \cdot (1024)^3 = 4\mathrm{Gi}$,
$2^{64} = 16\mathrm{Ei} \approx 16 \cdot 10^{18}$.

[2]Decimal vs. binary units: MB - Megabyte vs. MiB - Megabibyte (etc.)
kilo (K, Ki) – mega (M, Mi) – giga (G, Gi) – tera(T, Ti) – peta(P, Pi) – exa (E, Ei)

Hexadecimal Numbers

Numbers with base 16

$$h_n h_{n-1} \dots h_1 h_0$$

corresponds to the number

$$h_n \cdot 16^n + \dots + h_1 \cdot 16 + h_0.$$

notation in C++: prefix `0x`

`0xff` corresponds to 255.

Hex Nibbles

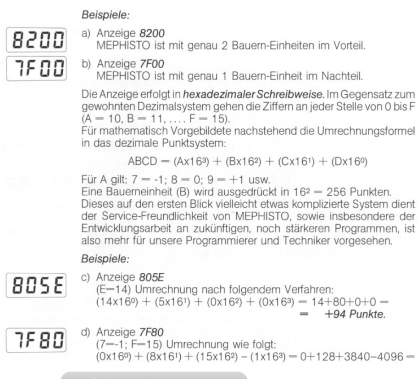
hex	bin	dec
0	0000	0
1	**0001**	**1**
2	**0010**	**2**
3	0011	3
4	**0100**	**4**
5	0101	5
6	0110	6
7	0111	7
8	**1000**	**8**
9	1001	9
a	1010	10
b	1011	11
c	1100	12
d	1101	13
e	1110	14
f	1111	15

Why Hexadecimal Numbers?

- A Hex-Nibble requires exactly 4 bits. Numbers 1, 2, 4 and 8 represent bits 0, 1, 2 and 3.
- “compact representation of binary numbers”

Why Hexadecimal Numbers?

“For programmers and technicians” (user manual of the chess computers *Mephisto II*, 1981)



Beispiele:

8200

a) Anzeige ***8200***
MEPHISTO ist mit genau 2 Bauern-Einheiten im Vorteil.

7F00

b) Anzeige ***7F00***
MEPHISTO ist mit genau 1 Bauern-Einheit im Nachteil.

Die Anzeige erfolgt in ***hexadezimaler Schreibweise.*** Im Gegensatz zum gewohnten Dezimalsystem gehen die Ziffern an jeder Stelle von 0 bis F (A = 10, B = 11, ... F = 15).
Für mathematisch Vorgebildete nachstehend die Umrechnungsformel in das dezimale Punktsystem:

$$\text{ABCD} = (\text{A}\text{x}16^3) + (\text{B}\text{x}16^2) + (\text{C}\text{x}16^1) + (\text{D}\text{x}16^0)$$

Für A gilt: 7 = -1; 8 = 0; 9 = +1 usw.
Eine Bauerneinheit (B) wird ausgedrückt in $16^2 = 256$ Punkten.
Dieses auf den ersten Blick vielleicht etwas komplizierte System dient der Service-Freundlichkeit von MEPHISTO, sowie insbesondere der Entwicklungsarbeit an zukünftigen, noch stärkeren Programmen, ist also mehr für unsere Programmierer und Techniker vorgesehen.

Beispiele:

805E

c) Anzeige ***805E***
(E=14) Umrechnung nach folgendem Verfahren:
$(14\text{x}16^0) + (5\text{x}16^1) + (0\text{x}16^2) + (0\text{x}16^3) = 14+80+0+0 =$
$=$ *+94 Punkte.*

7F80

d) Anzeige ***7F80***
(7=-1; F=15) Umrechnung wie folgt:
$(0\text{x}16^0) + (8\text{x}16^1) + (15\text{x}16^2) - (1\text{x}16^3) = 0+128+3840-4096 =$

http://www.zanchetta.net/default.aspx?Categorie=ECHIQUIERS&Page=documentations

Example: Hex-Colors

$$\#\underbrace{00}_{r}\underbrace{FF}_{g}\underbrace{00}_{b}$$

Why Hexadecimal Numbers?

The NZZ could have saved a lot of space ...

Domain of Type `int`

```
// Output the smallest and the largest value of type int.
#include <iostream>
#include <limits>

int main() {
  std::cout << "Minimum int value is "
            << std::numeric_limits<int>::min() << ".\n"
            << "Maximum int value is "
            << std::numeric_limits<int>::max() << ".\n";
  return 0;
}
```

```
Minimum int value is -2147483648.
Maximum int value is 2147483647.
```

Where do these numbers come from?

Domain of the Type `int`

- Representation with B bits. Domain comprises the 2^B integers:

$$\{-2^{B-1}, -2^{B-1}+1, \ldots, -1, 0, 1, \ldots, 2^{B-1}-2, 2^{B-1}-1\}$$

- On most platforms $B = 32$
- For the type **`int`** $\mathrm{C}{+}{+}$ guarantees $B \geq 16$
- Background: Section 2.2.8 (Binary Representation) in the lecture notes.

Over- and Underflow

- Arithmetic operations (`+`,`-`,`*`) can lead to numbers outside the valid domain.
- Results can be incorrect!

  `power8.cpp`: $15^8 = -1732076671$
- There is **no error message!**

The Type `unsigned int`

- Domain
$$\{0, 1, \ldots, 2^B - 1\}$$
- All arithmetic operations exist also for **`unsigned int`**.
- Literals: **`1u, 17u`** ...

Mixed Expressions

- Operators can have operands of different type (e.g. **`int`** and **`unsigned int`**).

```
17 + 17u
```

- Such mixed expressions are of the "more general" type **`unsigned int`**.
- **`int`**-operands are **converted** to **`unsigned int`**.

Conversion

`int` Value	Sign	`unsigned int` Value
x	≥ 0	x
x	< 0	$x + 2^B$

Due to a clever representation (two's complement), no addition is internally needed

Conversion "reversed"

The declaration

```
int a = 3u;
```

converts **`3u`** to **`int`**.
The value is preserved because it is in the domain of **`int`**; otherwise the result depends on the implementation.

Signed Numbers

Note: the remaining slides on signed numbers, computing with binary numbers, and the two's complement, are *not* relevant for the exam

Signed Number Representation

- (Hopefully) clear by now: binary number representation without sign, e.g.
$$[b_{31}b_{30}\ldots b_0]_u \quad \widehat{=} \quad b_{31}\cdot 2^{31} + b_{30}\cdot 2^{30} + \cdots + b_0$$
- Looking for a consistent solution

The representation with sign should coincide with the unsigned solution as much as possible. Positive numbers should arithmetically be treated equal in both systems.

Computing with Binary Numbers (4 digits)

Simple Addition

$$\begin{array}{r} 2 \\ +3 \\ \hline 5 \end{array} \qquad \begin{array}{r} 0010 \\ +0011 \\ \hline 0101_2 \end{array} \quad = 5_{10}$$

Simple Subtraction

$$\begin{array}{r} 5 \\ -3 \\ \hline 2 \end{array} \qquad \begin{array}{r} 0101 \\ -0011 \\ \hline 0010_2 \end{array} \quad = 2_{10}$$

Computing with Binary Numbers (4 digits)

Addition with Overflow

$$\begin{array}{r} 7 \\ +10 \\ \hline 17 \end{array} \qquad \begin{array}{r} 0111 \\ +1010 \\ \hline (1)0001_2 \end{array} \quad = 1_{10}(= 17 \bmod 16)$$

Subtraction with underflow

$$\begin{array}{r} 5 \\ +(-10) \\ \hline -5 \end{array} \qquad \begin{array}{r} 0101 \\ 1010 \\ \hline (\ldots 11)1011_2 \end{array} \quad = 11_{10}(= -5 \bmod 16)$$

Why this works

Modulo arithmetics: Compute on a circle[3]

$11 \equiv 23 \equiv -1 \equiv \ldots \mod 12$ $\quad$ $4 \equiv 16 \equiv \ldots \mod 12$ $\quad$ $3 \equiv 15 \equiv \ldots \mod 12$

[3]The arithmetics also work with decimal numbers (and for multiplication).

Negative Numbers (3 Digits)

	a	$-a$	
0	000	000	0
1	001	**111**	-1
2	010	**110**	-2
3	011	**101**	-3
4	**100**	**100**	-4
5	**101**		
6	**110**		
7	**111**		

The most significant bit decides about the sign *and* it contributes to the value.

Two's Complement

- Negation by bitwise negation and addition of 1

  $-2 = -[0010] = [1101] + [0001] = [1110]$

- Arithmetics of addition and subtraction **identical** to unsigned arithmetics

  $3 - 2 = 3 + (-2) = [0011] + [1110] = [0001]$

- Intuitive “wrap-around” conversion of negative numbers.

  $-n \to 2^B - n$

- Domain: $-2^{B-1} \ldots 2^{B-1} - 1$