11. Hashing

Hash Tables, Pre-Hashing, Hashing, Resolving Collisions using Chaining, Simple Uniform Hashing, Popular Hash Functions, Table-Doubling, Open Addressing: Probing [Ottman/Widmayer, Kap. 4.1-4.3.2, 4.3.4, Cormen et al, Kap. 11-11.4]

Motivating Example

Gloal: Efficient management of a table of all n ETH-students of
Possible Requirement: fast access (insertion, removal, find) of a dataset by name

Dictionary

Abstract Data Type (ADT) D to manage items[12] i with keys $k \in \mathcal{K}$ with operations

- **`D.insert`**(i): Insert or replace i in the dictionary D.
- **`D.delete`**(i): Delete i from the dictionary D. Not existing $\Rightarrow$ error message.
- **`D.search`**(k): Returns item with key k if it exists.

[12] Key-value pairs (k, v), in the following we consider mainly the keys

Dictionaries in Python

dictionary

```
fruits = {
  "banana": 2.95, "kiwi": 0.70,
  "pear": 4.20, "apple": 3.95
}
```

insert

```
fruits["melon"] = 3.95
```

update

```
fruits["banana"] = 1.90
```

find

```
print("banana", fruits["banana"])
print("melon in fruits", "melon" in
fruits)print("onion in fruits"
, "onion" in fruits)
```

remove

```
del fruits["strawberry"]
```

iterate

```
for name,price in fruits.items():
  print(name,"->",price)
```

Dictionaries in Java

dictionary	`Map<String,Double> fruits =`
	`  new HashMap<String,Double>();`
insert	`fruits.put("banana", 2.95);`
	`fruits.put("kiwi", 0.70);`
	`fruits.put("strawberry", 9.95);`
	`fruits.put("pear", 4.20);`
	`fruits.put("apple", 3.95);`
update	`fruits.put("banana", 2.90);`
find	`Out.println("banana " + fruits.get("banana"));`
remove	`fruits.remove("banana");`
iterate	`for (String s: fruits.keySet())`
	`  Out.println(s+" " + fruits.get(s));`

Motivation / Use

Perhaps **the** most popular data structure.

- Supported in many programming languages (C++, Java, Python, Ruby, Javascript, C# ...)
- Obvious use
  - Databases, Spreadsheets
  - Symbol tables in compilers and interpreters
- Less obvious
  - Substrin Search (Google, grep)
  - String commonalities (Document distance, DNA)
  - File Synchronisation
  - Cryptography: File-transfer and identification

1. Idea: Direct Access Table (Array)

Index	Item
0	-
1	-
2	-
3	[3,value(3)]
4	-
5	-
$\vdots$	$\vdots$
k	[k,value(k)]
$\vdots$	$\vdots$

Problems

1. Keys must be non-negative integers
2. Large key-range $\Rightarrow$ large array

Solution to the first problem: Pre-hashing

Prehashing: Map keys to positive integers using a function $ph : \mathcal{K} \to \mathbb{N}$

- Theoretically always possible because each key is stored as a bit-sequence in the computer
- Theoretically also: $x = y \Leftrightarrow ph(x) = ph(y)$
- Practically: APIs offer functions for pre-hashing. (Java: **`object.hashCode()`**, C++: **`std::hash<>`**, Python: **`hash(object)`**)
- APIs map the key from the key set to an integer with a restricted size.[13]

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[13] Therefore the implication $ph(x) = ph(y) \Rightarrow x = y$ does **not** hold any more for all x,y.

Prehashing Example : String

Mapping Name $s = s_1 s_2 \ldots s_{l_s}$ to key

$$ph(s) = \left(\sum_{i=0}^{l_s - 1} s_{l_s - i} \cdot b^i \right) \bmod 2^w$$

b so that different names map to different keys as far as possible.

b Word-size of the system (e.g. 32 or 64)

Example (Java) with $b = 31$, $w = 32$. Ascii-Values s_i.

Anna $\mapsto 2045632$
Jacqueline $\mapsto 2042089953442505 \bmod 2^{32} = 507919049$

Implementation Prehashing (String) in Java

$$ph_{b,m}(s) = \left(\sum_{i=0}^{l-1} s_{l-i+1} \cdot b^i\right) \bmod m$$

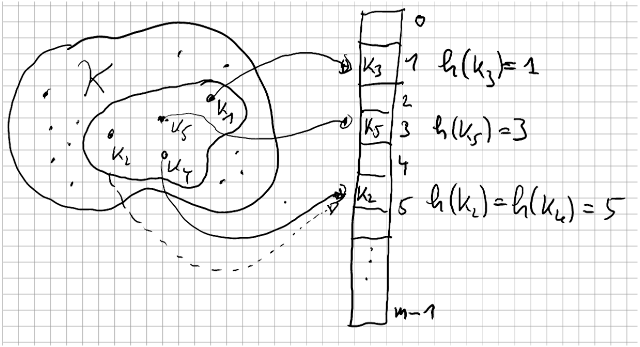
With $b = 31$ and $m = 2^{32}$ we get in Java[14]

```
int prehash(String s){
  int h = 0;
  for (int k = 0; k < s.length(); ++k){
    h = h * b + s.charAt(k);
  }
  return h;
}
```

[14] Try to understand why this works

Lösung zum zweiten Problem: Hashing

Reduce the universe. Map (hash-function) $h : \mathcal{K} \to \{0, ..., m-1\}$ ($m \approx n =$ number entries of the table)



Collision: $h(k_i) = h(k_j)$.

Nomenclature

Hash funtion h: Mapping from the set of keys $\mathcal{K}$ to the index set $\{0, 1, \ldots, m-1\}$ of an array (**hash table**).

$$h : \mathcal{K} \rightarrow \{0, 1, \ldots, m-1\}.$$

Normally $|\mathcal{K}| \gg m$. There are $k_1, k_2 \in \mathcal{K}$ with $h(k_1) = h(k_2)$ (**collision**).

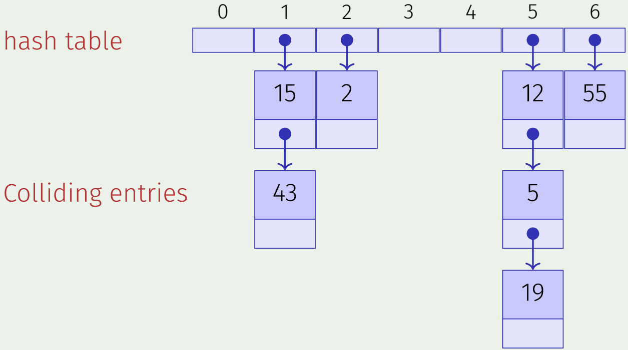
A hash function should map the set of keys as uniformly as possible to the hash table.

Resolving Collisions: Chaining

$m = 7, \mathcal{K} = \{0, \ldots, 500\}, h(k) = k \bmod m.$

Keys 12 , 55 , 5 , 15 , 2 , 19 , 43

Direct Chaining of the Colliding entries



Algorithm for Hashing with Chaining

- **`insert`**(i) Check if key k of item i is in list at position $h(k)$. If no, then append i to the end of the list. Otherwise replace element by i.
- **`find`**(k) Check if key k is in list at position $h(k)$. If yes, return the data associated to key k, otherwise return empty element **`null`**.
- **`delete`**(k) Search the list at position $h(k)$ for k. If successful, remove the list element.

Worst-case Analysis

Worst-case: all keys are mapped to the same index.
$\Rightarrow \Theta(n)$ per operation in the worst case. ☹

Simple Uniform Hashing

Strong Assumptions: Each key will be mapped to one of the m available slots

- with equal probability (Uniformity)
- and independent of where other keys are hashed (Independence).

Simple Uniform Hashing

Under the assumption of simple uniform hashing:
Expected length of a chain when n elements are inserted into a hash table with m elements

$$\mathbb{E}(\text{Länge Kette j}) = \mathbb{E}\left(\sum_{i=0}^{n-1} \mathbb{1}(k_i = j)\right) = \sum_{i=0}^{n-1} \mathbb{P}(k_i = j)$$
$$= \sum_{i=1}^{n} \frac{1}{m} = \frac{n}{m}$$

$\alpha = n/m$ is called **load factor** of the hash table.

Simple Uniform Hashing

Theorem 5

Let a hash table with chaining be filled with load-factor $\alpha = \frac{n}{m} < 1$. Under the assumption of simple uniform hashing, the next operation has expected costs of $\leq 1 + \alpha$.

Consequence: if the number slots m of the hash table is always at least proportional to the number of elements n of the hash table, $n \in \mathcal{O}(m) \Rightarrow$ Expected Running time of Insertion, Search and Deletion is $\mathcal{O}(1)$.

Advantages and Disadvantages of Chaining

Advantages

- Possible to overcommit: $\alpha > 1$ allowed
- Easy to remove keys.

Disadvantages

- Memory consumption of the chains-

An Example of a popular Hash Function

Division method

$$h(k) = k \bmod m$$

Ideal: m prime, not too close to powers of 2 or 10
But often: $m = 2^k - 1$ ($k \in \mathbb{N}$)
Other method: multiplication method (cf. Cormen et al, Kap. 11.3).

Table size increase

- We do not know beforehand how large n will be
- Require $m = \Theta(n)$ at all times.

Table size needs to be adapted. Hash-Function changes $\Rightarrow$ **rehashing**

- Allocate array A' with size $m' > m$
- Insert each entry of A into A' (with re-hashing the keys)
- Set $A \leftarrow A'$.
- Costs $\mathcal{O}(n + m + m')$.

How to choose m'?

Table size increase

- 1.Idea $n = m \Rightarrow m' \leftarrow m + 1$
  Increase for each insertion: Costs $\Theta(1 + 2 + 3 + \cdots + n) = \Theta(n^2)$ ☹
- 2.Idea $n = m \Rightarrow m' \leftarrow 2m$ Increase only if$m = 2^i$:
  $\Theta(1 + 2 + 4 + 8 + \cdots + n) = \Theta(n)$
  Few insertions cost linear time but on average we have $\Theta(1)$ ☺

Jede Operation vom Hashing mit Verketten hat erwartet amortisierte Kosten $\Theta(1)$.
($\Rightarrow$ Amortized Analysis)

Amortisierte Analyse

General procedure for dynamic arrays (e.g. Java: **`ArrayList`**, Python: **`List`**)

- The data structure provides, besides the data array, two numbers: size of the array (capacity m) and the number of used entries (size n)
- Double the size and copy entries when the list is full $n = m \Rightarrow m \leftarrow 2n$. Kosten $\Theta(m)$.
- Runtime costs for $n = 2^k$ insertion operations: $\Theta(1 + 2 + 4 + 8 + \cdots + 2^k) = \Theta(2^{k+1} - 1) = \Theta(n)$.

Costs per operation **averaged over all operations** = **amortized costs** = $\Theta(1)$ per insertion operation

Open Addressing

Store the colliding entries directly in the hash table using a **probing function** $s : \mathcal{K} \times \{0, 1, \ldots, m-1\} \to \{0, 1, \ldots, m-1\}$
Key table position along a **probing sequence**

$$S(k) := (s(k,0), s(k,1), \ldots, s(k,m-1)) \mod m$$

Probing sequence must for each $k \in \mathcal{K}$ be a permutation of $\{0, 1, \ldots, m-1\}$

Notational clarification: this method uses **open addressing**(meaning that the positions in the hashtable are not fixed) but it is a **closed hashing** procedure (because the entries stay in the hashtable)

Algorithms for open addressing

- **`insert`**(i) Search for kes k of i in the table according to $S(k)$. If k is not present, insert k at the first free position in the probing sequence. Otherwise error message.
- **`find`**(k) Traverse table entries according to $S(k)$. If k is found, return data associated to k. Otherwise return an empty element **`null`**.
- **`delete`**(k) Search k in the table according to $S(k)$. If k is found, replace it with a special key **`removed`**.

Linear Probing

$$s(k, j) = h(k) + j \Rightarrow S(k) = (h(k), h(k) + 1, \ldots, h(k) + m - 1) \mod m$$

$m = 7$, $\mathcal{K} = \{0, \ldots, 500\}$, $h(k) = k \bmod m$.

Key 12 , 55 , 5 , 15 , 2 , 19

0	1	2	3	4	5	6
5	15	2	19		12	55

Discussion

Example $\alpha = 0.95$

The unsuccessful search consideres 200 table entries on average! (here without derivation).

Disadvantage of the method?

Primary clustering: similar hash addresses have similar probing sequences $\Rightarrow$ long contiguous areas of used entries.

Quadratic Probing

$$s(k,j) = h(k) + \lceil j/2 \rceil^2 (-1)^{j+1}$$
$$S(k) = (h(k), h(k)+1, h(k)-1, h(k)+4, h(k)-4, \dots) \mod m$$

$m = 7, \mathcal{K} = \{0, \dots, 500\}, h(k) = k \mod m.$

Keys 12 , 55 , 5 , 15 , 2 , 19

0	1	2	3	4	5	6
19	15	2		5	12	55

Discussion

Example $\alpha = 0.95$

Unsuccessfuly search considers 22 entries on average (here without derivation)

Problems of this method?

Secondary clustering: Synonyms k and k' (with $h(k) = h(k')$) travers the same probing sequence.

Double Hashing

Two hash functions $h(k)$ and $h'(k)$. $s(k,j) = h(k) + j \cdot h'(k)$.
$S(k) = (h(k), h(k) + h'(k), h(k) + 2h'(k), \ldots, h(k) + (m-1)h'(k)) \mod m$

$m = 7$, $\mathcal{K} = \{0, \ldots, 500\}$, $h(k) = k \mod 7$, $h'(k) = 1 + k \mod 5$.

Keys 12 , 55 , 5 , 15 , 2 , 19

0	1	2	3	4	5	6
5	15	2	19		12	55

Double Hashing

- Probing sequence must permute all hash addresses. Thus $h'(k) \neq 0$ and $h'(k)$ may not divide m, for example guaranteed with m prime.
- h' should be as independent of h as possible (to avoid secondary clustering)

  Independence largely fulfilled by $h(k) = k \mod m$ and $h'(k) = 1 + k \mod (m-2)$ (m prime).

Uniform Hashing

Strong assumption: the probing sequence $S(k)$ of a key l is equaly likely to be any of the $m!$ permutations of $\{0, 1, \ldots, m-1\}$

(Double hashing is reasonably close)

Analysis of Uniform Hashing with Open Addressing

Theorem 6

Let an open-addressing hash table be filled with load-factor $\alpha = \frac{n}{m} < 1$. *Under the assumption of uniform hashing, the next operation has expected costs of* $\leq \frac{1}{1-\alpha}$.

Without Proof, cf. e.g. Cormen et al, Kap. 11.4