4. Algorithmen und Datenstrukturen

Algorithms and Data Structures, Overview
[Cormen et al, Kap. 1; Ottman/Widmayer, Kap. 1.1]

Algorithm

Algorithm

Well-defined procedure to compute **output** data from **input** data

Example Problem: Sorting

Input: A sequence of n numbers (comparable objects) $(a_1, a_2, \ldots, a_n)$

Example Problem: Sorting

Input: A sequence of n numbers (comparable objects) $(a_1, a_2, \ldots, a_n)$

Output: Permutation $(a'_1, a'_2, \ldots, a'_n)$ of the sequence $(a_i)_{1 \le i \le n}$, such that $a'_1 \le a'_2 \le \cdots \le a'_n$

Example Problem: Sorting

Input: A sequence of n numbers (comparable objects) $(a_1, a_2, \dots, a_n)$

Output: Permutation $(a'_1, a'_2, \dots, a'_n)$ of the sequence $(a_i)_{1 \leq i \leq n}$, such that $a'_1 \leq a'_2 \leq \dots \leq a'_n$

Possible input

$(1, 7, 3)$, $(15, 13, 12, -0.5)$, $(999, 998, 997, 996, \dots, 2, 1)$, (1), $()$ ...

Example Problem: Sorting

Input: A sequence of n numbers (comparable objects) $(a_1, a_2, \ldots, a_n)$
Output: Permutation $(a'_1, a'_2, \ldots, a'_n)$ of the sequence $(a_i)_{1 \leq i \leq n}$, such that $a'_1 \leq a'_2 \leq \cdots \leq a'_n$

Possible input

$(1, 7, 3)$, $(15, 13, 12, -0.5)$, $(999, 998, 997, 996, \ldots, 2, 1)$, (1), $()$ …

Every example represents a **problem instance**

The performance (speed) of an algorithm usually depends on the problem instance. Often there are "good" and "bad" instances.

Therefore we consider algorithms sometimes **"in the average"** and most often in the **"worst case"**.

Examples for algorithmic problems

- Tables and statistis: sorting, selection and searching

Examples for algorithmic problems

- Tables and statistis: sorting, selection and searching
- routing: shortest path algorithm, heap data structure

Examples for algorithmic problems

- Tables and statistis: sorting, selection and searching
- routing: shortest path algorithm, heap data structure
- DNA matching: Dynamic Programming

Examples for algorithmic problems

- Tables and statistis: sorting, selection and searching
- routing: shortest path algorithm, heap data structure
- DNA matching: Dynamic Programming
- evaluation order: Topological Sorting

Examples for algorithmic problems

- Tables and statistis: sorting, selection and searching
- routing: shortest path algorithm, heap data structure
- DNA matching: Dynamic Programming
- evaluation order: Topological Sorting
- autocomletion and spell-checking: Dictionaries / Trees

Examples for algorithmic problems

- Tables and statistis: sorting, selection and searching
- routing: shortest path algorithm, heap data structure
- DNA matching: Dynamic Programming
- evaluation order: Topological Sorting
- autocomletion and spell-checking: Dictionaries / Trees
- Fast Lookup : Hash-Tables

Examples for algorithmic problems

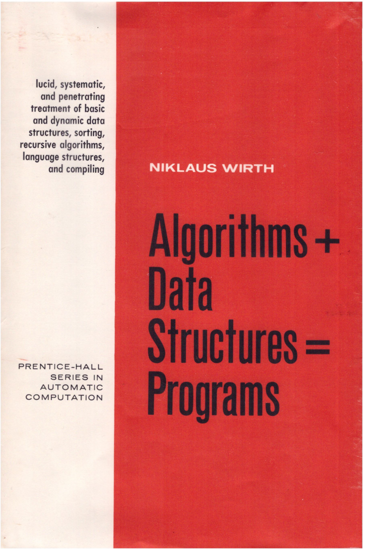
- Tables and statistis: sorting, selection and searching
- routing: shortest path algorithm, heap data structure
- DNA matching: Dynamic Programming
- evaluation order: Topological Sorting
- autocomletion and spell-checking: Dictionaries / Trees
- Fast Lookup : Hash-Tables
- The travelling Salesman: Dynamic Programming, Minimum Spanning Tree, Simulated Annealing

Characteristics

- Extremely large number of potential solutions
- Practical applicability

Data Structures

- A data structure is a particular way of **organizing data** in a computer so that they can be **used efficiently** (in the algorithms operating on them).
- Programs = algorithms + data structures.



Efficiency

- If computers were infinitely fast and had an infinite amount of memory …
- … then we would still need the theory of algorithms (only) for statements about correctness (and termination).

Efficiency

- If computers were infinitely fast and had an infinite amount of memory ...
- ... then we would still need the theory of algorithms (only) for statements about correctness (and termination).

Reality: resources are bounded and not free:

- Computing time $\rightarrow$ Efficiency
- Storage space $\rightarrow$ Efficiency

Actually, this course is nearly only about efficiency.

Hard problems.

- NP-complete problems: no known efficient solution (the existence of such a solution is very improbable – but it has not yet been proven that there is none!)
- Example: travelling salesman problem

This course is *mostly* about problems that can be solved efficiently (in polynomial time).

5. Efficiency of algorithms

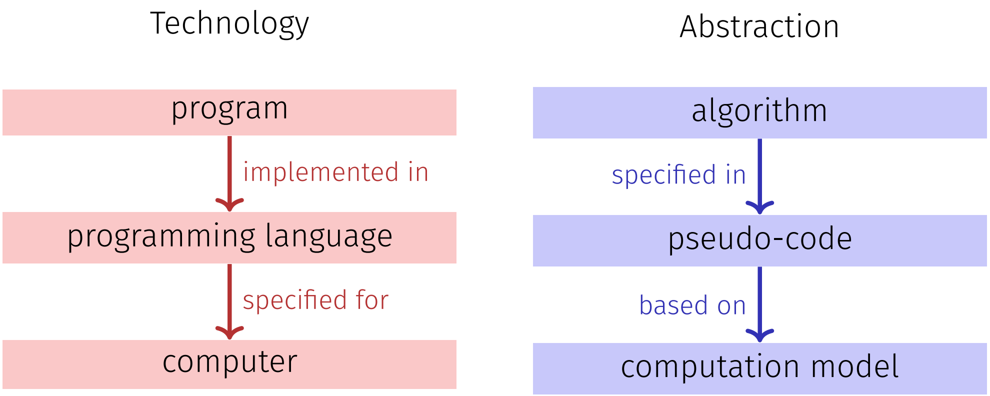
Efficiency of Algorithms, Random Access Machine Model, Function Growth, Asymptotics [Cormen et al, Kap. 2.2,3,4.2-4.4 | Ottman/Widmayer, Kap. 1.1]

Efficiency of Algorithms

Goals

- Quantify the runtime behavior of an algorithm independent of the machine.
- Compare efficiency of algorithms.
- Understand dependece on the input size.

Programs and Algorithms



Technology Model

Random Access Machine (RAM) Model

- Execution model: instructions are executed one after the other (on one processor core).

Technology Model

Random Access Machine (RAM) Model

- Execution model: instructions are executed one after the other (on one processor core).
- Memory model: constant access time (big array)

Technology Model

Random Access Machine (RAM) Model

- Execution model: instructions are executed one after the other (on one processor core).
- Memory model: constant access time (big array)
- Fundamental operations: computations ($+$,$-$,$\cdot$,...) comparisons, assignment / copy on machine words (registers), flow control (jumps)

Technology Model

Random Access Machine (RAM) Model

- Execution model: instructions are executed one after the other (on one processor core).
- Memory model: constant access time (big array)
- Fundamental operations: computations ($+,-,\cdot,\ldots$) comparisons, assignment / copy on machine words (registers), flow control (jumps)
- Unit cost model: fundamental operations provide a cost of 1.

Technology Model

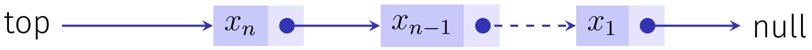
Random Access Machine (RAM) Model

- Execution model: instructions are executed one after the other (on one processor core).
- Memory model: constant access time (big array)
- Fundamental operations: computations ($+$,$-$,$\cdot$,...) comparisons, assignment / copy on machine words (registers), flow control (jumps)
- Unit cost model: fundamental operations provide a cost of 1.
- Data types: fundamental types like size-limited integer or floating point number.

For Dynamic Data Strcutures

Pointer Machine Model

- Objects bounded in size can be dynamically allocated in constant time
- Fields (with word-size) of the objects can be accessed in constant time 1.



Asymptotic behavior

An exact running time of an algorithm can normally not be predicted even for small input data.

- We consider the asymptotic behavior of the algorithm.
- And ignore all constant factors.

An operation with cost 20 is no worse than one with cost 1
Linear growth with gradient 5 is as good as linear growth with gradient 1.

5.2 Function growth

$\mathcal{O}$, Θ, Ω [Cormen et al, Kap. 3; Ottman/Widmayer, Kap. 1.1]

Superficially

Use the asymptotic notation to specify the execution time of algorithms.
We write $\Theta(n^2)$ and mean that the algorithm behaves for large n like n^2: when the problem size is doubled, the execution time multiplies by four.

More precise: asymptotic upper bound

provided: a function $g : \mathbb{N} \to \mathbb{R}$.

Definition:[1]

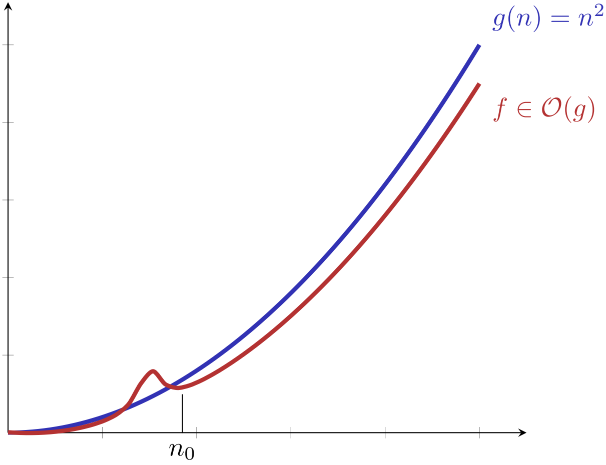
$$\begin{aligned}\mathcal{O}(g) = \{f : \mathbb{N} \to \mathbb{R}| \\ \exists\, c > 0, \exists n_0 \in \mathbb{N} : \\ \forall\, n \geq n_0 : 0 \leq f(n) \leq c \cdot g(n)\}\end{aligned}$$

Notation:

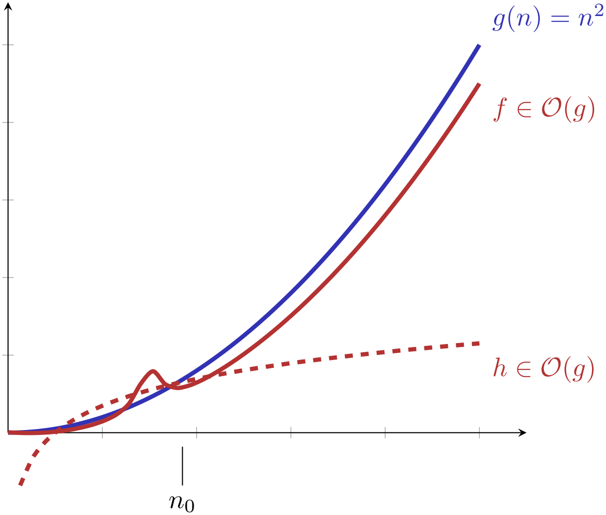
$$\mathcal{O}(g(n)) := \mathcal{O}(g(\cdot)) = \mathcal{O}(g).$$

[1]Ausgesprochen: Set of all functions $f : \mathbb{N} \to \mathbb{R}$ that satisfy: there is some (real valued) $c > 0$ and some $n_0 \in \mathbb{N}$ such that $0 \leq f(n) \leq n \cdot g(n)$ for all $n \geq n_0$.

Graphic



Graphic



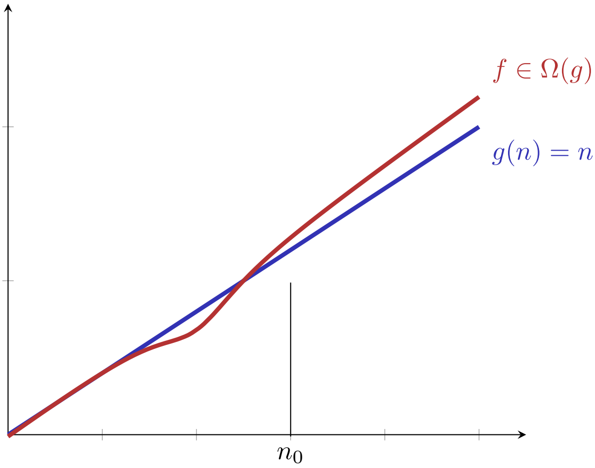
Converse: asymptotic lower bound

Given: a function $g : \mathbb{N} \to \mathbb{R}$.

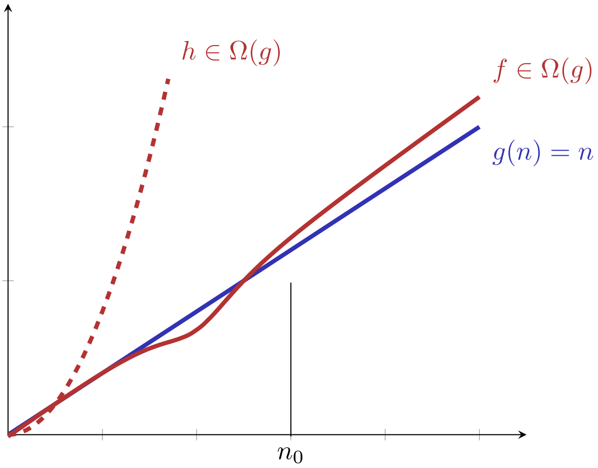
Definition:

$$\begin{aligned}\Omega(g) = \{f : \mathbb{N} \to \mathbb{R} \mid \\ &\exists\, c > 0, \exists n_0 \in \mathbb{N} : \\ &\forall\, n \geq n_0 : 0 \leq c \cdot g(n) \leq f(n)\}\end{aligned}$$

Example



Example



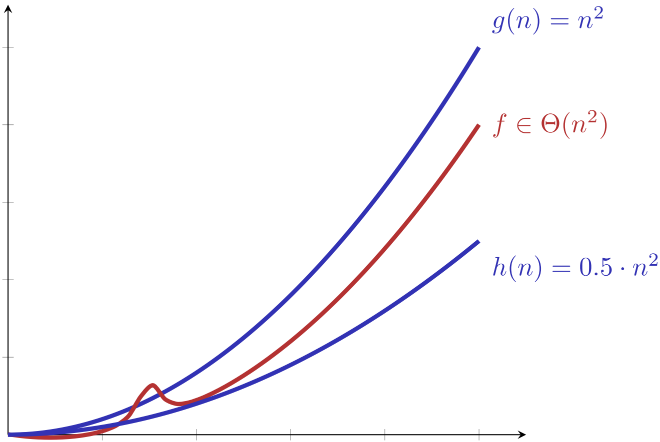
Asymptotic tight bound

Given: function $g : \mathbb{N} \to \mathbb{R}$.
Definition:

$$\Theta(g) := \Omega(g) \cap \mathcal{O}(g).$$

Simple, closed form: exercise.

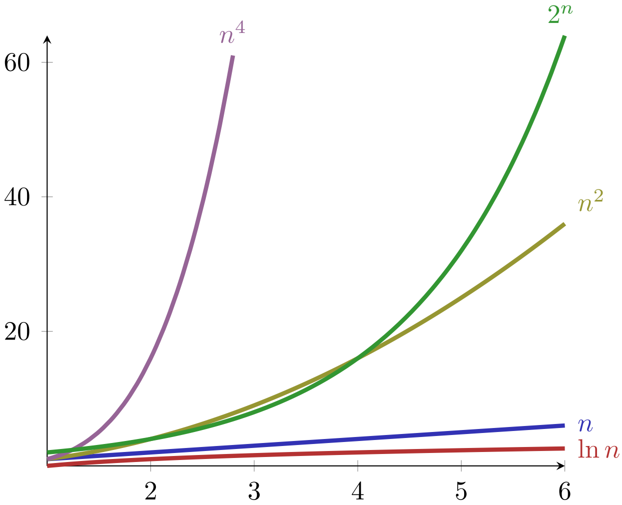
Example



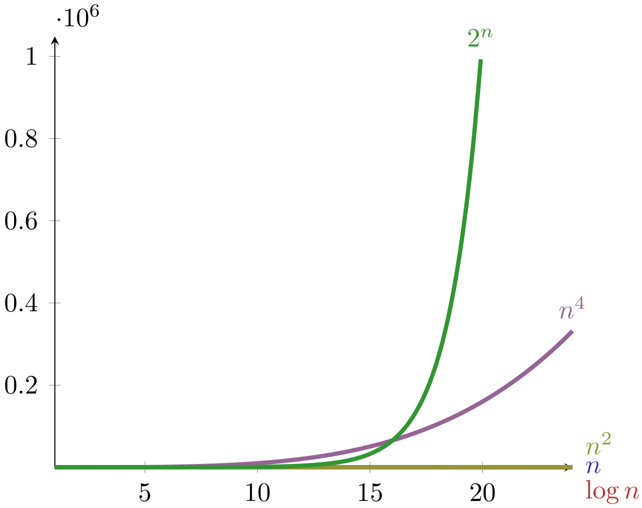
Notions of Growth

$\mathcal{O}(1)$	bounded	array access
$\mathcal{O}(\log\log n)$	double logarithmic	interpolated binary sorted sort
$\mathcal{O}(\log n)$	logarithmic	binary sorted search
$\mathcal{O}(\sqrt{n})$	like the square root	naive prime number test
$\mathcal{O}(n)$	linear	unsorted naive search
$\mathcal{O}(n\log n)$	superlinear / loglinear	good sorting algorithms
$\mathcal{O}(n^2)$	quadratic	simple sort algorithms
$\mathcal{O}(n^c)$	polynomial	matrix multiply
$\mathcal{O}(2^n)$	exponential	Travelling Salesman Dynamic Programming
$\mathcal{O}(n!)$	factorial	Travelling Salesman naively

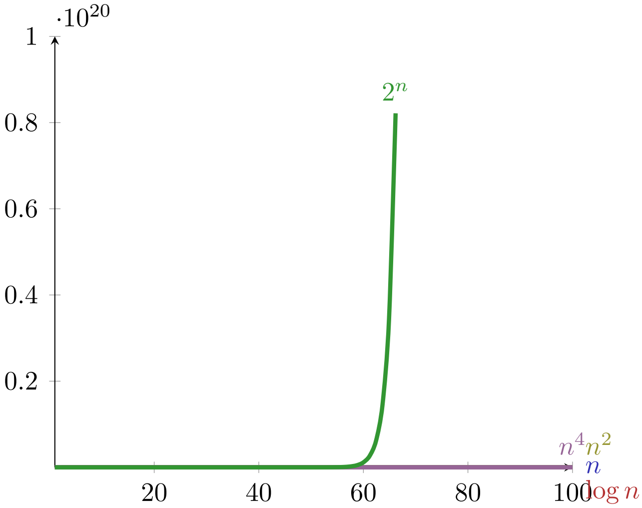
Small n



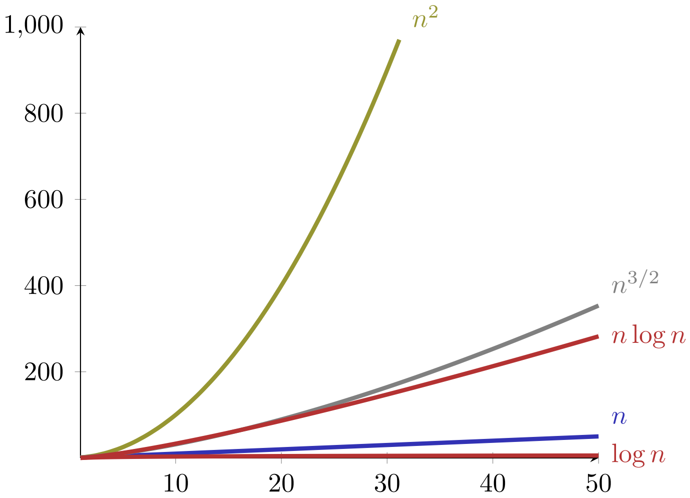
Larger n



“Large” n



Logarithms



Time Consumption

Assumption 1 Operation = $1\mu s$.

problem size	1	100	10000	10^6	10^9
$\log_2 n$	$1\mu s$				
n	$1\mu s$				
$n \log_2 n$	$1\mu s$				
n^2	$1\mu s$				
2^n	$1\mu s$				

Time Consumption

Assumption 1 Operation = $1\mu s$.

problem size	1	100	10000	10^6	10^9
$\log_2 n$	$1\mu s$				
n	$1\mu s$	$100\mu s$	$1/100s$	$1s$	17 minutes
$n \log_2 n$	$1\mu s$				
n^2	$1\mu s$				
2^n	$1\mu s$				

Time Consumption

Assumption 1 Operation = $1\mu s$.

problem size	1	100	10000	10^6	10^9
$\log_2 n$	$1\mu s$				
n	$1\mu s$	$100\mu s$	$1/100s$	$1s$	17 minutes
$n \log_2 n$	$1\mu s$				
n^2	$1\mu s$	$1/100s$	1.7 minutes	11.5 days	317 centuries
2^n	$1\mu s$				

Time Consumption

Assumption 1 Operation = $1\mu s$.

problem size	1	100	10000	10^6	10^9
$\log_2 n$	$1\mu s$	$7\mu s$	$13\mu s$	$20\mu s$	$30\mu s$
n	$1\mu s$	$100\mu s$	$1/100s$	$1s$	17 minutes
$n \log_2 n$	$1\mu s$				
n^2	$1\mu s$	$1/100s$	1.7 minutes	11.5 days	317 centuries
2^n	$1\mu s$				

Time Consumption

Assumption 1 Operation = $1\mu s$.

problem size	1	100	10000	10^6	10^9
$\log_2 n$	$1\mu s$	$7\mu s$	$13\mu s$	$20\mu s$	$30\mu s$
n	$1\mu s$	$100\mu s$	$1/100s$	$1s$	17 minutes
$n \log_2 n$	$1\mu s$	$700\mu s$	$13/100\mu s$	$20s$	8.5 hours
n^2	$1\mu s$	$1/100s$	1.7 minutes	11.5 days	317 centuries
2^n	$1\mu s$				

Time Consumption

Assumption 1 Operation = $1\mu s$.

problem size	1	100	10000	10^6	10^9
$\log_2 n$	$1\mu s$	$7\mu s$	$13\mu s$	$20\mu s$	$30\mu s$
n	$1\mu s$	$100\mu s$	$1/100s$	$1s$	17 minutes
$n \log_2 n$	$1\mu s$	$700\mu s$	$13/100\mu s$	$20s$	8.5 hours
n^2	$1\mu s$	$1/100s$	1.7 minutes	11.5 days	317 centuries
2^n	$1\mu s$	10^{14} centuries	$\approx \infty$	$\approx \infty$	$\approx \infty$

About the Notation

Common casual notation

$$f = \mathcal{O}(g)$$

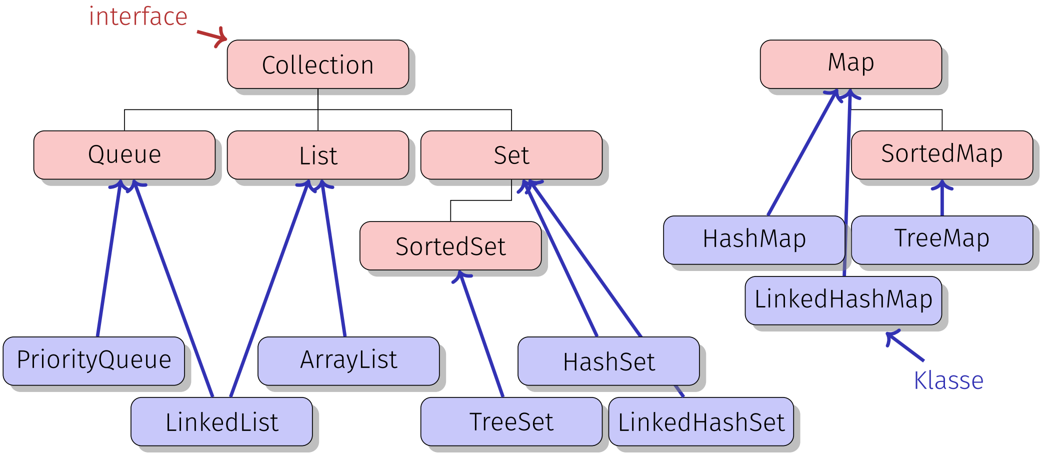
should be read as $f \in \mathcal{O}(g)$.
Clearly it holds that

$$f_1 = \mathcal{O}(g), f_2 = \mathcal{O}(g) \not\Rightarrow f_1 = f_2!$$

$n = \mathcal{O}(n^2), n^2 = \mathcal{O}(n^2)$ but naturally $n \neq n^2$.

We avoid this notation where it could lead to ambiguities.

Reminder: Java Collections / Maps

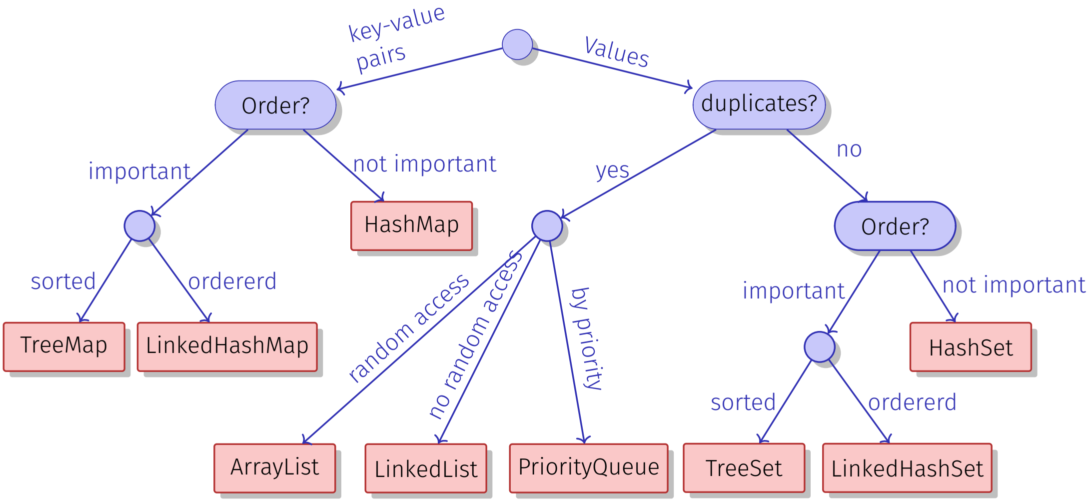


`ArrayList` versus `LinkedList`

run time measurements for 10000 operations (on [code] expert)

`ArrayList`	`LinkedList`
$469\mu s$	$1787\mu s$
$37900\mu s$	$761\mu s$
$1840\mu s$	$2050\mu s$
$426\mu s$	$110600\mu s$
$31ms$	$301ms$
$38ms$	$141ms$
$228ms$	$1080ms$
$648\mu s$	$757\mu s$
$58075\mu s$	$609\mu s$

Reminder: Decision



Asymptotic Runtimes (Java)

With our new language ($\Omega, \mathcal{O}, \Theta$), we can now **state the behavior of the data structures and their algorithms more precisely**

Asymptotic running times (Anticipation!)

Data structure	Random Access	Insert	Next	Insert After Element	Search
`ArrayList`	$\Theta(1)$	$\Theta(1)\ A$	$\Theta(1)$	$\Theta(n)$	$\Theta(n)$
`LinkedList`	$\Theta(n)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(n)$
`TreeSet`	–	$\Theta(\log n)$	$\Theta(\log n)$	–	$\Theta(\log n)$
`HashSet`	–	$\Theta(1)\ P$	–	–	$\Theta(1)\ P$

A = amortized, P=expected, otherwise worst case

Asymptotic Runtimes (Python)

Asymptotic running times

Data structure	Random Access	Insert	Iteration	Insert After Element	Search `x in S`
`list`	$\Theta(1)$	$\Theta(1)\ A$	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
`set`	–	$\Theta(1)\ P$	$\Theta(n)$	–	$\Theta(1)\ P$
`dict`	–	$\Theta(1)\ P$	$\Theta(n)$	–	$\Theta(1)\ P$

A = amortized, P=expected, otherwise worst case