17. Dynamic Programming II

Editing Distance, Bellman-Ford Algorithm
[Cormen et al, Kap. 24.1]]

Minimal Editing Distance

Editing distance of two sequences $A_n = (a_1, \ldots, a_n)$, $B_m = (b_1, \ldots, b_m)$.

Editing operations:

- Insertion of a character
- Deletion of a character
- Replacement of a character

Question: how many editing operations at least required in order to transform string A into string B.

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Minimal Editing Distance

Wanted: cheapest character-wise transformation $A_n \to B_m$ with costs

operation	Levenshtein	LCS[24]	general
Insert c	1	1	ins(c)
Delete c	1	1	del(c)
Replace $c \to c'$	$\mathbb{1}(c \neq c')$	$\infty \cdot \mathbb{1}(c \neq c')$	repl(c, c')

Beispiel

T	I	G	E	R		T	I	_	G	E	R		T→Z	+E	-R
Z	I	E	G	E		Z	I	E	G	E	_		Z→T	-E	+R

[24]Longest common subsequence – A special case of an editing problem

DP

0. $E(n, m)$ = mimimum number edit operations (ED cost) $a_{1\dots n} \to b_{1\dots m}$
1. Subproblems $E(i, j)$ = ED von $a_{1\dots i}$. $b_{1\dots j}$. #SP $= n \cdot m$
2. Guess Costs$\Theta(1)$

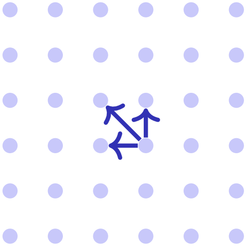
   - $a_{1..i} \to a_{1\dots i-1}$ (delete)
   - $a_{1..i} \to a_{1\dots i} b_j$ (insert)
   - $a_{1..i} \to a_{1\dots i-1} b_j$ (replace)

3. Rekursion

$$E(i,j) = \min \begin{cases} \text{del}(a_i) + E(i-1, j), \\ \text{ins}(b_j) + E(i, j-1), \\ \text{repl}(a_i, b_j) + E(i-1, j-1) \end{cases}$$

DP

4. Dependencies



$\Rightarrow$ Computation from left top to bottom right. Row- or column-wise.

5. Solution in $E(n, m)$

Example (Levenshtein Distance)

$$E[i,j] \leftarrow \min\left\{E[i-1,j]+1, E[i,j-1]+1, E[i-1,j-1]+\mathbb{1}(a_i \neq b_j)\right\}$$

	∅	Z	I	E	G	E
∅	0	1	2	3	4	5
T	1	1	2	3	4	5
I	2	2	1	2	3	4
G	3	3	2	2	2	3
E	4	4	3	2	3	2
R	5	5	4	3	3	3

Editing steps: from bottom right to top left, following the recursion.
Bottom-Up description of the algorithm: exercise

Bottom-Up DP algorithm ED

Dimension of the table? Semantics?

1.

Bottom-Up DP algorithm ED

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1. Table $E[0,\ldots,m][0,\ldots,n]$. $E[i,j]$: minimal edit distance of the strings $(a_1,\ldots,a_i)$ and $(b_1,\ldots,b_j)$

Bottom-Up DP algorithm ED

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Computation of an entry

2.

Bottom-Up DP algorithm ED

Dimension of the table? Semantics?

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Computation of an entry

2. $E[0,i] \leftarrow i \; \forall 0 \le i \le m$, $E[j,0] \leftarrow i \; \forall 0 \le j \le n$. Computation of $E[i,j]$ otherwise via $E[i,j] = \min\{\text{del}(a_i)+E(i-1,j), \text{ins}(b_j)+E(i,j-1), \text{repl}(a_i,b_j)+E(i-1,j-1)\}$

Bottom-Up DP algorithm ED

3. **Computation order**

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   Rows increasing and within columns increasing (or the other way round).

Bottom-Up DP algorithm ED

3\.

Computation order

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Reconstruction of a solution?

4\.

Bottom-Up DP algorithm ED

3.

Computation order

Rows increasing and within columns increasing (or the other way round).

4.

Reconstruction of a solution?

Start with $j = m$, $i = n$. If $E[i, j] = \text{repl}(a_i, b_j) + E(i-1, j-1)$ then output $a_i \to b_j$ and continue with $(j, i) \leftarrow (j-1, i-1)$; otherwise, if $E[i, j] = \text{del}(a_i) + E(i-1, j)$ output $\text{del}(a_i)$ and continue with $j \leftarrow j-1$ otherwise, if $E[i, j] = \text{ins}(b_j) + E(i, j-1)$, continue with $i \leftarrow i-1$. Terminate for $i = 0$ and $j = 0$.

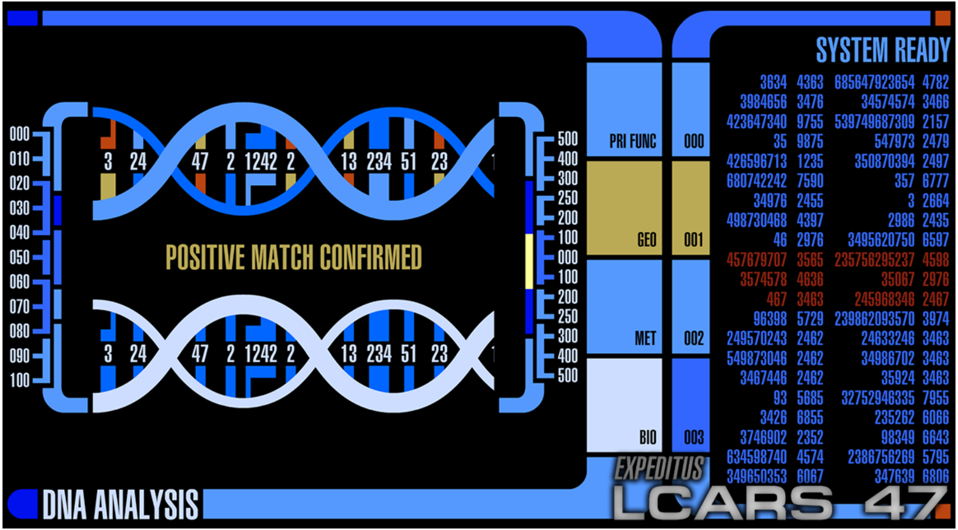
Analysis ED

- Number table entries: $(m+1)\cdot(n+1)$.
- Constant number of assignments and comparisons each. Number steps: $\mathcal{O}(mn)$
- Determination of solition: decrease i or j. Maximally $\mathcal{O}(n+m)$ steps.

Runtime overal:

$$\mathcal{O}(mn).$$

DNA - Comparison (Star Trek)



DNA - Comparison

- DNA consists of sequences of four different nucleotides **A**denine **G**uanine **T**hymine **C**ytosine
- DNA sequences (genes) thus can be described with strings of A, G, T and C.
- Possible comparison of two genes: determine the **longest common subsequence**

The longest common subsequence problem is a special case of the minimal edit distance problem.

Reminder: Shortest Path Algorithm

1. Initialise d_s and π_s: $d_s[v] = \infty$, $\pi_s[v] =$ null for each $v \in V$
2. Set $d_s[s] \leftarrow 0$
3. Choose an edge $(u, v) \in E$

   Relaxiere (u, v):
   if $d_s[v] > d[u] + c(u, v)$ then
   $d_s[v] \leftarrow d_s[u] + c(u, v)$
   $\pi_s[v] \leftarrow u$
4. Repeat 3 until nothing can be relaxed any more.
   (until $d_s[v] \leq d_s[u] + c(u, v) \quad \forall (u, v) \in E$)

Dynamic Programming Approach (Bellman)

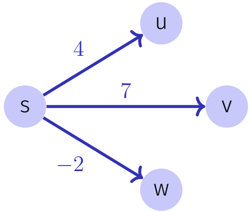
Induction over number of edges $d_s[i,v]$: Shortest path from s to v via maximally i edges.

$$d_s[i,v] = \min\{d_s[i-1,v], \min_{(u,v)\in E}(d_s[i-1,u] + c(u,v))$$

$$d_s[0,s] = 0, d_s[0,v] = \infty \ \forall v \neq s.$$

Dynamic Programming Approach (Bellman)

	s	$\cdots$	v	$\cdots$	w
0	0	∞	∞	∞	∞
1	0	∞	7	∞	-2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1$	0	$\cdots$	$\cdots$	$\cdots$	$\cdots$



Algorithm: Iterate over last row until the relaxation steps do not provide any further changes, maximally $n-1$ iterations. If still changes, then there is no shortest path.

Algorithm Bellman-Ford(G, s)

Input: Graph $G = (V, E, c)$, starting point $s \in V$
Output: If return value true, minimal weights d for all shortest paths from s, otherwise no shortest path.

foreach $u \in V$ **do**
  $d_s[u] \leftarrow \infty$; $\pi_s[u] \leftarrow$ null
$d_s[s] \leftarrow 0$;
for $i \leftarrow 1$ **to** $|V|$ **do**
  $f \leftarrow$ false
  foreach $(u, v) \in E$ **do**
    $f \leftarrow f \vee \mathrm{Relax}(u, v)$
  if $f =$ false **then return** true
return false;