3. Searching

Linear Search, Binary Search [Ottman/Widmayer, Kap. 3.2, Cormen et al, Kap. 2: Problems 2.1-3,2.2-3,2.3-5]

The Search Problem

Provided

- A set of data sets

examples

telephone book, dictionary, symbol table

- Each dataset has a key k.
- Keys are comparable: unique answer to the question $k_1 \leq k_2$ for keys k_1, k_2.

Task: find data set by key k.

Search in Array

Provided

- Array A with n elements $(A[1], \dots, A[n])$.
- Key b

Wanted: index k, $1 \leq k \leq n$ with $A[k] = b$ or "not found".

22	20	32	10	35	24	42	38	28	41
1	2	3	4	5	6	7	8	9	10

Linear Search

Traverse the array from $A[1]$ to $A[n]$.

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- *Best case:* 1 comparison.

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$$\frac{1}{n}\sum_{i=1}^{n} i = \frac{n+1}{2}.$$

Search in a Sorted Array

Provided

- Sorted array A with n elements $(A[1], \dots, A[n])$ with $A[1] \leq A[2] \leq \cdots \leq A[n]$.
- Key b

Wanted: index k, $1 \leq k \leq n$ with $A[k] = b$ or "not found".

10	20	22	24	28	32	35	38	41	42
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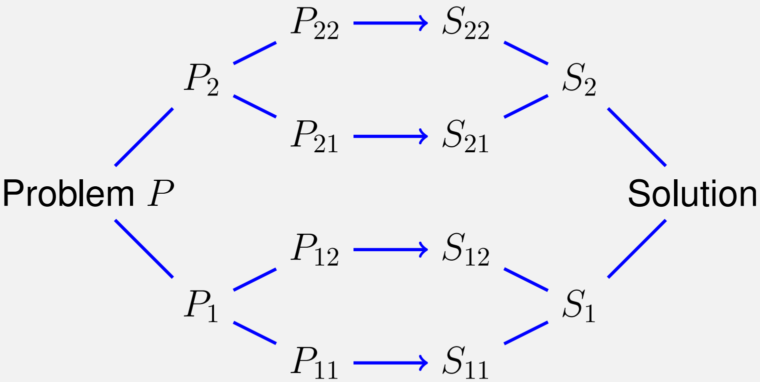
divide et impera

Divide and Conquer

Divide the problem into subproblems that contribute to the simplified computation of the overal problem.

divide et impera



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Search $b = 23$.

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$b < 28$

Divide and Conquer!

Search $b = 23$.

10	20	22	24	28	32	35	38	41	42	$b < 28$
1	2	3	4	5	6	7	8	9	10	

10	20	22	24	28	32	35	38	41	42	$b > 20$
1	2	3	4	5	6	7	8	9	10	

Divide and Conquer!

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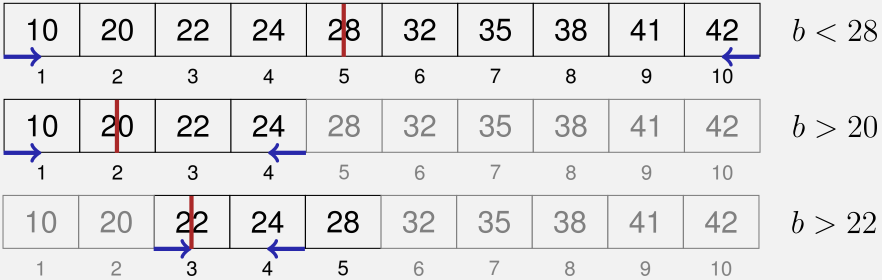

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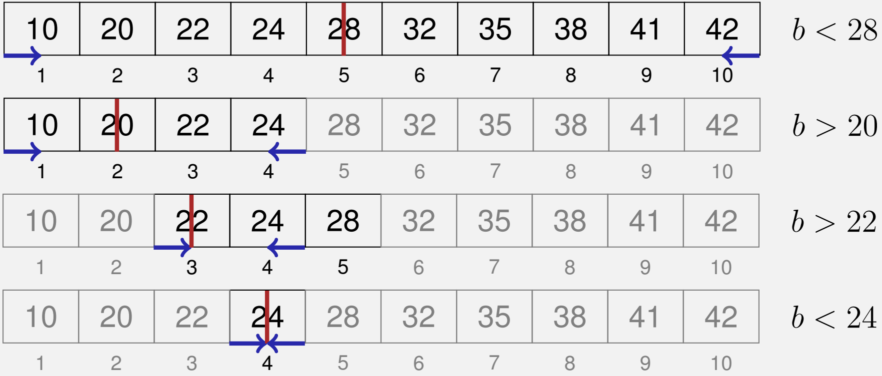

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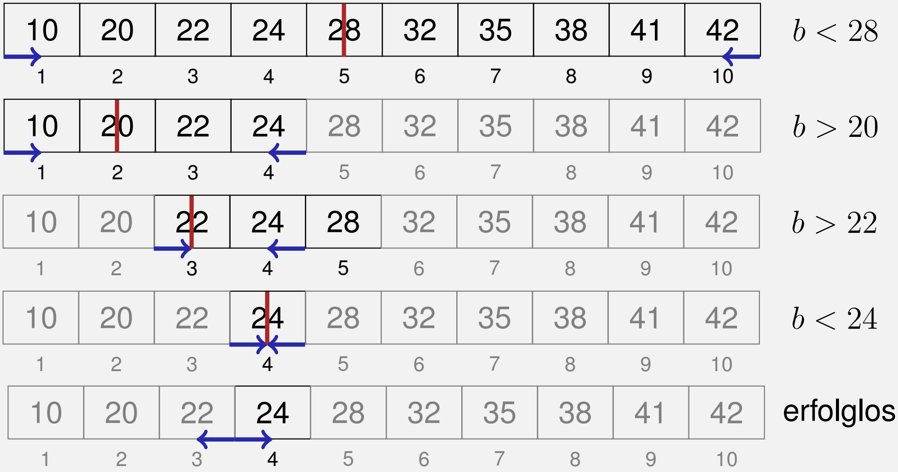

Binary Search Algorithm BSearch$(A[l..r], b)$

Input: Sorted array A of n keys. Key b. Bounds $1 \leq l \leq r \leq n$ or $l > r$ beliebig.
Output: Index of the found element. 0, if not found.

$m \leftarrow \lfloor (l + r)/2 \rfloor$
if $l > r$ **then** // Unsuccessful search
  return *NotFound*
else if $b = A[m]$ **then** // found
  return m
else if $b < A[m]$ **then** // element to the left
  return BSearch$(A[l..m-1], b)$
else // $b > A[m]$: element to the right
  return BSearch$(A[m+1..r], b)$

Analysis (worst case)

Recurrence ($n = 2^k$)

$$T(n) = \begin{cases} d & \text{falls } n = 1, \\ T(n/2) + c & \text{falls } n > 1. \end{cases}$$

Compute: [2]

$$T(n) = T\left(\frac{n}{2}\right) + c$$

[2] Try to find a closed form of T by applying the recurrence repeatedly (starting with $T(n)$).

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$$T(n) = T\left(\frac{n}{2}\right) + c = T\left(\frac{n}{4}\right) + 2c$$

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Compute: [2]

$$\begin{aligned} T(n) &= T\left(\frac{n}{2}\right) + c = T\left(\frac{n}{4}\right) + 2c = \dots \\ &= T\left(\frac{n}{2^i}\right) + i \cdot c \end{aligned}$$

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Analysis (worst case)

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Compute: [2]

$$\begin{aligned} T(n) &= T\left(\frac{n}{2}\right) + c = T\left(\frac{n}{4}\right) + 2c = \ldots \\ &= T\left(\frac{n}{2^i}\right) + i \cdot c \\ &= T\left(\frac{n}{n}\right) + \log_2 n \cdot c = d + c \cdot \log_2 n \in \Theta(\log n) \end{aligned}$$

---

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Analysis (worst case)

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Proof by induction:

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Proof by induction:

- Base clause: $T(1) = d$.

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$$T(n) = \begin{cases} d & \text{if } n = 1, \\ T(n/2) + c & \text{if } n > 1. \end{cases}$$

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- Base clause: $T(1) = d$.
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Guess : $T(n) = d + c \cdot \log_2 n$

Proof by induction:

- Base clause: $T(1) = d$.
- Hypothesis: $T(n/2) = d + c \cdot \log_2 n/2$
- Step: $(n/2 \to n)$

$$T(n) = T(n/2) + c = d + c \cdot (\log_2 n - 1) + c = d + c \log_2 n.$$

Result

Theorem

The binary sorted search algorithm requires $\Theta(\log n)$ *fundamental operations.*

4. Sorting

Simple Sorting, Quicksort, Mergesort

Problem

Input: An array $A = (A[1], ..., A[n])$ with length n.

Output: a permutation A' of A, that is sorted: $A'[i] \leq A'[j]$ for all $1 \leq i \leq j \leq n$.

Selection Sort

5 6 2 8 4 1 $(i = 1)$

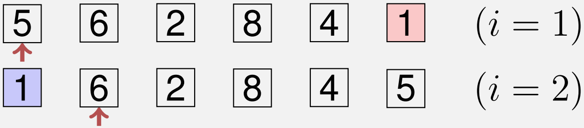
- Selection of the smallest element by search in the unsorted part $A[i..n]$ of the array.

Selection Sort

5 6 2 8 4 1 $(i = 1)$

- Selection of the smallest element by search in the unsorted part $A[i..n]$ of the array.
- Swap the smallest element with the first element of the unsorted part.

Selection Sort



- Selection of the smallest element by search in the unsorted part $A[i..n]$ of the array.
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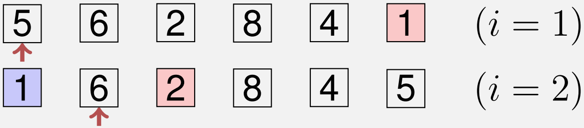
- Unsorted part decreases in size by one element $(i \to i + 1)$. Repeat until all is sorted. $(i = n)$

Selection Sort



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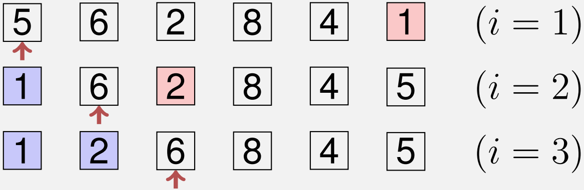

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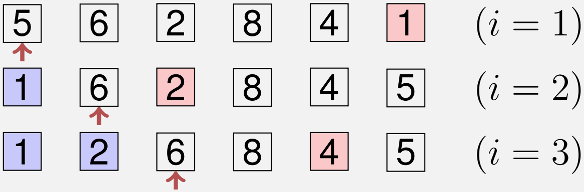

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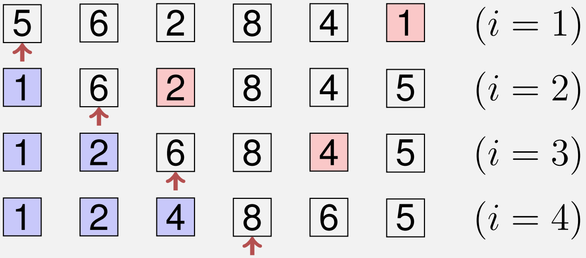

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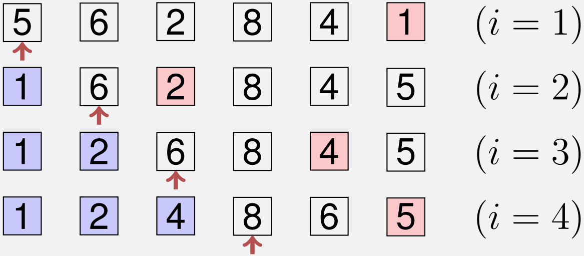

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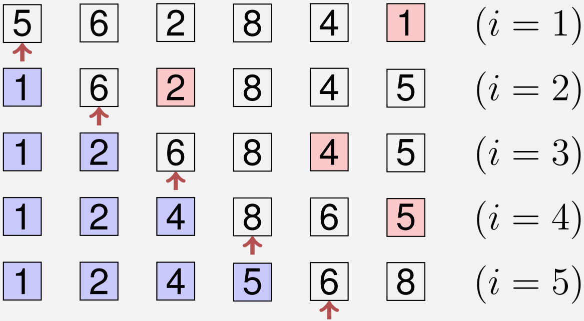

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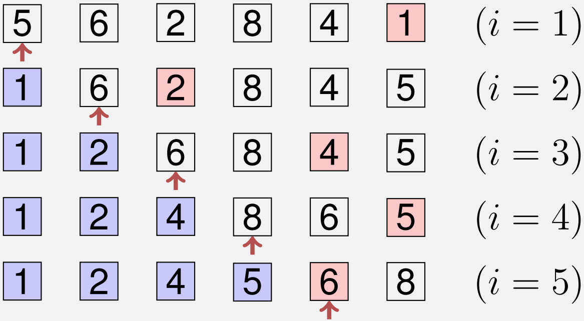

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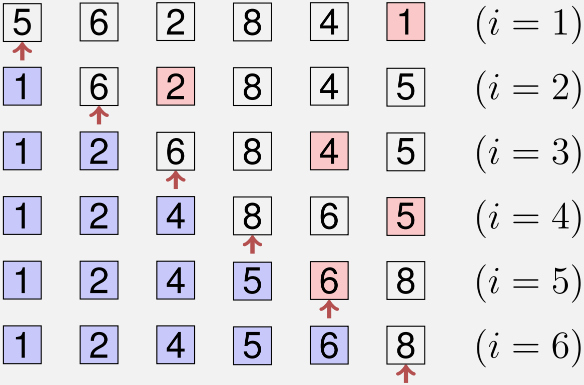

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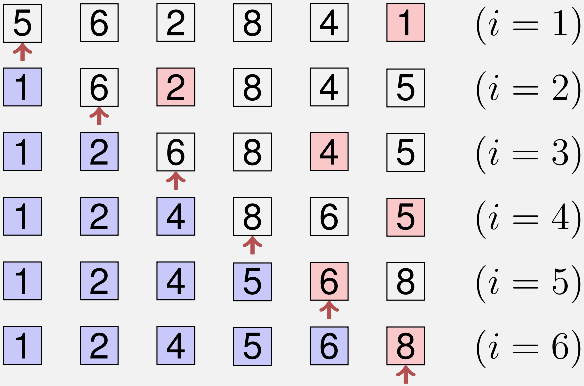

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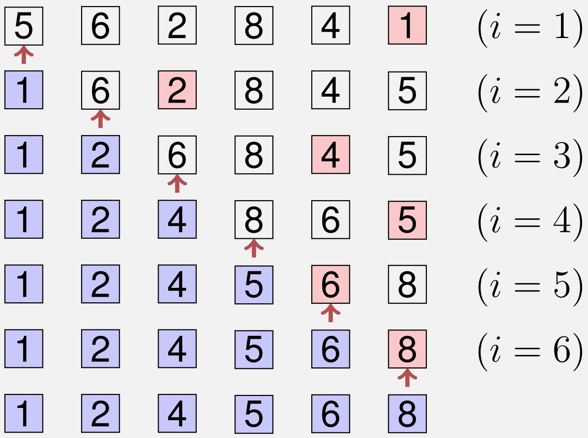

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Algorithm: Selection Sort

Input: Array $A = (A[1], \ldots, A[n])$, $n \geq 0$.
Output: Sorted Array A

```
for i ← 1 to n − 1 do
    p ← i
    for j ← i + 1 to n do
        if A[j] < A[p] then
            p ← j;
    swap(A[i], A[p])
```

Analysis

Number comparisons in worst case:

Analysis

Number comparisons in worst case: $\Theta(n^2)$.

Number swaps in the worst case:

Analysis

Number comparisons in worst case: $\Theta(n^2)$.

Number swaps in the worst case: $n - 1 = \Theta(n)$

4.1 Mergesort

[Ottman/Widmayer, Kap. 2.4, Cormen et al, Kap. 2.3],

Mergesort

Divide and Conquer!

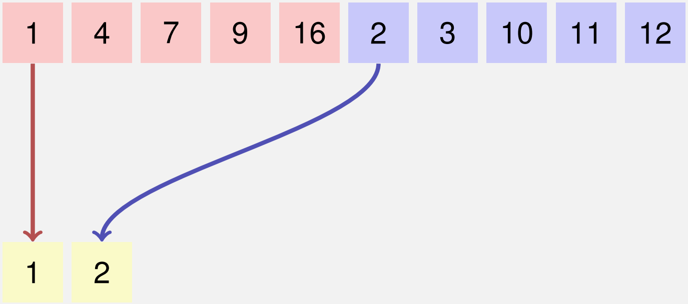
- Assumption: two halves of the array A are already sorted.
- Minimum of A can be evaluated with two comparisons.
- Iteratively: merge the two presorted halves of A in $\mathcal{O}(n)$.

Merge

1	4	7	9	16	2	3	10	11	12

Merge

Merge



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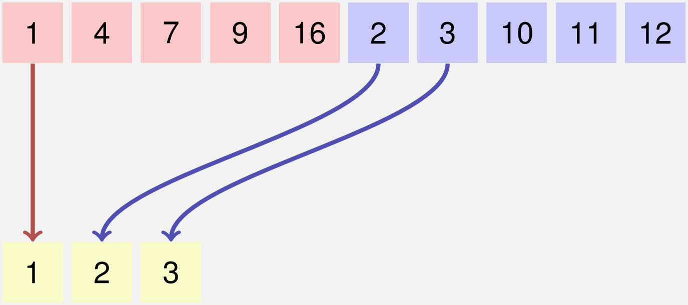

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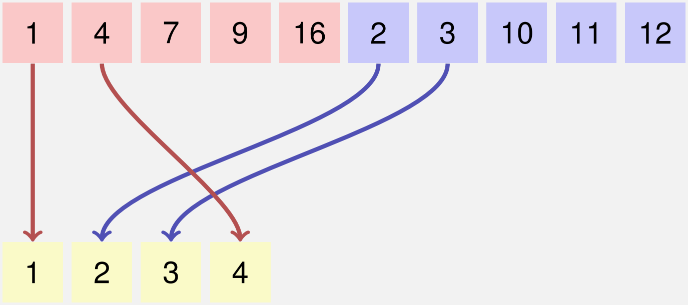

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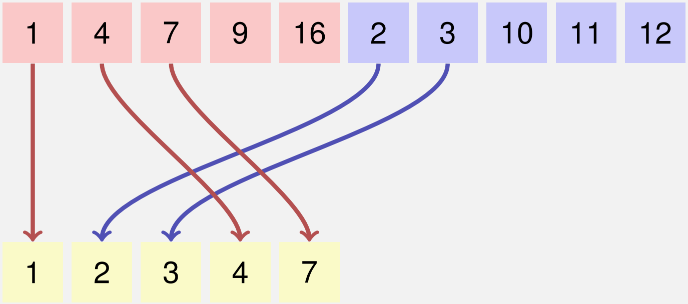

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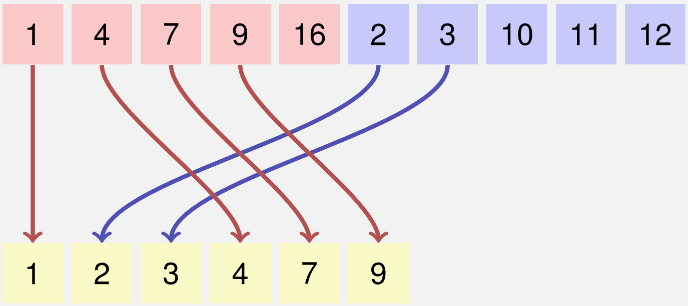

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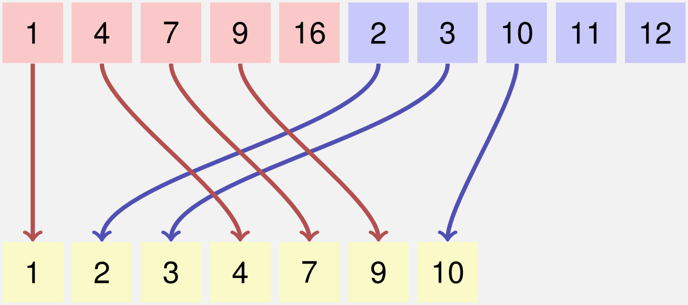

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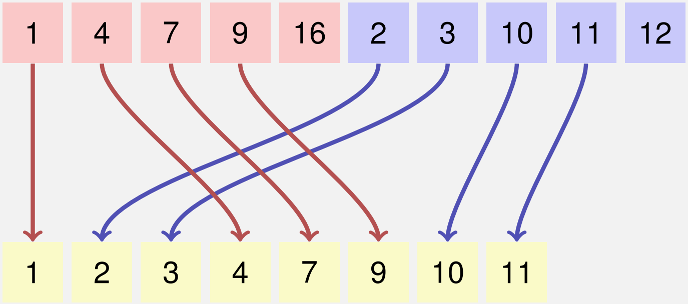

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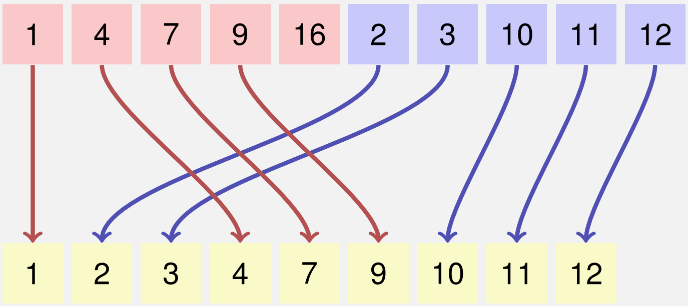

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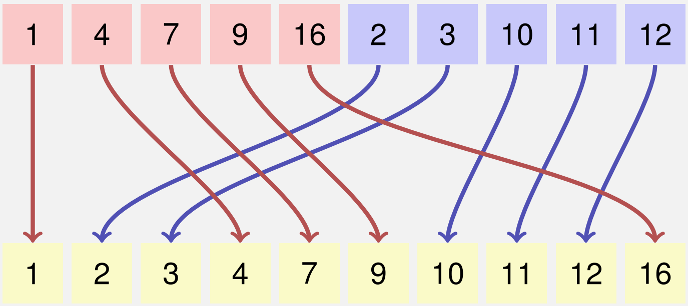

Algorithm Merge(A, l, m, r)

Input: Array A with length n, indexes $1 \leq l \leq m \leq r \leq n$.
$A[l, \ldots, m]$, $A[m+1, \ldots, r]$ sorted

Output: $A[l, \ldots, r]$ sorted

```
B ← new Array(r − l + 1)
i ← l; j ← m + 1; k ← 1
while i ≤ m and j ≤ r do
    if A[i] ≤ A[j] then  B[k] ← A[i]; i ← i + 1
    else  B[k] ← A[j]; j ← j + 1
    k ← k + 1;
while i ≤ m do  B[k] ← A[i]; i ← i + 1; k ← k + 1
while j ≤ r do  B[k] ← A[j]; j ← j + 1; k ← k + 1
for k ← l to r do A[k] ← B[k − l + 1]
```

Mergesort

5	2	6	1	8	4	3	9

Mergesort

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Split

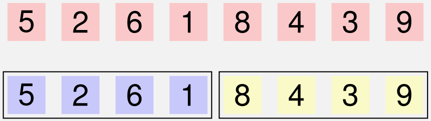
Mergesort

5 2 6 1 8 4 3 9

[5 2 6 1] [8 4 3 9]

Split

Mergesort



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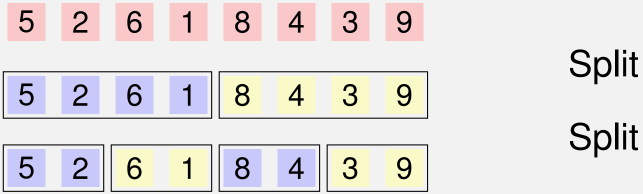
Mergesort



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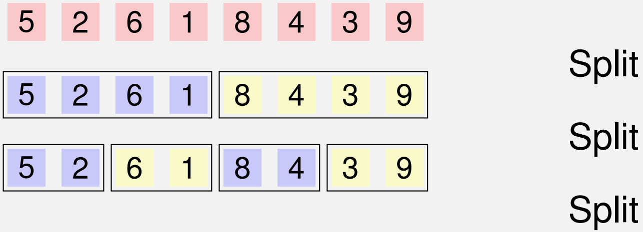

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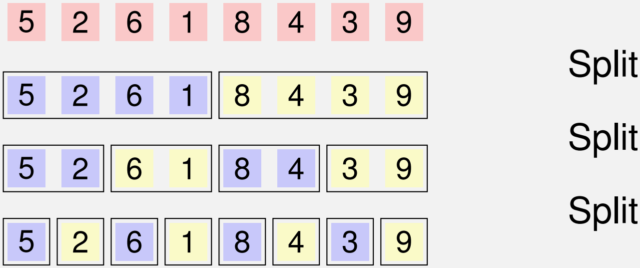

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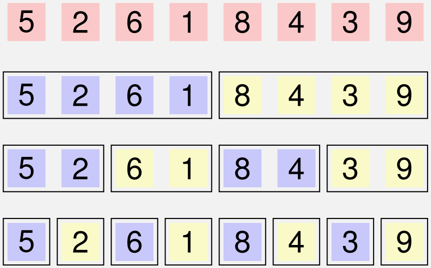


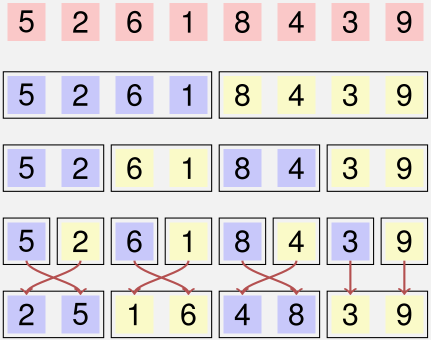
Split

Split

Split

Merge

Mergesort



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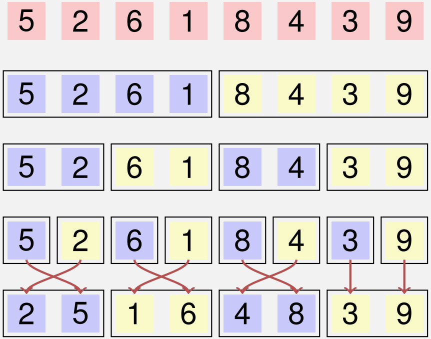

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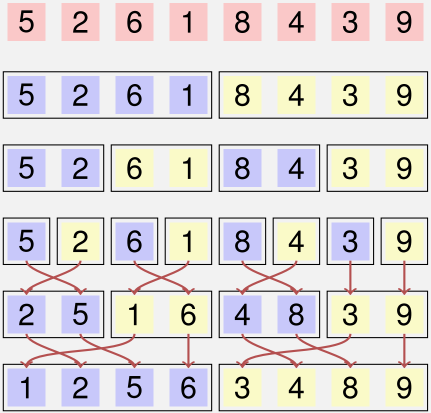

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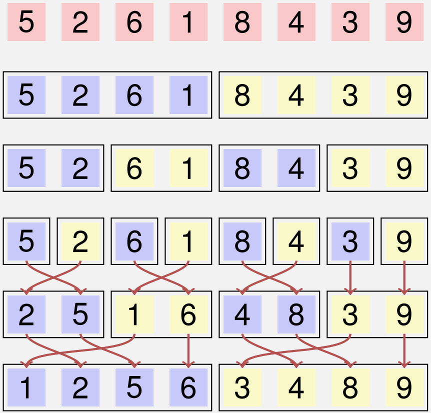

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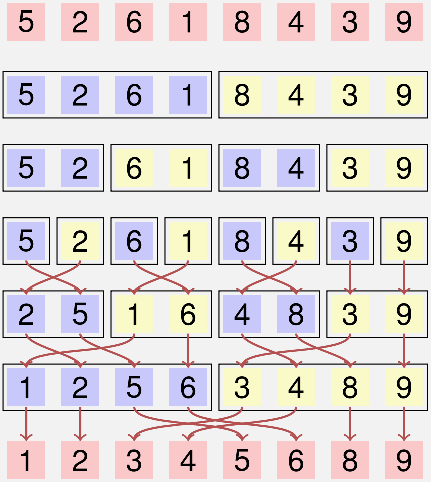

Split

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Algorithm (recursive 2-way) Mergesort(A, l, r)

Input: Array A with length n. $1 \leq l \leq r \leq n$
Output: Array $A[l, \ldots, r]$ sorted.

if $l < r$ **then**
    $m \leftarrow \lfloor (l+r)/2 \rfloor$ // middle position
    Mergesort(A, l, m) // sort lower half
    Mergesort($A, m+1, r$) // sort higher half
    Merge(A, l, m, r) // Merge subsequences

Analysis

Recursion equation for the number of comparisons and key movements:

$$T(n) = T\left(\left\lceil \frac{n}{2} \right\rceil\right) + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \Theta(n)$$

Analysis

Recursion equation for the number of comparisons and key movements:

$$T(n) = T(\left\lceil \frac{n}{2} \right\rceil) + T(\left\lfloor \frac{n}{2} \right\rfloor) + \Theta(n) \in \Theta(n \log n)$$

Derivation for $n = 2^k$

Let $n = 2^k$, $k > 0$. Recurrence

$$T(n) = \begin{cases} d & \text{if } n = 1 \\ 2T(n/2) + cn & \text{if } n > 1 \end{cases}$$

Apply recursively

$$\begin{aligned} T(n) &= 2T(n/2) + cn = 2(2T(n/4) + cn/2) + cn \\ &= 2(2(T(n/8) + cn/4) + cn/2) + cn = \ldots \\ &= 2(2(\ldots(2(2T(n/2^k) + cn/2^{k-1})\ldots) + cn/2^2) + cn/2^1) + cn \\ &= 2^k T(1) + \underbrace{2^{k-1}cn/2^{k-1} + 2^{k-2}cn/2^{k-2} + \ldots + 2^{k-k}cn/2^{k-k}}_{k \text{terms}} \\ &= nd + cnk = nd + cn\log_2 n \in \Theta(n \log n). \end{aligned}$$

4.2 Quicksort

[Ottman/Widmayer, Kap. 2.2, Cormen et al, Kap. 7]

Quicksort

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! Pivot and Partition!

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p	$>$	$\leq$	$\leq$	$>$	$>$	$\leq$	$\leq$	$>$	$\leq$

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$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	p	$>$	$>$	$>$	$>$
1					r				n

Algorithm Partition($A[l..r], p$)

Input: Array A, that contains the pivot p in the interval $[l, r]$ at least once.
Output: Array A partitioned in $[l..r]$ around p. Returns position of p.

while $l \leq r$ **do**
- **while** $A[l] < p$ **do**
  - $l \leftarrow l + 1$
- **while** $A[r] > p$ **do**
  - $r \leftarrow r - 1$
- swap($A[l]$, $A[r]$)
- **if** $A[l] = A[r]$ **then**
  - $l \leftarrow l + 1$

return l-1

Algorithm Quicksort($A[l, \ldots, r]$)

Input: Array A with length n. $1 \leq l \leq r \leq n$.
Output: Array A, sorted between l and r.

if $l < r$ **then**
- Choose pivot $p \in A[l, \ldots, r]$
- $k \leftarrow$ Partition($A[l, \ldots, r], p$)
- Quicksort($A[l, \ldots, k-1]$)
- Quicksort($A[k+1, \ldots, r]$)

Choice of the pivot.

The minimum is a bad pivot: worst case $\Theta(?)$

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p_1	p_2	p_3							

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The minimum is a bad pivot: worst case $\Theta(?)$

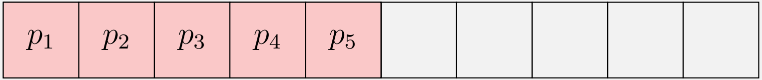
Choice of the pivot.

The minimum is a bad pivot: worst case $\Theta(n^2)$

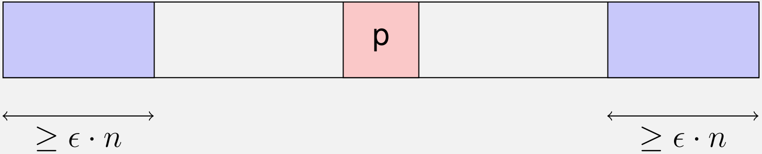
p_1	p_2	p_3	p_4	p_5					

Choice of the pivot.

The minimum is a bad pivot: worst case $\Theta(n^2)$

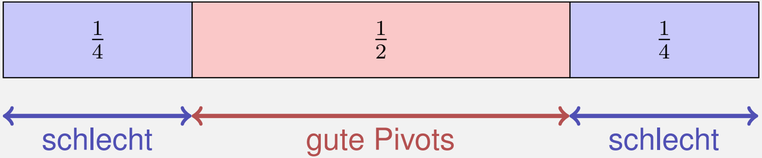


A good pivot has a linear number of elements on both sides.



Choice of the Pivot?

Randomness to our rescue (Tony Hoare, 1961). In each step choose a random pivot.



Probability for a good pivot in one trial: $\frac{1}{2} =: \rho$.

Probability for a good pivot after k trials: $(1-\rho)^{k-1} \cdot \rho$.

Expected number of trials[3]: $1/\rho = 2$

[3] Expected value of the geometric distribution:

Quicksort (arbitrary pivot)

2 4 5 6 8 3 7 9 1

Quicksort (arbitrary pivot)

2 4 5 6 8 3 7 9 1

Quicksort (arbitrary pivot)

2	4	5	6	8	3	7	9	1
2	1	3	6	8	5	7	9	4

Quicksort (arbitrary pivot)

2	4	5	6	8	3	7	9	1
2	1	3	6	8	5	7	9	4

Quicksort (arbitrary pivot)

2	4	5	6	8	3	7	9	1
2	1	3	6	8	5	7	9	4
1	2	3	4	5	8	7	9	6

Quicksort (arbitrary pivot)

2 4 5 6 8 3 7 9 1

2 1 3 6 8 5 7 9 4

1 2 3 4 5 8 7 9 6

Quicksort (arbitrary pivot)

2	4	5	6	8	3	7	9	1
2	1	3	6	8	5	7	9	4
1	2	3	4	5	8	7	9	6
1	2	3	4	5	6	7	9	8

Quicksort (arbitrary pivot)

2	4	5	6	8	3	7	9	1
2	1	3	6	8	5	7	9	4
1	2	3	4	5	8	7	9	6
1	2	3	4	5	6	7	9	8

Quicksort (arbitrary pivot)

2	4	5	6	8	3	7	9	1
2	1	3	6	8	5	7	9	4
1	2	3	4	5	8	7	9	6
1	2	3	4	5	6	7	9	8
1	2	3	4	5	6	7	8	9

Quicksort (arbitrary pivot)

Analysis: number comparisons

Worst case.

Analysis: number comparisons

Worst case. Pivot = min or max; number comparisons:

$$T(n) = T(n-1) + c \cdot n,\ T(1) = 0 \quad \Rightarrow \quad T(n) \in \Theta(n^2)$$

Analysis (randomized quicksort)

Theorem

On average randomized quicksort requires $\mathcal{O}(n \cdot \log n)$ *comparisons.*

(without proof.)

Practical Considerations.

- Practically the pivot is often the median of three elements. For example: Median3($A[l], A[r], A[\lfloor l+r/2 \rfloor]$).