Computer Science II

Course at D-BAUG, ETH Zurich

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SS 2019

1. Introduction

Algorithms and Data Structures, Correctness, a First Example

Goals of the course

- Understand the design and analysis of fundamental algorithms and data structures.
- Understand how an algorithmic problem is mapped to a sufficiently efficient computer program.

Contents

data structures / algorithms

The notion invariant, cost model, Landau notation

algorithms design, induction

searching, selection and sorting

dictionaries: hashing and search trees, balanced trees

dynamic programming

fundamental graph algorithms, shortest paths, maximum flow

Software Engineering

Python Introduction

Python Datastructures

1.1 Algorithms

[Cormen et al, Kap. 1;Ottman/Widmayer, Kap. 1.1]

Algorithm

Algorithm: well defined computing procedure to compute *output* data from *input* data

example problem

Input: A sequence of n numbers $(a_1, a_2, \ldots, a_n)$

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Possible input

$(1, 7, 3)$, $(15, 13, 12, -0.5)$, (1) ...

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$(1, 7, 3)$, $(15, 13, 12, -0.5)$, (1) ...

Every example represents a *problem instance*

The performance (speed) of an algorithm usually depends on the problem instance. Often there are “good” and “bad” instances.

Examples for algorithmic problems

- Tables and statistis: sorting, selection and searching

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- autocomletion and spell-checking: Dictionaries / Trees

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- Fast Lookup : Hash-Tables

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- routing: shortest path algorithm, heap data structure
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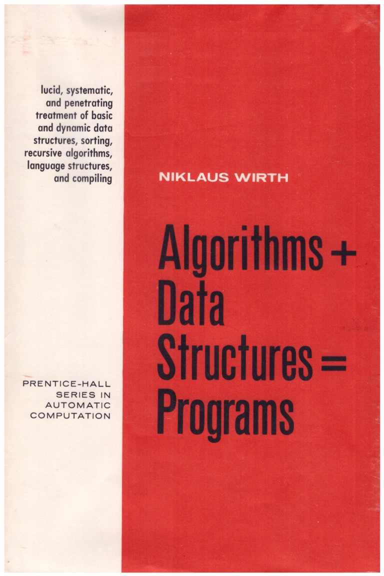
- evaluation order: Topological Sorting
- autocomletion and spell-checking: Dictionaries / Trees
- Fast Lookup : Hash-Tables
- The travelling Salesman: Dynamic Programming, Minimum Spanning Tree, Simulated Annealing

Characteristics

- Extremely large number of potential solutions
- Practical applicability

Data Structures

- A data structure is a particular way of *organizing data* in a computer so that they can be *used efficiently* (in the algorithms operating on them).
- Programs = algorithms + data structures.



Efficiency

Illusion:

- If computers were infinitely fast and had an infinite amount of memory ...
- ... then we would still need the theory of algorithms (only) for statements about correctness (and termination).

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Reality: resources are bounded and not free:

- Computing time $\rightarrow$ Efficiency
- Storage space $\rightarrow$ Efficiency

Actually, this course is nearly only about efficiency.

2. Efficiency of algorithms

Efficiency of Algorithms, Random Access Machine Model, Function Growth, Asymptotics [Cormen et al, Kap. 2.2,3,4.2-4.4 | Ottman/Widmayer, Kap. 1.1]

Efficiency of Algorithms

Goals

- Quantify the runtime behavior of an algorithm independent of the machine.
- Compare efficiency of algorithms.
- Understand dependece on the input size.

Programs and Algorithms

Technology

program

$\downarrow$ implemented in

programming language

$\downarrow$ specified for

computer

Abstraction

algorithm

specified in $\downarrow$

pseudo-code

based on $\downarrow$

computation model

Technology Model

Random Access Machine (RAM)

- Execution model: instructions are executed one after the other (on one processor core).

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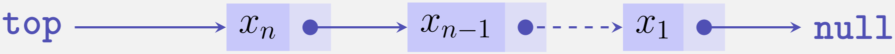
- Data types: fundamental types like size-limited integer or floating point number.

Pointer Machine Model

We assume

- Objects bounded in size can be dynamically allocated in constant time
- Fields (with word-size) of the objects can be accessed in constant time 1.



Asymptotic behavior

An exact running time of an algorithm can normally not be predicted even for small input data.

- We consider the asymptotic behavior of the algorithm.
- And ignore all constant factors.

Example

An operation with cost 20 is no worse than one with cost 1
Linear growth with gradient 5 is as good as linear growth with gradient 1.

2.2 Function growth

$\mathcal{O}$, Θ, Ω [Cormen et al, Kap. 3; Ottman/Widmayer, Kap. 1.1]

Superficially

Use the asymptotic notation to specify the execution time of algorithms.

We write $\Theta(n^2)$ and mean that the algorithm behaves for large n like n^2: when the problem size is doubled, the execution time multiplies by four.

More precise: asymptotic upper bound

provided: a function $g : \mathbb{N} \to \mathbb{R}$.

Definition:[1]

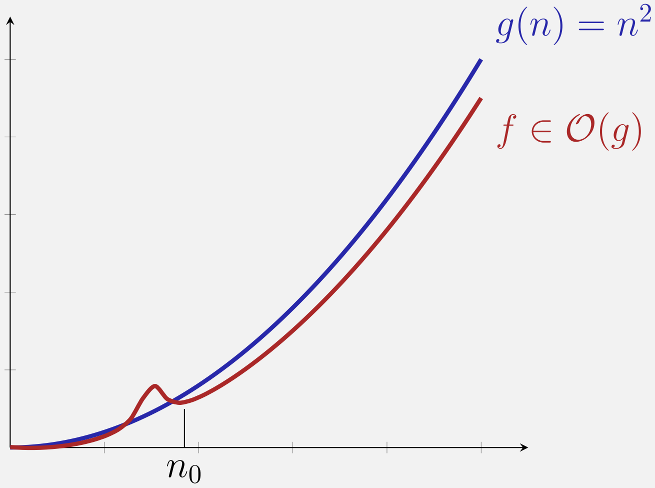
$$\begin{aligned}\mathcal{O}(g) = \{f : \mathbb{N} \to \mathbb{R} |\\ \exists\, c > 0, \exists n_0 \in \mathbb{N} :\\ \forall\, n \geq n_0 : 0 \leq f(n) \leq c \cdot g(n)\}\end{aligned}$$

Notation:

$$\mathcal{O}(g(n)) := \mathcal{O}(g(\cdot)) = \mathcal{O}(g).$$

[1] Ausgesprochen: Set of all functions $f : \mathbb{N} \to \mathbb{R}$ that satisfy: there is some (real valued) $c > 0$ and some $n_0 \in \mathbb{N}$ such that $0 \leq f(n) \leq n \cdot g(n)$ for all $n \geq n_0$.

Graphic



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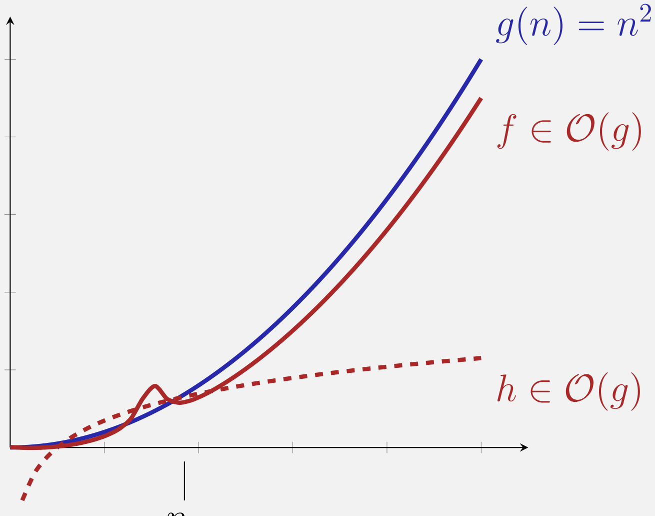

Examples

$$\mathcal{O}(g) = \{f : \mathbb{N} \to \mathbb{R} |\ \exists c > 0, \exists n_0 \in \mathbb{N} : \forall n \geq n_0 : 0 \leq f(n) \leq c \cdot g(n)\}$$

$f(n)$	$f \in \mathcal{O}(?)$	Example
$3n + 4$		
$2n$		
$n^2 + 100n$		
$n + \sqrt{n}$		

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$n^2 + 100n$		
$n + \sqrt{n}$		

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$2n$	$\mathcal{O}(n)$	$c = 2, n_0 = 0$
$n^2 + 100n$	$\mathcal{O}(n^2)$	$c = 2, n_0 = 100$
$n + \sqrt{n}$		

Examples

$$\mathcal{O}(g) = \{f : \mathbb{N} \to \mathbb{R} \mid \exists c > 0, \exists n_0 \in \mathbb{N} : \forall n \geq n_0 : 0 \leq f(n) \leq c \cdot g(n)\}$$

$f(n)$	$f \in \mathcal{O}(?)$	Example
$3n + 4$	$\mathcal{O}(n)$	$c = 4, n_0 = 4$
$2n$	$\mathcal{O}(n)$	$c = 2, n_0 = 0$
$n^2 + 100n$	$\mathcal{O}(n^2)$	$c = 2, n_0 = 100$
$n + \sqrt{n}$	$\mathcal{O}(n)$	$c = 2, n_0 = 1$

Property

$$f_1 \in \mathcal{O}(g), f_2 \in \mathcal{O}(g) \Rightarrow f_1 + f_2 \in \mathcal{O}(g)$$

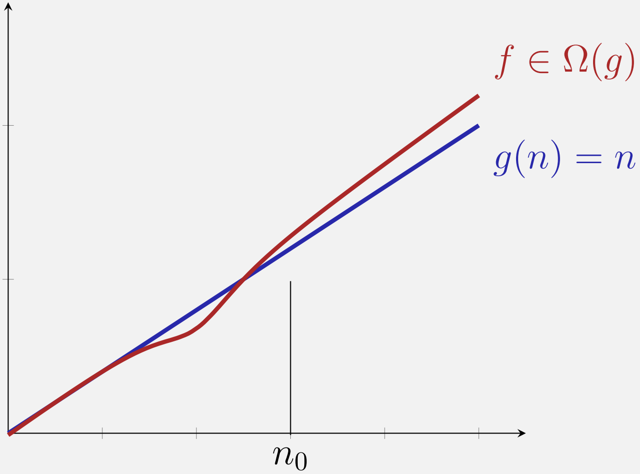
Converse: asymptotic lower bound

Given: a function $g : \mathbb{N} \to \mathbb{R}$.

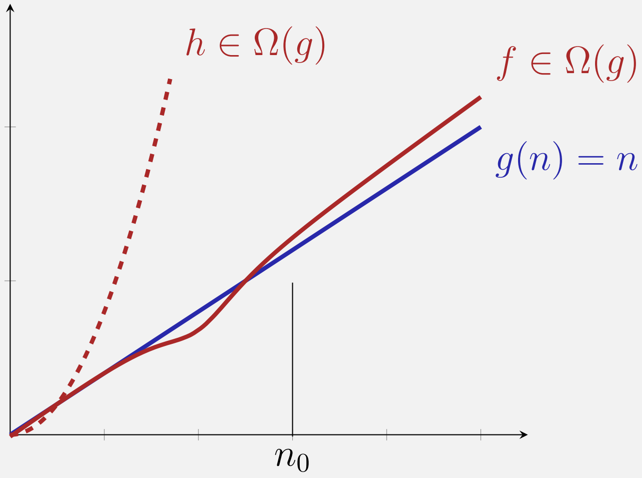
Definition:

$$\begin{aligned}\Omega(g) = \{f : \mathbb{N} \to \mathbb{R} | \\ &\exists\, c > 0, \exists n_0 \in \mathbb{N} : \\ &\forall\, n \geq n_0 : 0 \leq c \cdot g(n) \leq f(n)\}\end{aligned}$$

Example



Example



Asymptotic tight bound

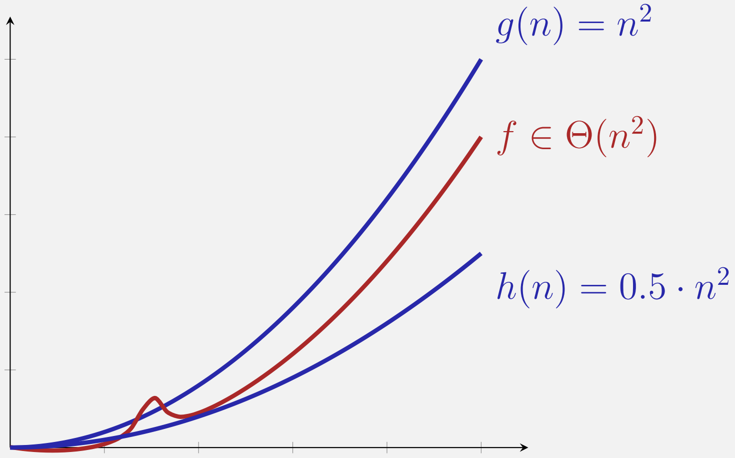
Given: function $g : \mathbb{N} \to \mathbb{R}$.

Definition:

$$\Theta(g) := \Omega(g) \cap \mathcal{O}(g).$$

Simple, closed form: exercise.

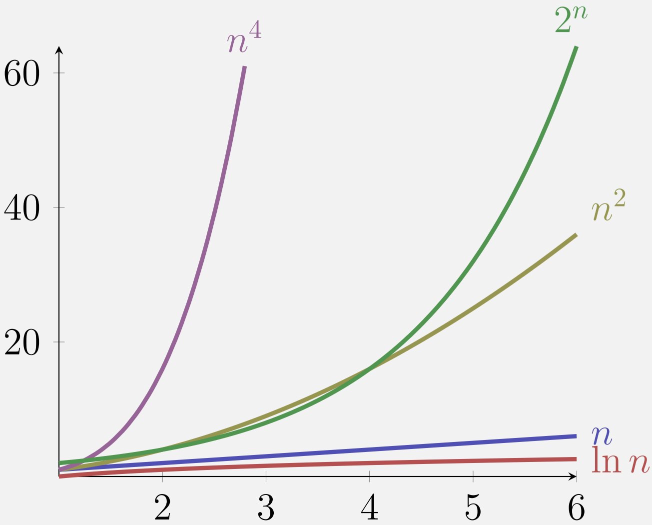
Example



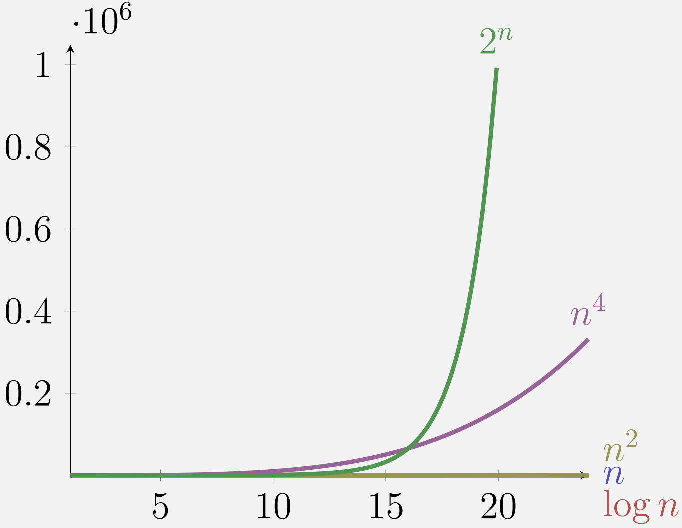
Notions of Growth

$\mathcal{O}(1)$	bounded	array access
$\mathcal{O}(\log\log n)$	double logarithmic	interpolated binary sorted sort
$\mathcal{O}(\log n)$	logarithmic	binary sorted search
$\mathcal{O}(\sqrt{n})$	like the square root	naive prime number test
$\mathcal{O}(n)$	linear	unsorted naive search
$\mathcal{O}(n\log n)$	superlinear / loglinear	good sorting algorithms
$\mathcal{O}(n^2)$	quadratic	simple sort algorithms
$\mathcal{O}(n^c)$	polynomial	matrix multiply
$\mathcal{O}(2^n)$	exponential	Travelling Salesman Dynamic Programming
$\mathcal{O}(n!)$	factorial	Travelling Salesman naively

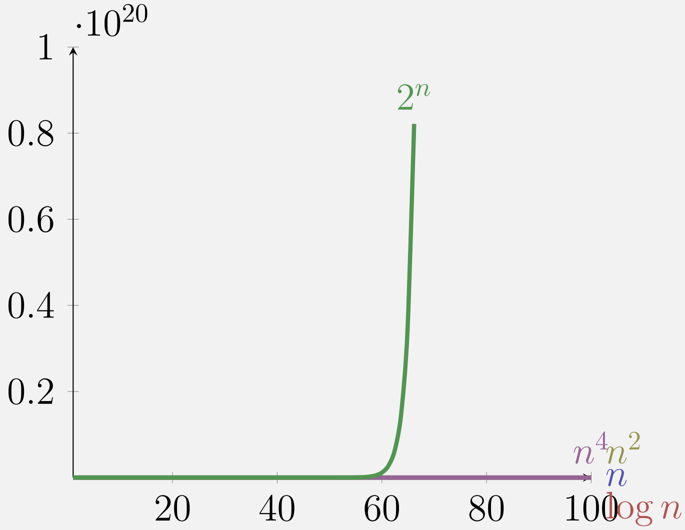
Small n



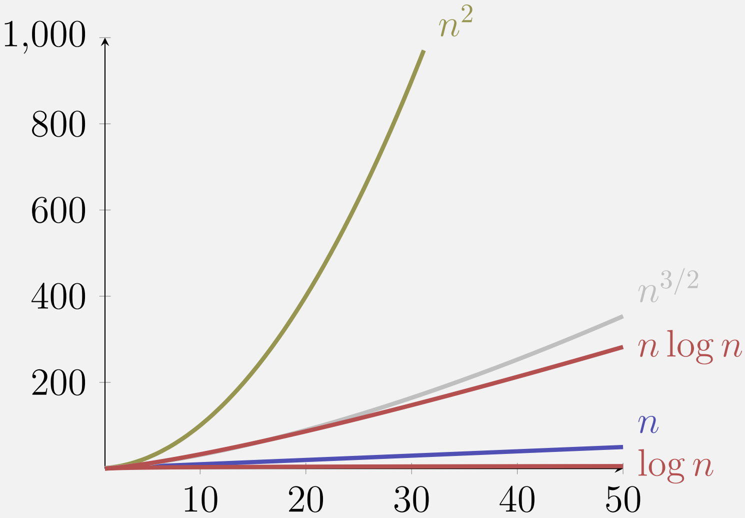
Larger n



“Large” n



Logarithms



Time Consumption

Assumption 1 Operation = $1\mu s$.

problem size	1	100	10000	10^6	10^9
$\log_2 n$	$1\mu s$				
n	$1\mu s$				
$n \log_2 n$	$1\mu s$				
n^2	$1\mu s$				
2^n	$1\mu s$				

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problem size	1	100	10000	10^6	10^9
$\log_2 n$	$1\mu s$				
n	$1\mu s$	$100\mu s$	$1/100s$	$1s$	17 minutes
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n^2	$1\mu s$	$1/100s$	1.7 minutes	11.5 days	317 centuries
2^n	$1\mu s$				

Time Consumption

Assumption 1 Operation = $1\mu s$.

problem size	1	100	10000	10^6	10^9
$\log_2 n$	$1\mu s$	$7\mu s$	$13\mu s$	$20\mu s$	$30\mu s$
n	$1\mu s$	$100\mu s$	$1/100s$	$1s$	17 minutes
$n\log_2 n$	$1\mu s$				
n^2	$1\mu s$	$1/100s$	1.7 minutes	11.5 days	317 centuries
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n	$1\mu s$	$100\mu s$	$1/100s$	$1s$	17 minutes
$n\log_2 n$	$1\mu s$	$700\mu s$	$13/100\mu s$	$20s$	8.5 hours
n^2	$1\mu s$	$1/100s$	1.7 minutes	11.5 days	317 centuries
2^n	$1\mu s$				

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problem size	1	100	10000	10^6	10^9
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$n \log_2 n$	$1\mu s$	$700\mu s$	$13/100\mu s$	$20s$	8.5 hours
n^2	$1\mu s$	$1/100s$	1.7 minutes	11.5 days	317 centuries
2^n	$1\mu s$	10^{14} centuries	$\approx \infty$	$\approx \infty$	$\approx \infty$

About the Notation

Common casual notation

$$f = \mathcal{O}(g)$$

should be read as $f \in \mathcal{O}(g)$.

Clearly it holds that

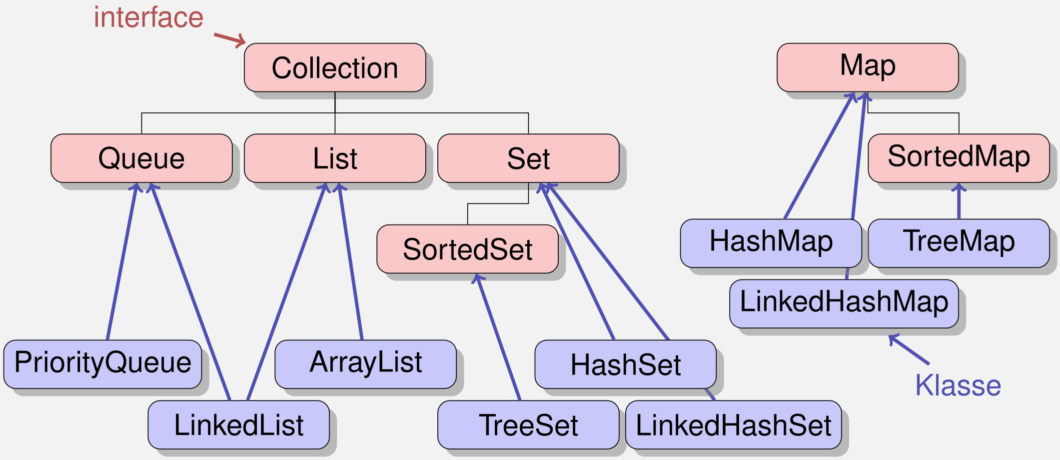
$$f_1 = \mathcal{O}(g), f_2 = \mathcal{O}(g) \not\Rightarrow f_1 = f_2!$$

Beispiel

$n = \mathcal{O}(n^2), n^2 = \mathcal{O}(n^2)$ but naturally $n \neq n^2$.

We avoid this notation where it could lead to ambiguities.

Reminder: Java Collections / Maps

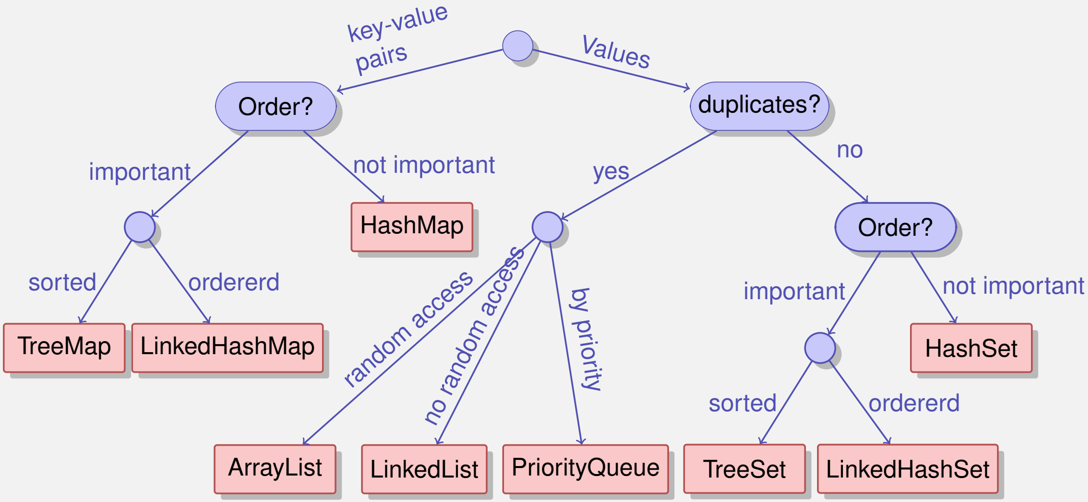


`ArrayList` versus `LinkedList`

run time measurements for 10000 operations (on [code] expert)

`ArrayList`	`LinkedList`
469μs	1787μs
37900μs	**761**μs
1840μs	2050μs
426μs	110600μs
31ms	301ms
38ms	141ms
228ms	1080ms
648μs	757μs
58075μs	**609**μs

Reminder: Decision



Asymptotic Runtimes

With our new language ($\Omega, \mathcal{O}, \Theta$), we can now *state the behavior of the data structures and their algorithms more precisely*

Asymptotic running times (Anticipation!)

Data structure	Random Access	Insert	Next	Insert After Element	Search
`ArrayList`	$\Theta(1)$	$\Theta(1)\ A$	$\Theta(1)$	$\Theta(n)$	$\Theta(n)$
`LinkedList`	$\Theta(n)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(n)$
`TreeSet`	–	$\Theta(\log n)$	$\Theta(\log n)$	–	$\Theta(\log n)$
`HashSet`	–	$\Theta(1)\ P$	–	–	$\Theta(1)\ P$

A = amortized, P=expected, otherwise worst case