Informatik II

Übung 13

FS 2019

Program Today

1. Feedback of last exercise
2. Repetition of Lecture
3. In-Class-Exercise (practical)

1. Feedback of last exercise

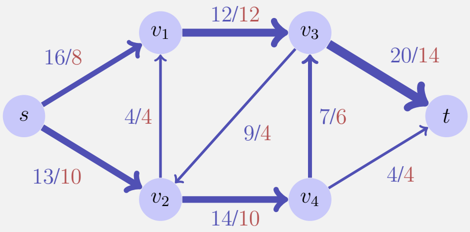
2. Repetition of Lecture

Flow

A *Flow* $f : V \times V \to \mathbb{R}$ fulfills the following conditions:

- *Bounded Capacity*:
  For all $u, v \in V$: $f(u,v) \le c(u,v)$.
- *Skew Symmetry*:
  For all $u, v \in V$: $f(u,v) = -f(v,u)$.
- *Conservation of flow*:
  For all $u \in V \setminus \{s,t\}$:

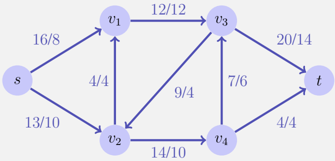
$$\sum_{v \in V} f(u,v) = 0.$$

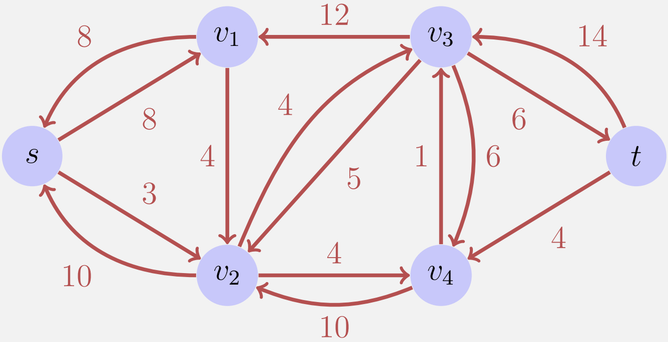


Value of the flow:
$|f| = \sum_{v \in V} f(s,v)$.
Here $|f| = 18$.

Rest Network

Rest network G_f provided by the edges with positive rest capacity:





Rest networks provide the same kind of properties as flow networks with the exception of permitting antiparallel capacity-edges

Augmenting Paths

expansion path p: simple path from s to t in the rest network G_f.

Rest capacity $c_f(p) = \min\{c_f(u, v) : (u, v)$ edge in $p\}$

Max-Flow Min-Cut Theorem

Theorem

Let f be a flow in a flow network $G = (V, E, c)$ with source s and sink t. The following statementsa are equivalent:

1. *f is a maximal flow in G*
2. *The rest network G_f does not provide any expansion paths*
3. *It holds that $|f| = c(S, T)$ for a cut (S, T) of G.*

Algorithm Ford-Fulkerson(G, s, t)

Input: Flow network $G = (V, E, c)$
Output: Maximal flow f.

for $(u, v) \in E$ **do**
    $f(u, v) \leftarrow 0$

while Exists path $p : s \rightsquigarrow t$ in rest network G_f **do**
    $c_f(p) \leftarrow \min\{c_f(u, v) : (u, v) \in p\}$
    foreach $(u, v) \in p$ **do**
        $f(u, v) \leftarrow f(u, v) + c_f(p)$
        $f(v, u) \leftarrow f(v, u) - c_f(p)$

Practical Consideration

In an implementation of the Ford-Fulkerson algorithm the negative flow egdes are usually not stored because their value always equals the negated value of the antiparallel edge.

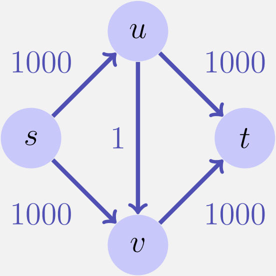
$$f(u,v) \leftarrow f(u,v) + c_f(p)$$
$$f(v,u) \leftarrow f(v,u) - c_f(p)$$

is then transformed to

if $(u,v) \in E$ **then**
  $f(u,v) \leftarrow f(u,v) + c_f(p)$
else
  $f(v,u) \leftarrow f(v,u) - c_f(p)$

Analysis

- For an integer flow, the algorithms requires maximally $|f_{\max}|$ iterations of the while loop (because the flow increases minimally by 1). Search a single increasing path (e.g. with DFS or BFS) $\mathcal{O}(|E|)$ Therefore overal running time in $\mathcal{O}(f_{\max}|E|)$.



With an unlucky choice the algorithm may require up to 2000 iterations here.

Edmonds-Karp Algorithm

Choose in the Ford-Fulkerson-Method for finding a path in G_f the expansion path of shortest possible length (e.g. with BFS)

Edmonds-Karp Algorithm

Theorem

When the Edmonds-Karp algorithm is applied to some integer valued flow network $G = (V, E)$ with source s and sink t then the number of flow increases applied by the algorithm is in $\mathcal{O}(|V| \cdot |E|)$.

$\Rightarrow$ *Overal asymptotic runtime:* $\mathcal{O}(|V| \cdot |E|^2)$

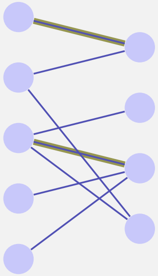
[Without proof]

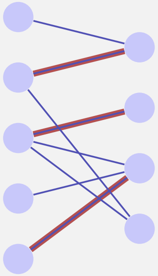
Application: maximal bipartite matching

Given: bipartite undirected graph $G = (V, E)$.

Matching M: $M \subseteq E$ such that $|\{m \in M : v \in m\}| \leq 1$ for all $v \in V$.

Maximal Matching M: Matching M, such that $|M| \geq |M'|$ for each matching M'.

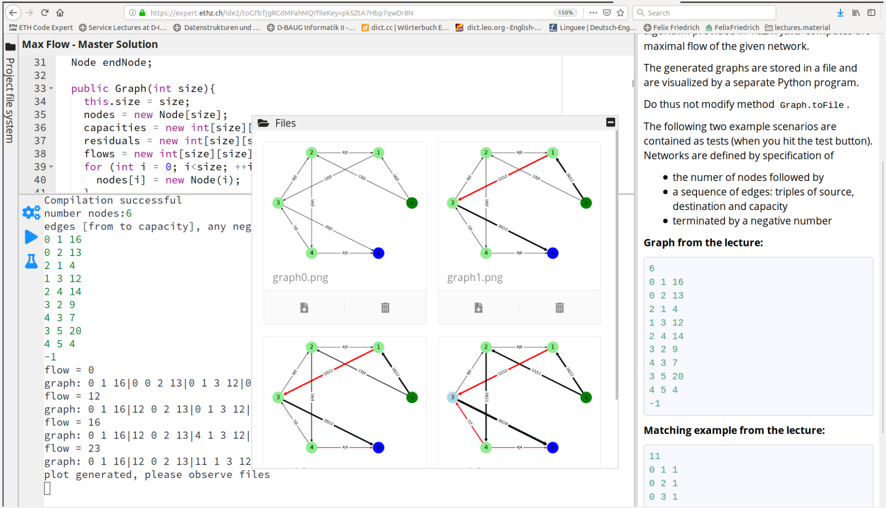




3. In-Class-Exercise (practical)

Implementation of Max-Flow

Max-Flow Implementation



Questions?