Informatik II

Exercise 1

FS 2019

Program for Today

1. Repetition Theory
   - Problem and Algorithm
   - Aysmptotic Running Time
2. Programming Exercise
   - Toss an Unfair Dice

Warm-up

- What is a problem?

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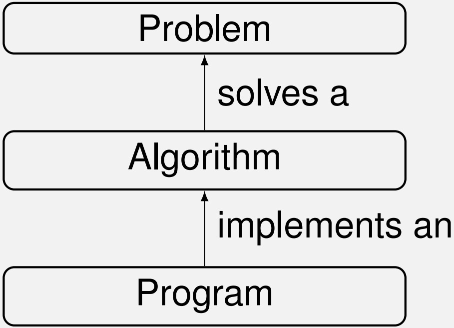
- What is a problem?
- What is an algorithm?
    - ➜ well-defined computing procedure to compute output data from input data.
- What is a program?
    - ➜ Concrete implementation of an algorithm

Warm-up



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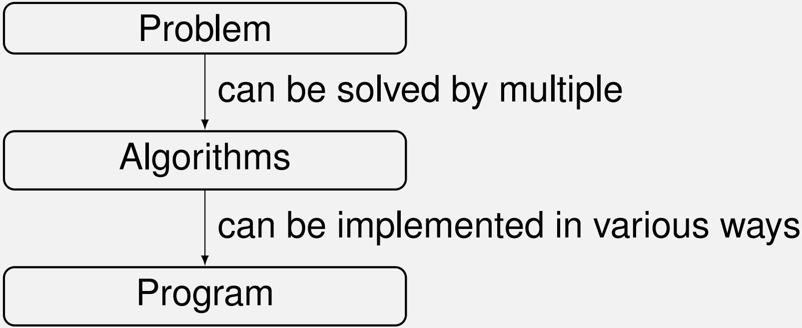

Problem
can be solved by multiple
Algorithms
can be implemented in various ways
Program

Efficiency

Problem	Complexity	Minimal (asymptotic) cost over all algorithms that solve the problem.
Algorithm	Cost	Number of elementary operations
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➜ Estimation of cost or computing time depending on the input size, denoted by n.

Asymptotic behavior

- What are $\Omega(g(n))$, $\Theta(g(n))$, $\mathcal{O}(g(n))$?

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→ Sets of functions!

Repetition, sets A, B:

subset	$A \subseteq B$
proper subset	$A \subsetneq B$
intersection	$A \cap B$

Asymptotic behavior

Given: function $f : \mathbb{N} \to \mathbb{R}$.

Definition:

$$\mathcal{O}(g) = \{f : \mathbb{N} \to \mathbb{R} | \exists c > 0, n_0 \in \mathbb{N} : 0 \leq f(n) \leq c \cdot g(n) \; \forall n \geq n_0\}$$
$$\Omega(g) = \{f : \mathbb{N} \to \mathbb{R} | \exists c > 0, n_0 \in \mathbb{N} : 0 \leq c \cdot g(n) \leq f(n) \; \forall n \geq n_0\}$$
$$\Theta(g) = \mathcal{O}(g) \cap \Omega(g)$$

Useful information for the exercise

Theorem

1. $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0 \Rightarrow f \in \mathcal{O}(g), \mathcal{O}(f) \subsetneq \mathcal{O}(g)$.
2. $\lim_{n\to\infty} \frac{f(n)}{g(n)} = C > 0$ *(C constant)* $\Rightarrow f \in \Theta(g)$.
3. $\frac{f(n)}{g(n)} \underset{n\to\infty}{\longrightarrow} \infty \Rightarrow g \in \mathcal{O}(f), \mathcal{O}(g) \subsetneq \mathcal{O}(f)$.

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Beispiel

1. $\lim_{n\to\infty} \frac{n}{n^2} = 0 \Rightarrow n \in \mathcal{O}(n^2), \mathcal{O}(n) \subsetneq \mathcal{O}(n^2)$.
2. $\lim_{n\to\infty} \frac{2n}{n} = 2 > 0 \Rightarrow 2n \in \Theta(n)$.
3. $\frac{n^2}{n} \underset{n\to\infty}{\to} \infty \Rightarrow n \in \mathcal{O}(n^2), \mathcal{O}(n) \subsetneq \mathcal{O}(n^2)$.

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- $\mathcal{O}(n) \subseteq \mathcal{O}(n^2)$ is correct
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Quiz

$1 \in \mathcal{O}(15)$?

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$1 \in \mathcal{O}(15)$? ✓ better $1 \in \mathcal{O}(1)$

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n^2	$n \rightarrow 3.16 \cdot n$	$n \rightarrow 10 \cdot n$
2^n	$n \rightarrow n + 3.32$	$n \rightarrow n + 6.64$

[1] To see this, you set $f(n') = c \cdot f(n)$ ($c = 10$ or $c = 100$) and solve for n'

Asymptotic Running Times with Θ

```
void run(int n){
  for (int i = 1; i<n; ++i)
    for (int j = 1; j<n; ++j)
      op();
}
```

How often is `op()` called?

Asymptotic Running Times with mit Θ

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Asymptotic Running Times with Θ

```
void run(int n){
  for(int i = 1; i <= n; ++i)
    for(int j = 1; j*j <= n; ++j)
      for(int k = n; k >= 2; --k)
        op();
}
```

How often is `op()` called?

2. Programming Exercise

Unfair Dice

Dice Simulation

Given: Simulation of the uniformly distributed random variable using `Math.Random()`

We want: Simulation of a *fair* die

Dice Simulation

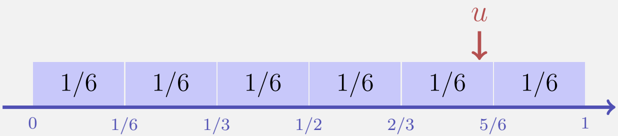
`Math.Random()` returns $U \in [0,1)$ with

$$\mathbb{P}(U \in [l,r)) = r - l.$$

`Dice()` should return $Y \in \{1,\ldots,6\}$, such that

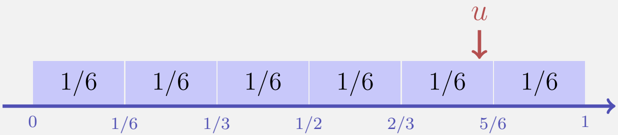
$$\mathbb{P}(Y = k) = 1/6 \text{ for all } k \in \{1,\ldots,6\}$$

Cumbersome, but Correct




```
static int Dice(){
  double u = Math.random();
  if (u<1.0/6) return 1;
  else if (u<1.0/3) return 2;
  else if (u<1.0/2) return 3;
  else if (u<2.0/3) return 4;
  else if (u<5.0/6) return 5;
  else return 6;
}
```

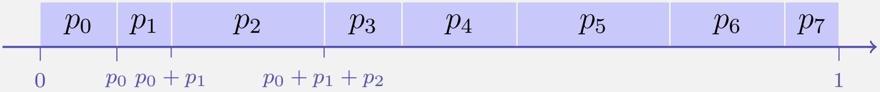
Easier




```
static int Dice(){
  double u = Math.random();
  return (int)(u*6+1);
}
```

Toss an Unfair Dice

Given: Probabilitiy vector $p = (p_0, \dots, p_{n-1})$ with $\sum_{i=0}^{n-1} p_i = 1$ and $p_i \geq 0$ $(0 \leq i < n)$.

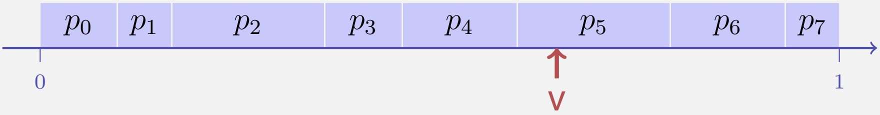


Wanted: `Sample(p)` returning j $(0 \leq j < n)$ with probability p_j.

Formal

Using an existing random number generator, a number v is drawn uniformly distributed from the interval $[0, 1)$. From this number v an integer $0 \leq S(p, v) < n$ is generated according to the following rule

$$S(p, v) = \min\{0 \leq i < n : \sum_{k=0}^{i} p_k > v\}$$



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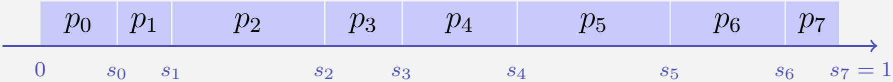
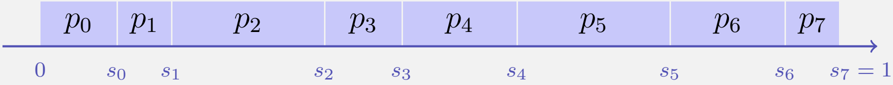



```
static int Sample(double[] p){
  double u = Math.random();
  if (u<p[0]) return 0;
  if (u<p[0]+p[1]) return 1;
  if (u<p[0]+p[1]+p[2]) return 2;
  if (u<p[0]+p[1]+p[2]+p[3]) return 3;
  ...
}
```

Zu umständlich: wir brauchen eine Schleife!

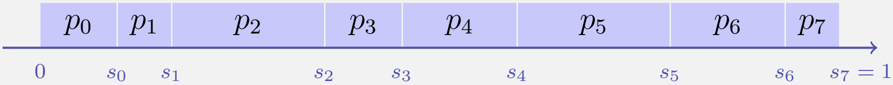
A Tiny Trick




```
static int Sample(double[] p){
  double u = Math.random();
  if (u<p[0]) return 0;
  u -= p[0];
  if (u<p[1]) return 1;
  u -= p[1];
  if (u<p[2]) return 2;
  ...
}
```

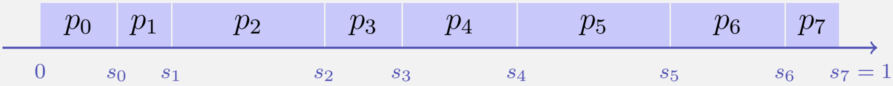
We do not have to compute the sums of the p_i

In a Loop




```
static int Sample(double[] p){
  double u = Math.random();
  for (int k = 0; k < p.length-1; ++k){
    if (u<p[k]) return k;
    u -= p[k];
  }
  return p.length-1;
}
```

More Compact




```
static int Sample(double[] p){
  double u = Math.random();
  int k=0;
  while (k < p.length && u>0){
    u -= p[k++];
  }
  return k-1;
}
```

Questions or Comments?