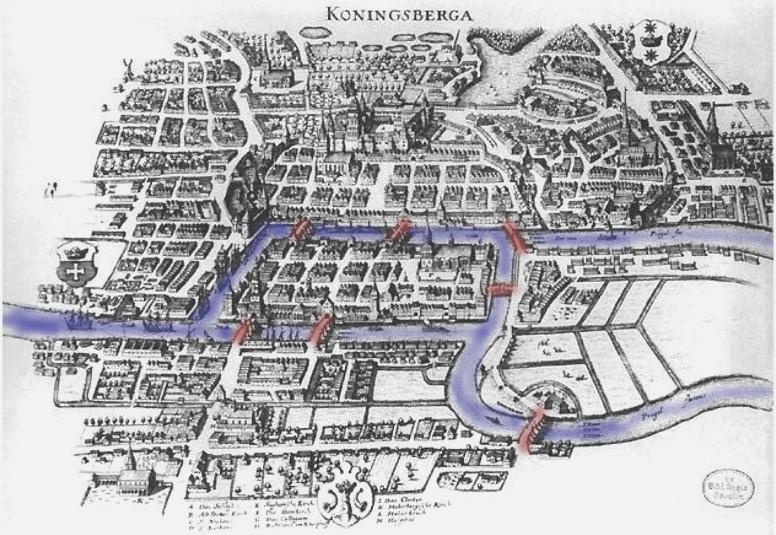
# 12. Graphs

Notation, Representation, Graph Traversal (DFS, BFS), Topological Sorting, Ottman/Widmayer, Kap. 9.1 - 9.4,Cormen et al, Kap. 22

# Königsberg 1736



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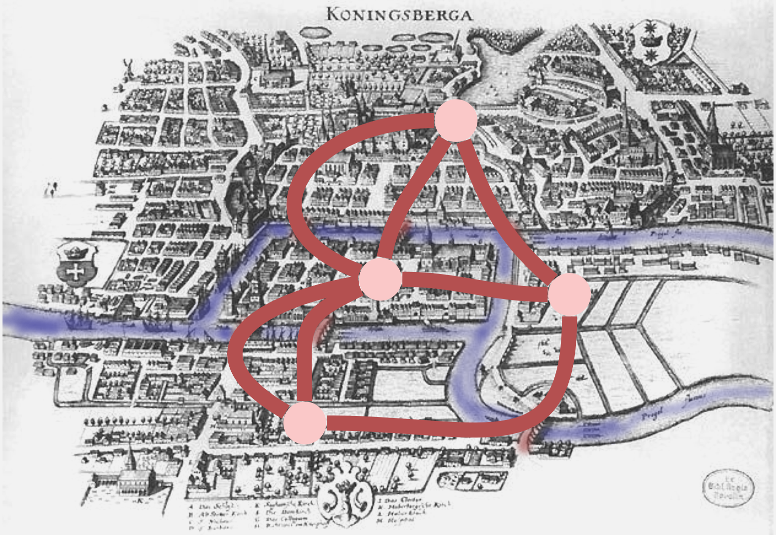

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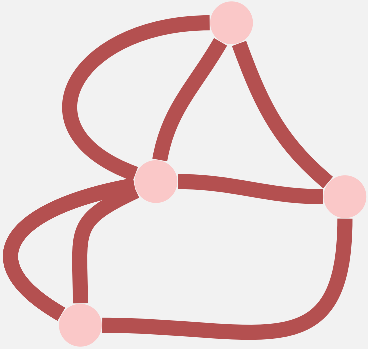
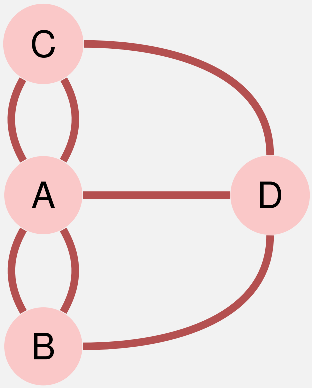
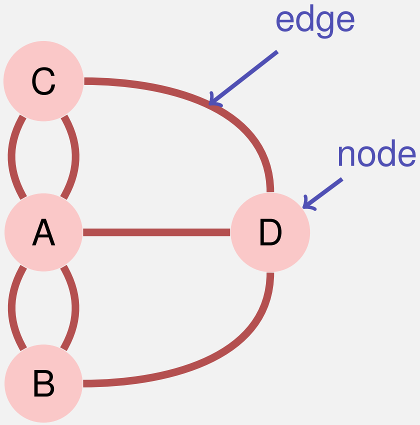


# [Multi]Graph

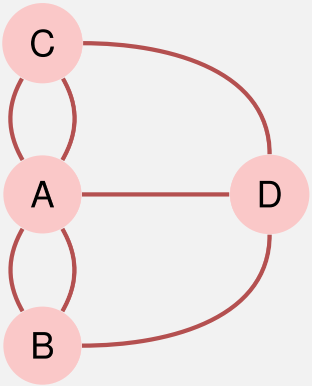


# [Multi]Graph



# Cycles

- Is there a cycle through the town (the graph) that uses each bridge (each edge) exactly once?



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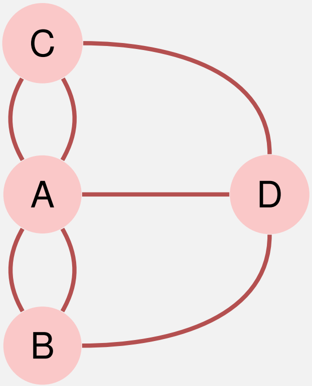

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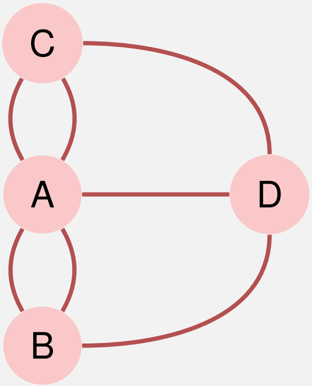

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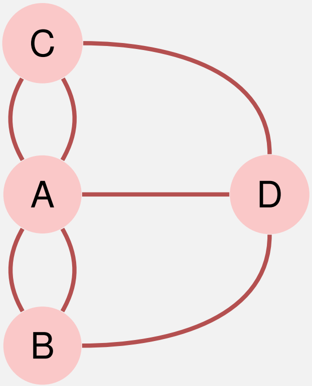
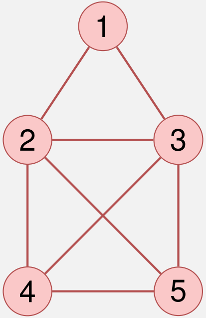
- Eulerian path $\Leftrightarrow$ each node provides an even number of edges (each node is of an *even degree*).

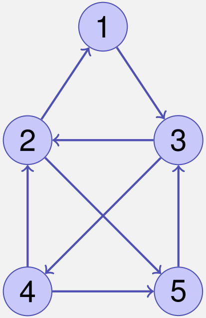
  "$\Rightarrow$" ist straightforward, "$\Leftarrow$" ist a bit more difficult



# Notation




undirected



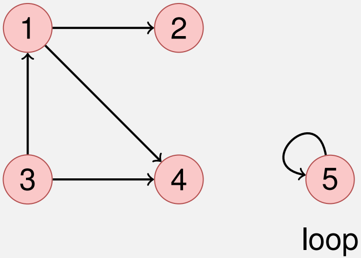

directed

$$\begin{aligned} V =& \{1, 2, 3, 4, 5\} \\ E =& \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \\ & \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\} \end{aligned}$$

$$\begin{aligned} V =& \{1, 2, 3, 4, 5\} \\ E =& \{(1, 3), (2, 1), (2, 5), (3, 2), \\ & (3, 4), (4, 2), (4, 5), (5, 3)\} \end{aligned}$$

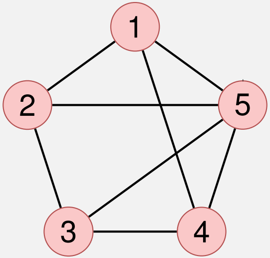
# Notation

A *directed graph* consists of a set $V = \{v_1, \ldots, v_n\}$ of nodes (*Vertices*) *and a set* $E \subseteq V \times V$ *of* Edges. The same edges may not be contained more than once.



# Notation

An *undirected graph* consists of a set $V = \{v_1, \ldots, v_n\}$ of nodes a and a set $E \subseteq \{\{u, v\} | u, v \in V\}$ of edges. Edges may bot be contained more than once.[8]

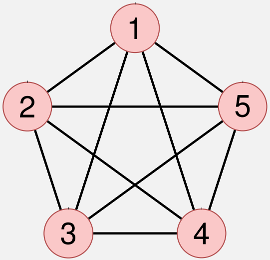



undirected graph

[8] As opposed to the introductory example – it is then called multi-graph.

# Notation

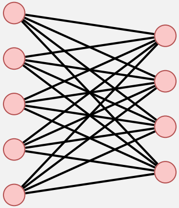
An undirected graph $G = (V, E)$ without loops where $E$ comprises all edges between pairwise different nodes is called *complete*.




a complete undirected graph

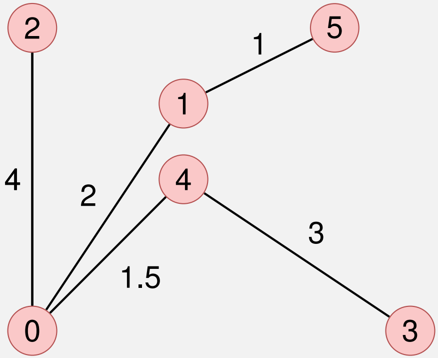
# Notation

A graph where $V$ can be partitioned into disjoint sets $U$ and $W$ such that each $e \in E$ provides a node in $U$ and a node in $W$ is called *bipartite*.



# Notation

A *weighted graph* $G = (V, E, c)$ is a graph $G = (V, E)$ with an *edge weight function* $c : E \to \mathbb{R}$. $c(e)$ is called *weight* of the edge $e$.



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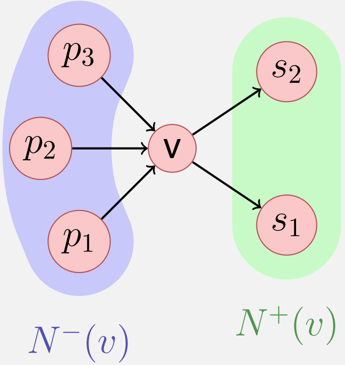
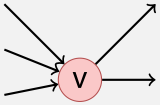
- *Predecessors* of $v \in V$: $N^-(v) := \{u \in V | (u, v) \in E\}$.
  *Successors*: $N^+(v) := \{u \in V | (v, u) \in E\}$



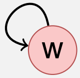
# Notation

For directed graphs $G = (V, E)$

- *In-Degree*: $\deg^-(v) = |N^-(v)|$,
  *Out-Degree*: $\deg^+(v) = |N^+(v)|$




$\deg^-(v) = 3, \deg^+(v) = 2$




$\deg^-(w) = 1, \deg^+(w) = 1$

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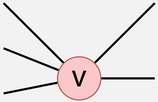
- *Neighbourhood* of $v \in V$: $N(v) = \{w \in V | \{v, w\} \in E\}$
- *Degree* of $v$: $\deg(v) = |N(v)|$ with a special case for the loops: increase the degree by $2$.




$\deg(v) = 5$


$\deg(w) = 2$

# Relationship between node degrees and number of edges

For each graph $G = (V, E)$ it holds

1. $\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|$, for $G$ directed
2. $\sum_{v \in V} \deg(v) = 2|E|$, for $G$ undirected.

# Paths

- *Path*: a sequence of nodes $\langle v_1, \ldots, v_{k+1} \rangle$ such that for each $i \in \{1 \ldots k\}$ there is an edge from $v_i$ to $v_{i+1}$ .

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- *Simple path*: path without repeating vertices

# Connectedness

- An undirected graph is called *connected*, if for eacheach pair $v, w \in V$ there is a connecting path.
- A directed graph is called *strongly connected*, if for each pair $v, w \in V$ there is a connecting path.
- A directed graph is called *weakly connected*, if the corresponding undirected graph is connected.

# Simple Observations

- generally: $0 \leq |E| \in \mathcal{O}(|V|^2)$
- connected graph: $|E| \in \Omega(|V|)$
- complete graph: $|E| = \frac{|V| \cdot (|V|-1)}{2}$ (undirected)
- Maximally $|E| = |V|^2$ (directed ),$|E| = \frac{|V| \cdot (|V|+1)}{2}$ (undirected)

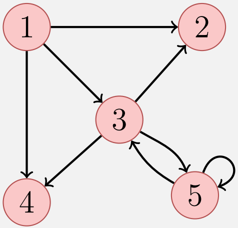
# Cycles

- *Cycle*: path $\langle v_1, \ldots, v_{k+1} \rangle$ with $v_1 = v_{k+1}$
- *Simple cycle*: Cycle with pairwise different $v_1, \ldots, v_k$, that does not use an edge more than once.
- *Acyclic*: graph without any cycles.

Conclusion: undirected graphs cannot contain cycles with length 2 (loops have length 1)

# Representation using a Matrix

Graph $G = (V, E)$ with nodes $v_1 \dots, v_n$ stored as *adjacency matrix* $A_G = (a_{ij})_{1 \leq i,j \leq n}$ with entries from $\{0, 1\}$. $a_{ij} = 1$ if and only if edge from $v_i$ to $v_j$.

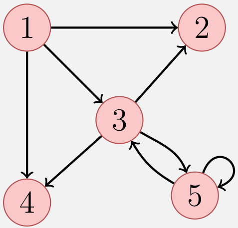


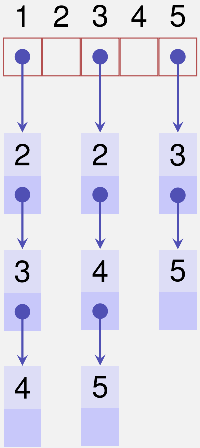

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Memory consumption $\Theta(|V|^2)$. $A_G$ is symmetric, if $G$ undirected.

# Representation with a List

Many graphs $G = (V, E)$ with nodes $v_1, \dots, v_n$ provide much less than $n^2$ edges. Representation with *adjacency list*: Array $A[1], \dots, A[n]$, $A_i$ comprises a linked list of nodes in $N^+(v_i)$.




Memory Consumption $\Theta(|V| + |E|)$.

# Runtimes of simple Operations

| **Operation** | Matrix | List |
|---|---|---|
| Find neighbours/successors of $v \in V$ | | |
| find $v \in V$ without neighbour/successor | | |
| $(u, v) \in E$ ? | | |
| Insert edge | | |
| Delete edge | | |

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| Find neighbours/successors of $v \in V$ | $\Theta(n)$ | |
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| Find neighbours/successors of $v \in V$ | $\Theta(n)$ | $\Theta(\deg^+ v)$ |
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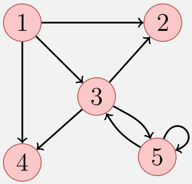
# Adjacency Matrix Product




$$B := A_G^2 = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 2 \end{pmatrix}$$

# Interpretation

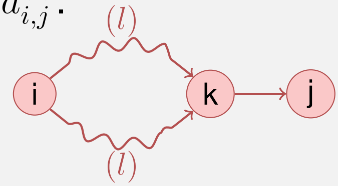
**Theorem**

*Let* $G = (V, E)$ *be a graph and* $k \in \mathbb{N}$. *Then the element* $a_{i,j}^{(k)}$ *of the matrix* $(a_{i,j}^{(k)})_{1 \leq i,j \leq n} = (A_G)^k$ *provides the number of paths with length* $k$ *from* $v_i$ *to* $v_j$ *.*

# Proof

By Induction.

**Base case:** straightforward for $k = 1$. $a_{i,j} = a_{i,j}^{(1)}$.
**Hypothesis:** claim is true for all $k \leq l$
**Step ($l \to l+1$):**




$$a_{i,j}^{(l+1)} = \sum_{k=1}^{n} a_{i,k}^{(l)} \cdot a_{k,j}$$

$a_{k,j} = 1$ iff egde $k$ to $j$, 0 otherwise. Sum counts the number paths of length $l$ from node $v_i$ to all nodes $v_k$ that provide a direct direction to node $v_j$, i.e. all paths with length $l + 1$.

# Example: Shortest Path

*Question:* is there a path from $i$ to $j$? How long is the shortest path?

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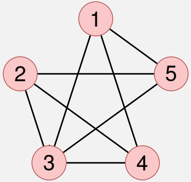
*Question:* is there a path from $i$ to $j$? How long is the shortest path?

*Answer:* exponentiate $A_G$ until for some $k < n$ it holds that $a_{i,j}^{(k)} > 0$.
$k$ provides the path length of the shortest path. If $a_{i,j}^{(k)} = 0$ for all $1 \leq k < n$, then there is no path from $i$ to $j$.

# Example: Number triangles

*Question:* How many triangular path does an undirected graph contain?



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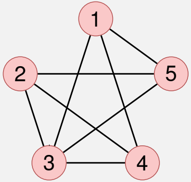
*Question:* How many triangular path does an undirected graph contain?

*Answer:* Remove all cycles (diagonal entries). Compute $A_G^3$. $a_{ii}^{(3)}$ determines the number of paths of length $3$ that contain $i$.




$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}^3 = \begin{pmatrix} 4 & 4 & 8 & 8 & 8 \\ 4 & 4 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 & 8 \\ 8 & 8 & 8 & 4 & 4 \\ 8 & 8 & 8 & 4 & 4 \end{pmatrix}$$

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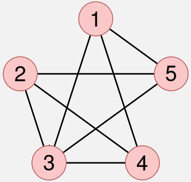
*Answer:* Remove all cycles (diagonal entries). Compute $A_G^3$. $a_{ii}^{(3)}$ determines the number of paths of length $3$ that contain $i$. There are $6$ different permutations of a triangular path. Thus for the number of triangles: $\sum_{i=1}^{n} a_{ii}^{(3)}/6$.

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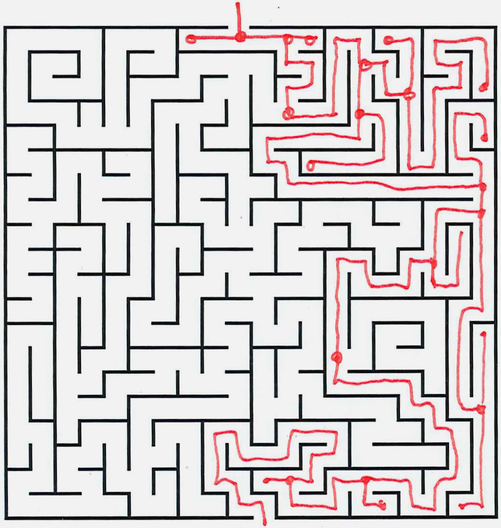
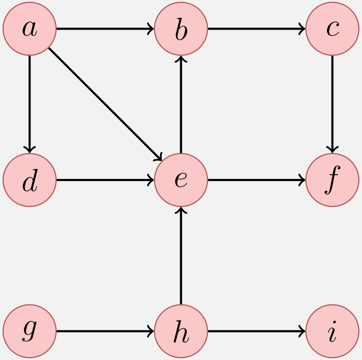
$\Rightarrow 24/6 = 4$ Dreiecke.

# Depth First Search



# Graph Traversal: Depth First Search

Follow the path into its depth until nothing is left to visit.




Adjazenzliste

- $a \rightarrow b \rightarrow d \rightarrow e$
- $b \rightarrow c$
- $c \rightarrow f$
- $d \rightarrow e$
- $e \rightarrow b \rightarrow f$
- $f$
- $g \rightarrow h$
- $h \rightarrow i$
- $i$

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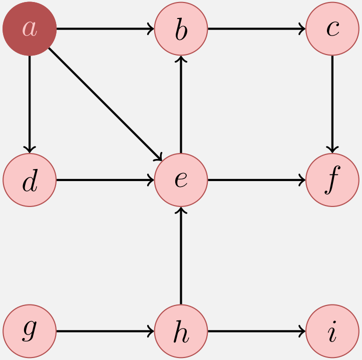


Adjazenzliste

- $a \rightarrow b \rightarrow d \rightarrow e$
- $b \rightarrow c$
- $c \rightarrow f$
- $d \rightarrow e$
- $e \rightarrow b \rightarrow f$
- $f$
- $g \rightarrow h$
- $h \rightarrow i$
- $i$

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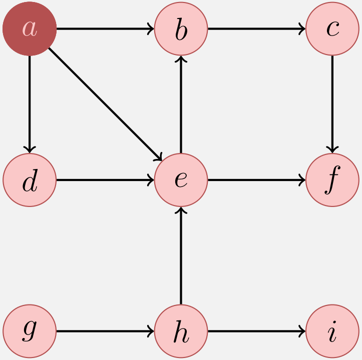


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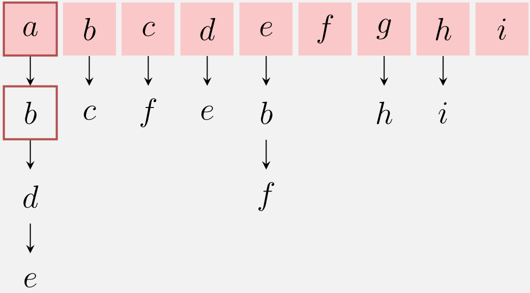

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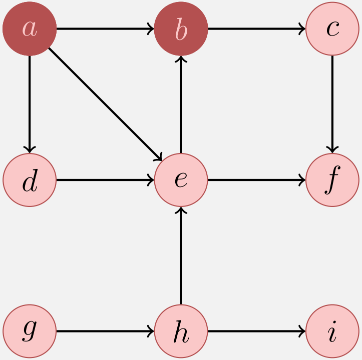


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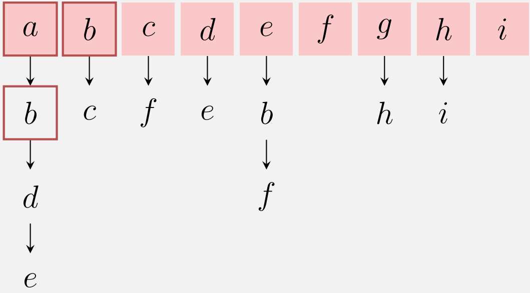

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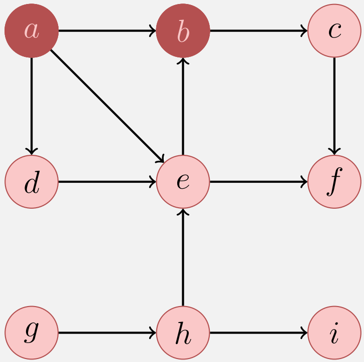


Adjazenzliste

| $a$ | $b$ | $c$ | $d$ | $e$ | $f$ | $g$ | $h$ | $i$ |
|---|---|---|---|---|---|---|---|---|
| $b$ | $c$ | $f$ | $e$ | $b$ | | $h$ | $i$ | |
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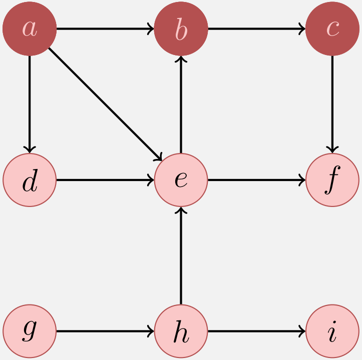


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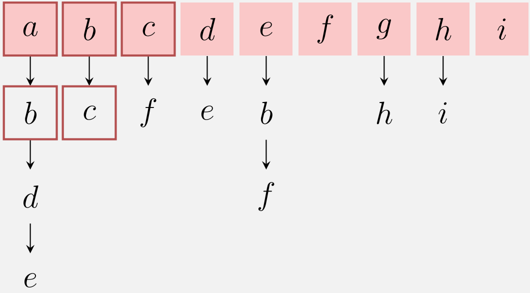

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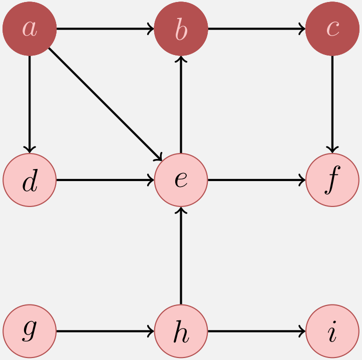


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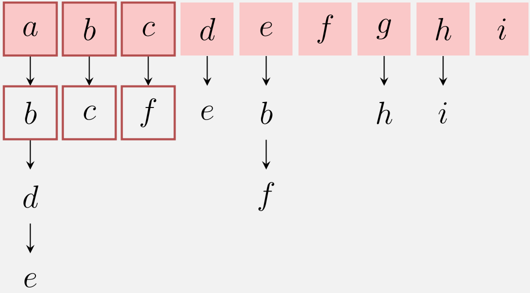

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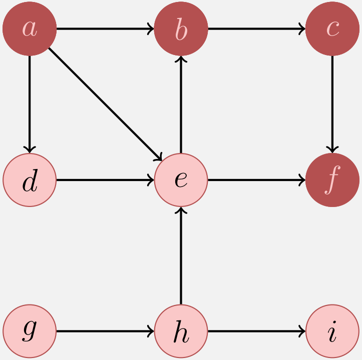


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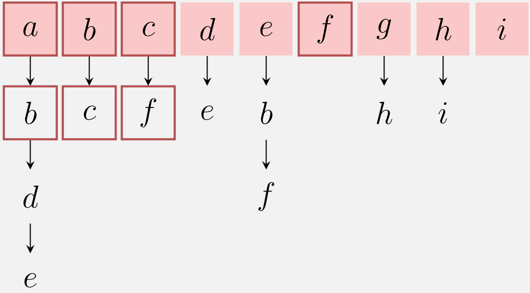

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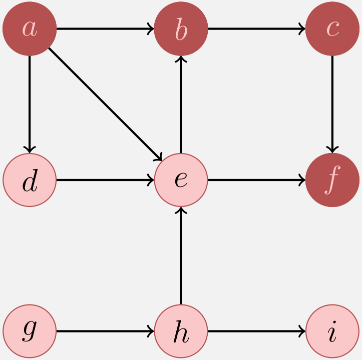


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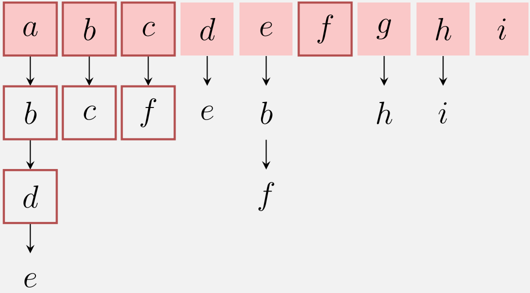

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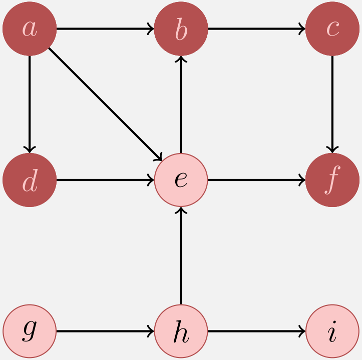


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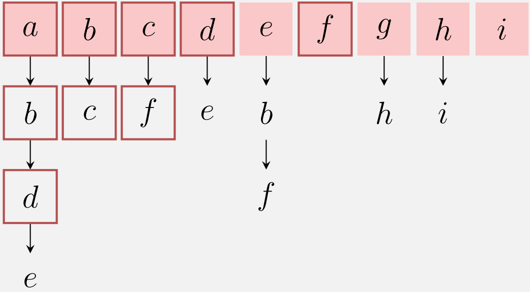

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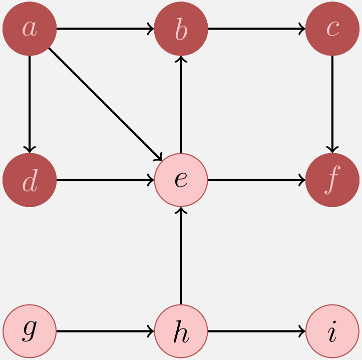


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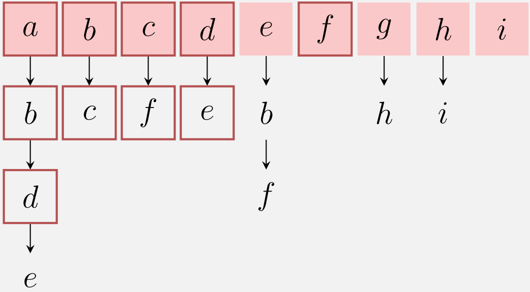

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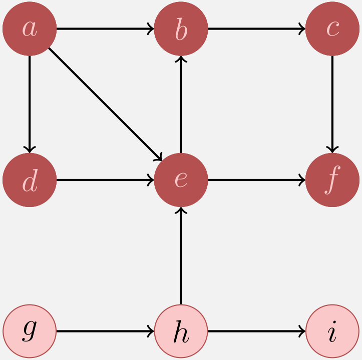


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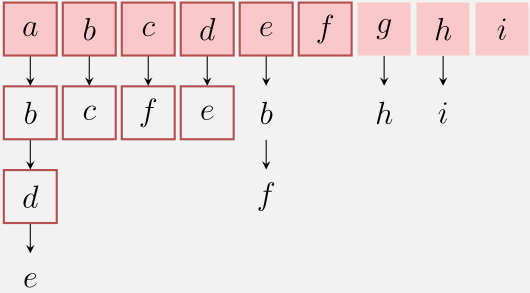

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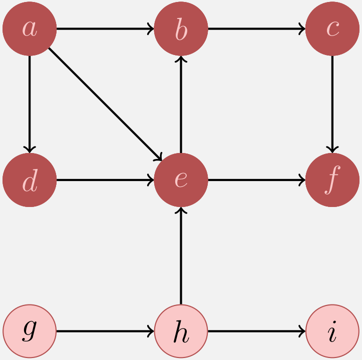


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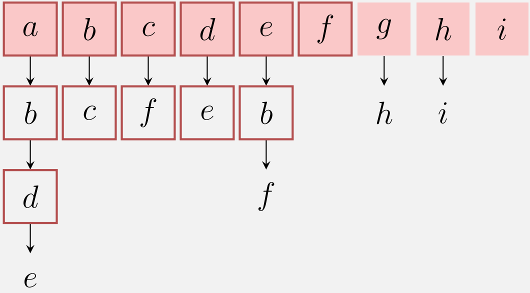

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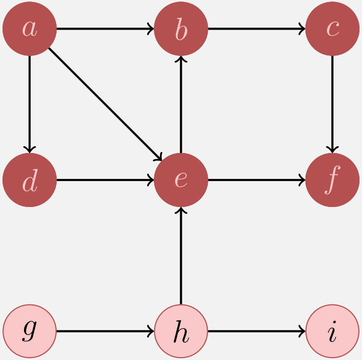


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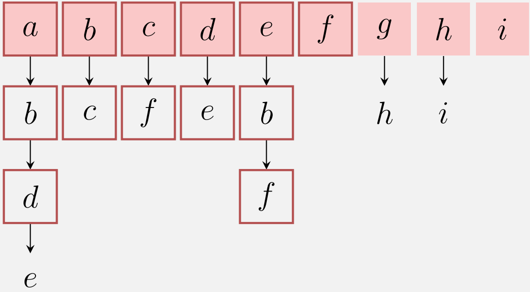

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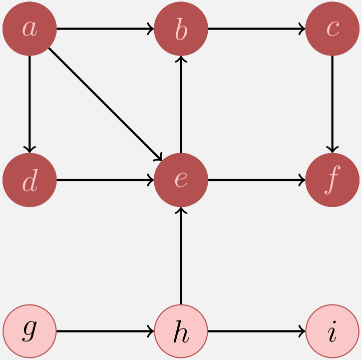


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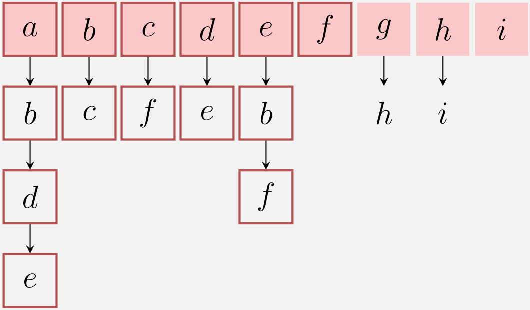

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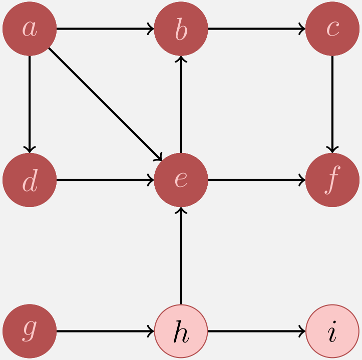


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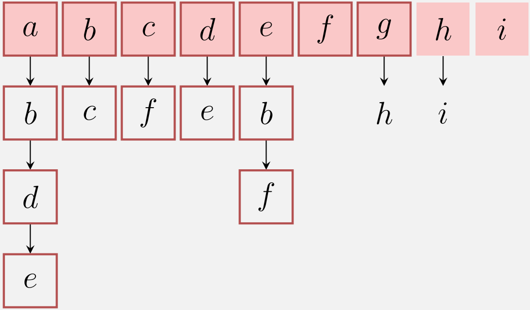

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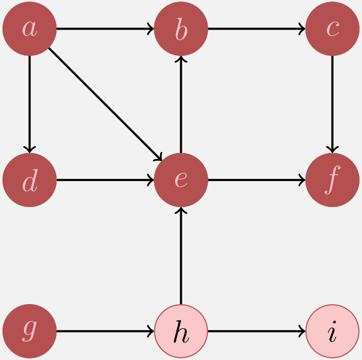


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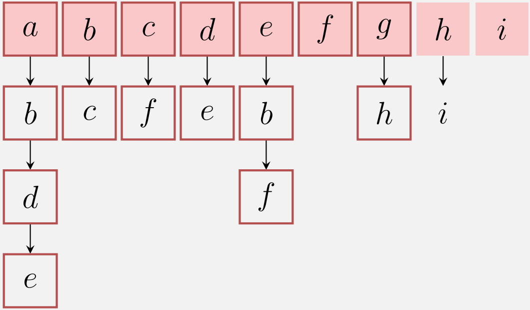

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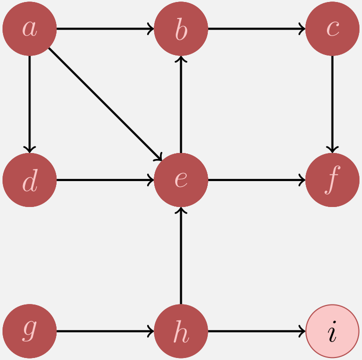


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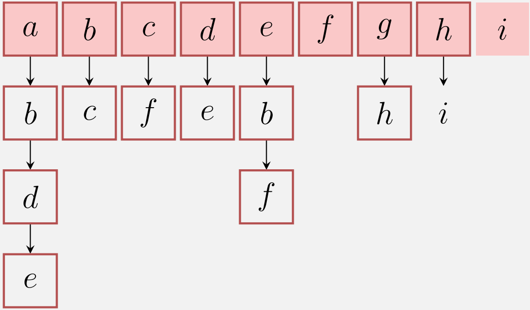

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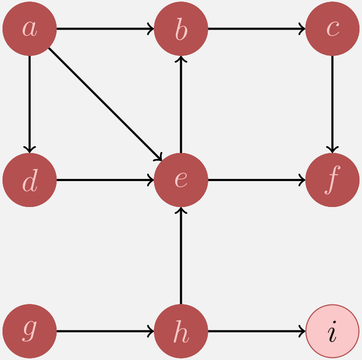


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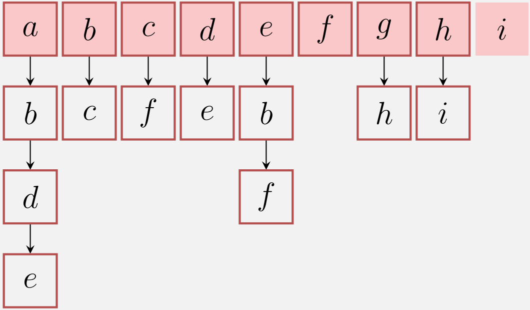

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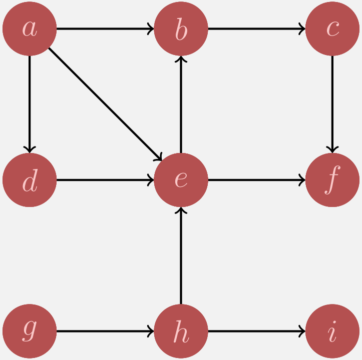
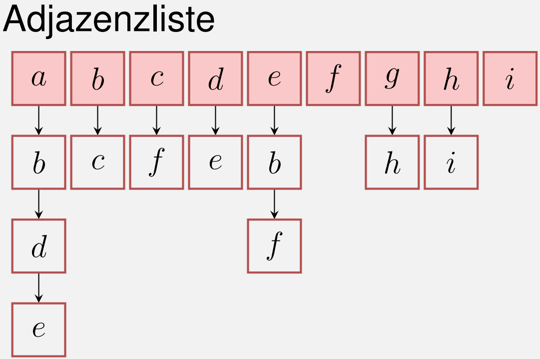



Order $a, b, c, f, d, e, g, h, i$



# Algorithm Depth First visit DFS-Visit($G, v$)

**Input :** graph $G = (V, E)$, Knoten $v$.

Mark $v$ visited
**foreach** $w \in N^+(v)$ **do**
    **if** $\neg(w$ visited$)$ **then**
        DFS-Visit$(G, w)$

Depth First Search starting from node $v$. Running time (without recursion): $\Theta(\deg^+ v)$

# Algorithm Depth First visit DFS-Visit($G$)

**Input :** graph $G = (V, E)$

**foreach** $v \in V$ **do**
  Mark $v$ not visited

**foreach** $v \in V$ **do**
  **if** $\neg(v$ visited$)$ **then**
    DFS-Visit(G,v)

Depth First Search for all nodes of a graph. Running time: $\Theta(|V| + \sum_{v \in V}(\deg^+(v) + 1)) = \Theta(|V| + |E|)$.

# Iterative DFS-Visit($G, v$)

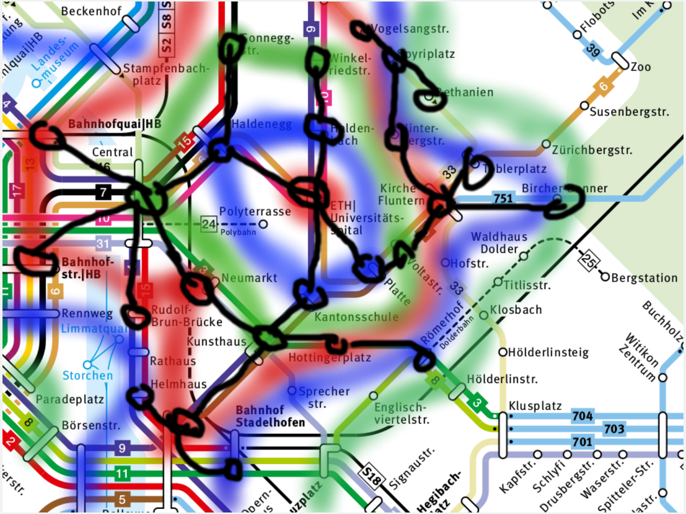
**Input :** graph $G = (V, E)$

Stack $S \leftarrow \emptyset$; push$(S, v)$
**while** $S \neq \emptyset$ **do**
  $w \leftarrow$ pop$(S)$
  **if** $\neg(w$ visited$)$ **then**
    mark $w$ visited
    **foreach** $(w, c) \in E$ **do** // (in reverse order, potentially)
      **if** $\neg(c$ visited$)$ **then**
        push$(S, c)$

Stack size up to $|E|$, for each node an extra of $\Theta(\deg^+(w) + 1)$ operations. Overal: $\Theta(|V| + |E|)$

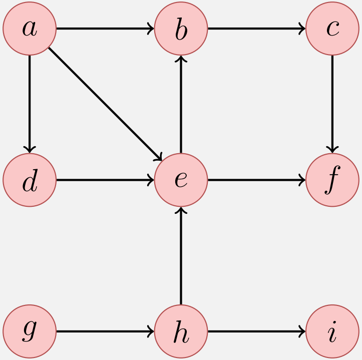
Including all calls from the above main program: $\Theta(|V| + |E|)$

# Breadth First Search



# Graph Traversal: Breadth First Search

Follow the path in breadth and only then descend into depth.




Adjazenzliste

$a \rightarrow b \rightarrow d \rightarrow e$

$b \rightarrow c$

$c \rightarrow f$

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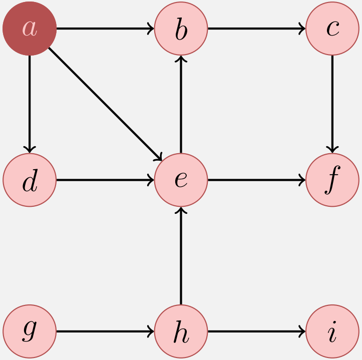


Adjazenzliste

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|---|---|---|---|---|---|---|---|---|
| $b$ | $c$ | $f$ | $e$ | $b$ | | $h$ | $i$ | |
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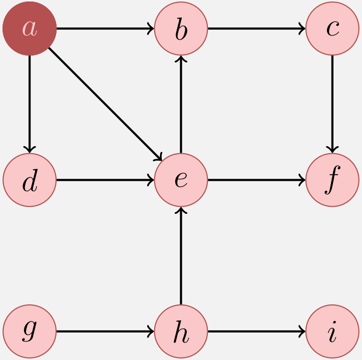


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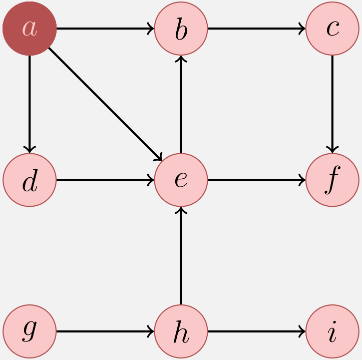


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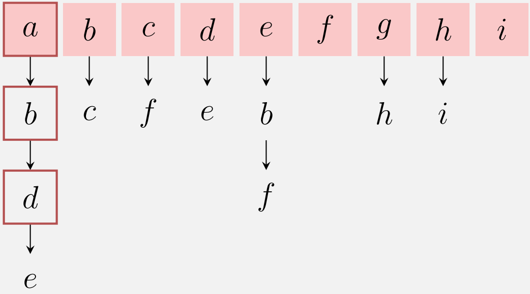

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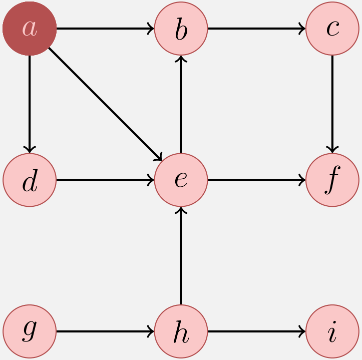


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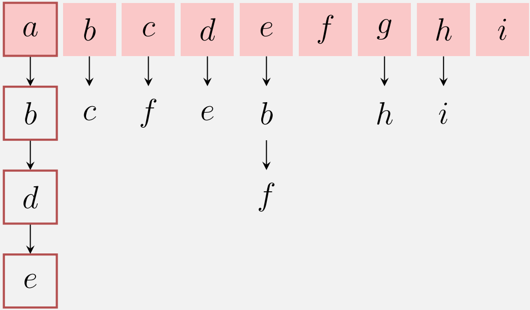

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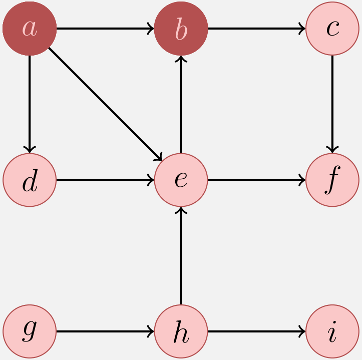


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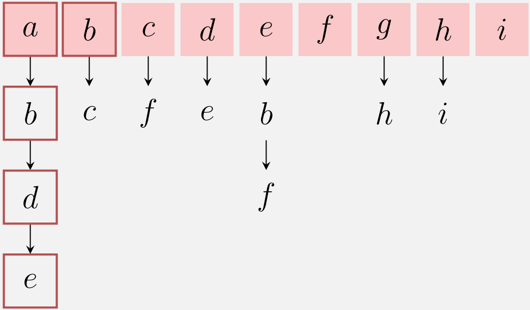

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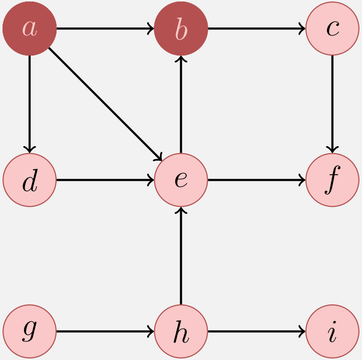


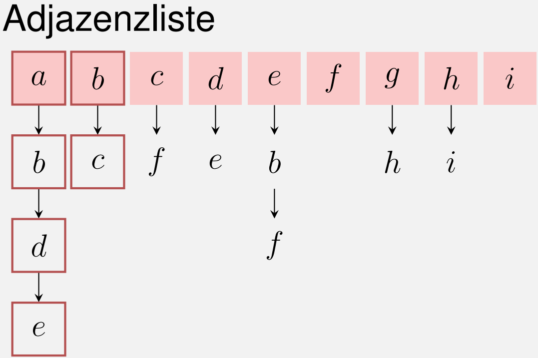

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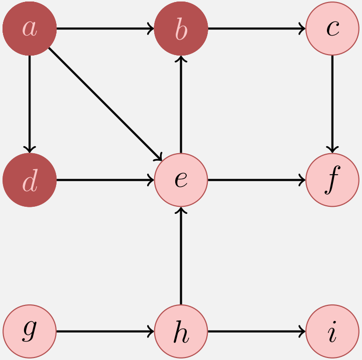


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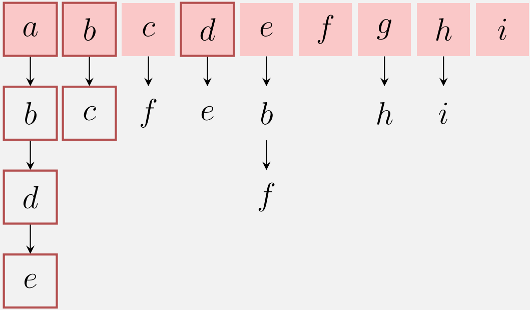

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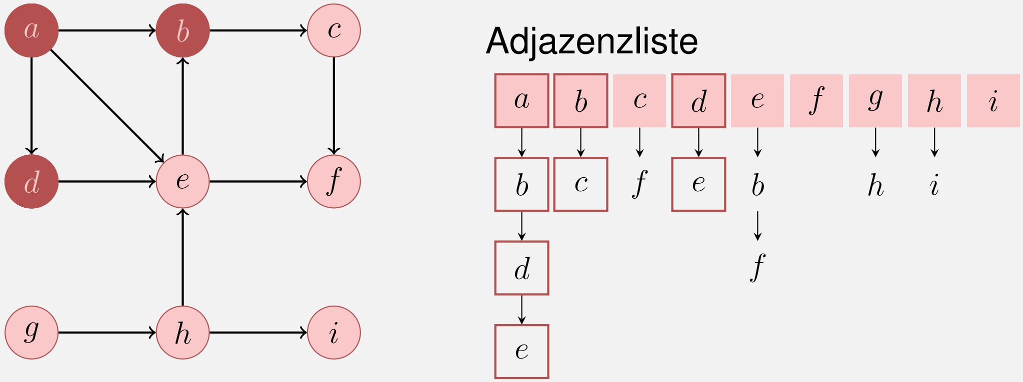

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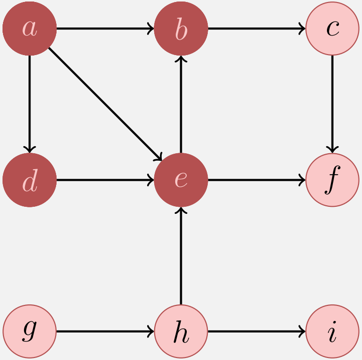


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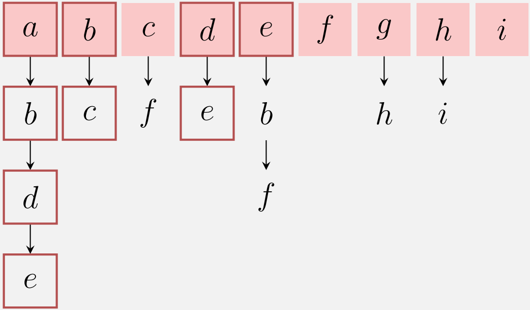

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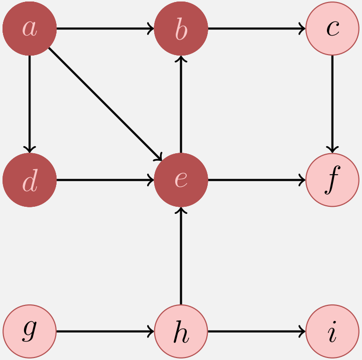


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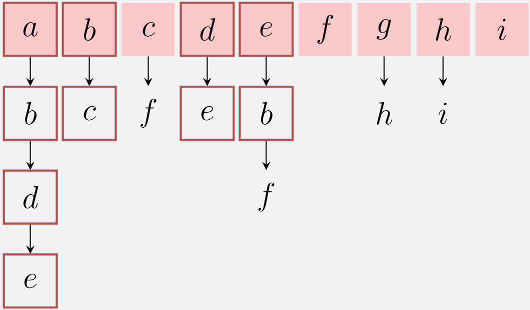

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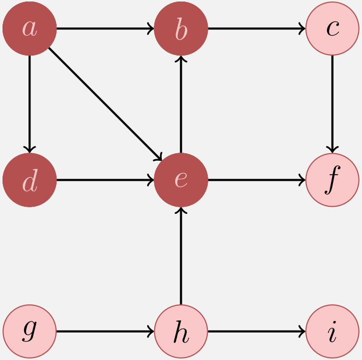


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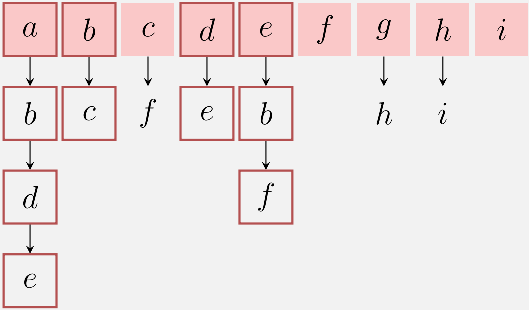

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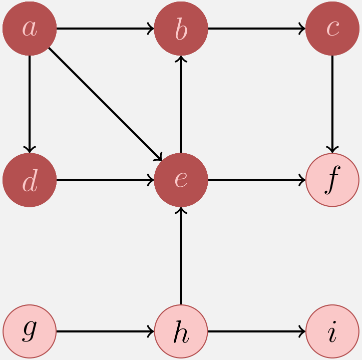


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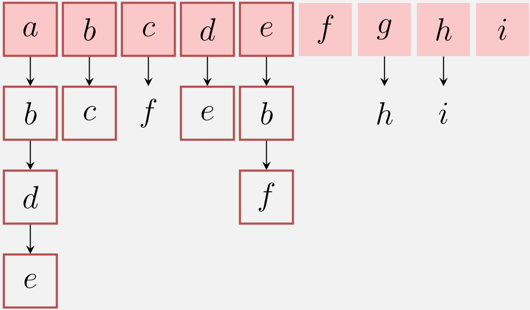

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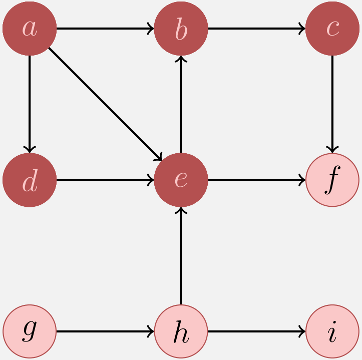
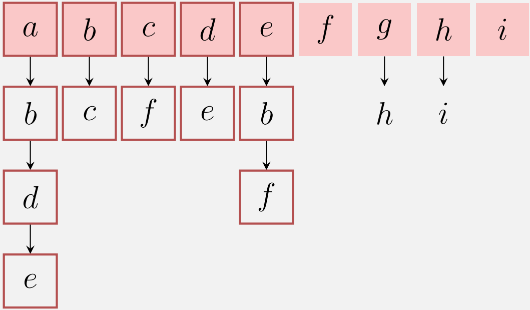
Follow the path in breadth and only then descend into depth.




Adjazenzliste



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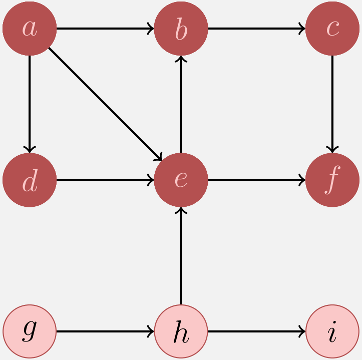


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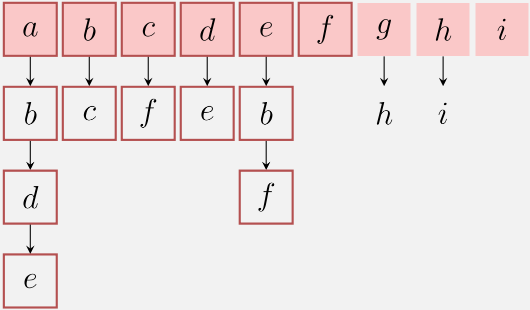

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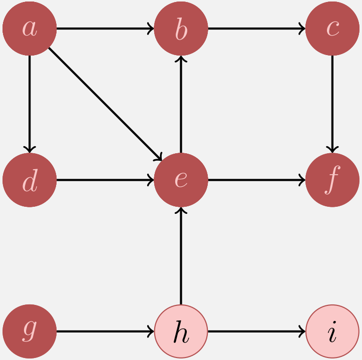


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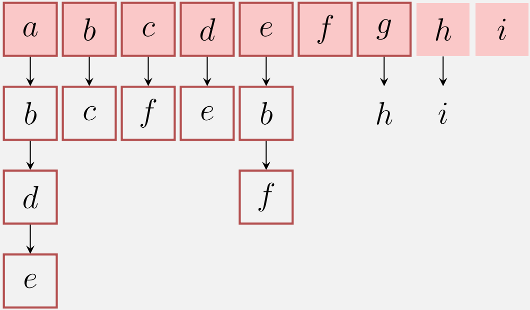

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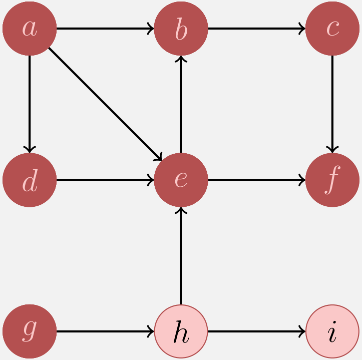


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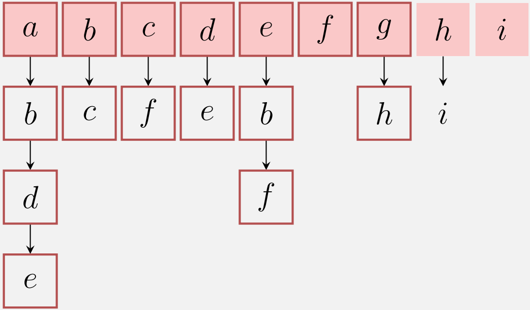

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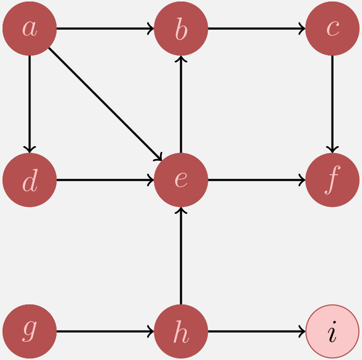


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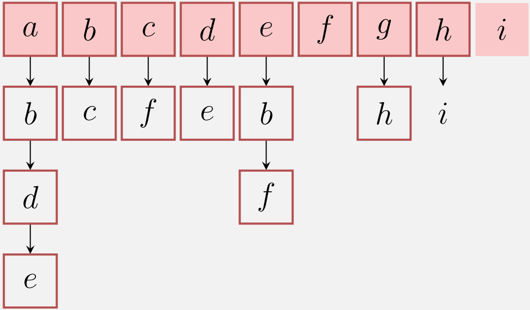

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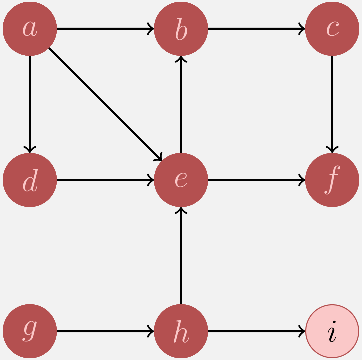


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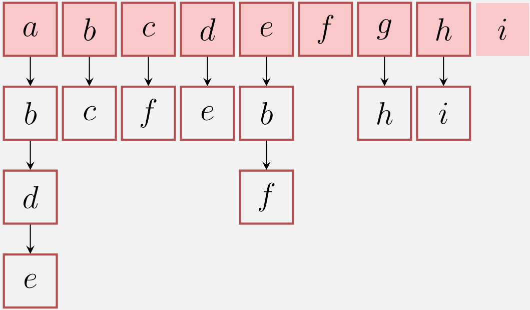

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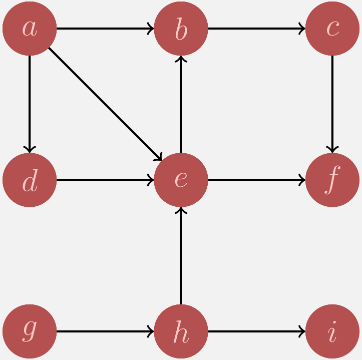
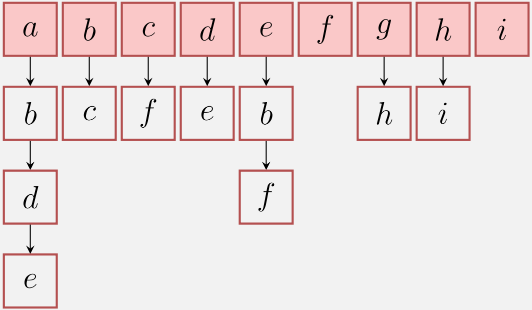



Order $a, b, d, e, c, f, g, h, i$

Adjazenzliste



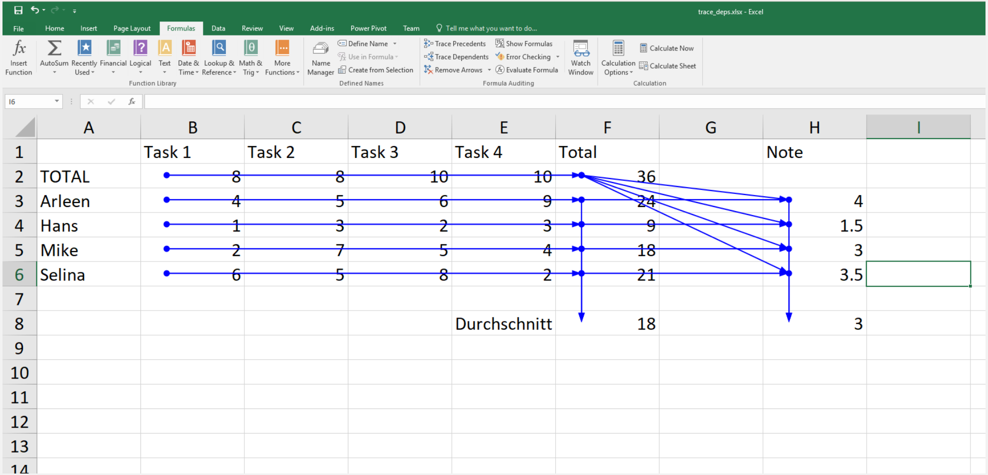
# Iterative BFS-Visit($G, v$)

**Input :** graph $G = (V, E)$

Queue $Q \leftarrow \emptyset$
Mark $v$ as active
enqueue$(Q, v)$
**while** $Q \neq \emptyset$ **do**
    $w \leftarrow$ dequeue$(Q)$
    mark $w$ visited
    **foreach** $c \in N^+(w)$ **do**
        **if** $\neg(c$ visited $\vee$ $c$ active$)$ **then**
            Mark $c$ as active
            enqueue$(Q, c)$

- Algorithm requires extra space of $\mathcal{O}(|V|)$.
- Running time including main program: $\Theta(|V| + |E|)$.

# Topological Sorting




Evaluation Order?

# Topological Sorting

*Topological Sorting* of an acyclic directed graph $G = (V, E)$:

Bijective mapping

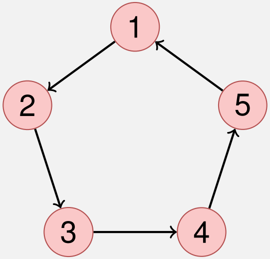
$$\mathrm{ord} : V \to \{1, \dots, |V|\}$$

such that

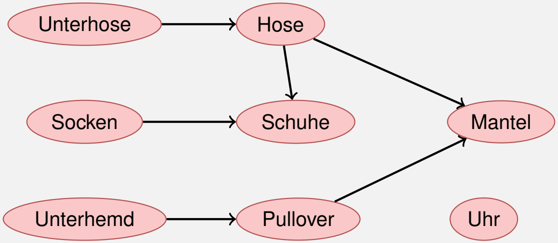
$$\mathrm{ord}(v) < \mathrm{ord}(w)\ \forall\ (v, w) \in E.$$

Identify $i$ with Element $v_i := \mathrm{ord}^{1}(i)$. Topological sorting $\widehat{=}$ $\langle v_1, \dots, v_{|V|} \rangle$.

# (Counter-)Examples




Cyclic graph: cannot be sorted topologically.




A possible toplogical sorting of the graph:
Unterhemd,Pullover,Unterhose,Uhr,Hose,Mantel,Socken,Schuhe

# Observation

**Theorem**

*A directed graph* $G = (V, E)$ *permits a topological sorting if and only if it is acyclic.*

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*A directed graph $G = (V, E)$ permits a topological sorting if and only if it is acyclic.*

Proof "⇒": If $G$ contains a cycle it cannot permit a topological sorting, because in a cycle $\langle v_{i_1}, \ldots, v_{i_m} \rangle$ it would hold that $v_{i_1} < \cdots < v_{i_m} < v_{i_1}$.

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- Base case ($n = 1$): Graph with a single node without loop can be sorted topologically, set$\mathrm{ord}(v_1) = 1$.

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  2. Graph without node $v_q$ and without its edges can be topologically sorted by the hypothesis. Now use this sorting and set $\mathrm{ord}(v_i) \leftarrow \mathrm{ord}(v_i) + 1$ for all $i \neq q$ and set $\mathrm{ord}(v_q) \leftarrow 1$.

# Preliminary Sketch of an Algorithm

Graph $G = (V, E)$. $d \leftarrow 1$

1. Traverse backwards starting from any node until a node $v_q$ with in-degree $0$ is found.

Worst case runtime:

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Worst case runtime: $\Theta(|V|^2)$.

# Improvement

Idea?

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Compute the in-degree of all nodes in advance and traverse the nodes with in-degree 0 while correcting the in-degrees of following nodes.

# Algorithm Topological-Sort($G$)

**Input :** graph $G = (V, E)$.
**Output :** Topological sorting $\mathrm{ord}$

```
Stack S ← ∅
foreach v ∈ V do A[v] ← 0
foreach (v, w) ∈ E do A[w] ← A[w] + 1 // Compute in-degrees
foreach v ∈ V with A[v] = 0 do push(S, v) // Memorize nodes with in-degree 0
i ← 1
while S ≠ ∅ do
    v ← pop(S); ord[v] ← i; i ← i + 1 // Choose node with in-degree 0
    foreach (v, w) ∈ E do // Decrease in-degree of successors
        A[w] ← A[w] − 1
        if A[w] = 0 then push(S, w)
if i = |V| + 1 then return ord else return "Cycle Detected"
```

# Algorithm Correctness

**Theorem**

*Let $G = (V, E)$ be a directed acyclic graph. Algorithm TopologicalSort($G$) computes a topological sorting $\mathrm{ord}$ for $G$ with runtime $\Theta(|V| + |E|)$.*

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Proof: follows from previous theorem:

1. Decreasing the in-degree corresponds with node removal.
2. In the algorithm it holds for each node $v$ with $A[v] = 0$ that either the node has in-degree 0 or that previously all predecessors have been assigned a value $\mathrm{ord}[u] \leftarrow i$ and thus $\mathrm{ord}[v] > \mathrm{ord}[u]$ for all predecessors $u$ of $v$. Nodes are put to the stack only once.
3. Runtime: inspection of the algorithm (with some arguments like with graph traversal)

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*Let $G = (V, E)$ be a directed graph containing a cycle. Algorithm TopologicalSort($G$) terminates within $\Theta(|V| + |E|)$ steps and detects a cycle.*

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Proof: let $\langle v_{i_1}, \dots, v_{i_k}\rangle$ be a cycle in $G$. In each step of the algorithm remains $A[v_{i_j}] \geq 1$ for all $j = 1, \dots, k$. Thus $k$ nodes are never pushed on the stack und therefore at the end it holds that $i \leq V + 1 - k$.

The runtime of the second part of the algorithm can become shorter. But the computation of the in-degree costs already $\Theta(|V| + |E|)$.