Educational Objectives

- You understand how a solution to a recursive problem can be implemented in Java.
- You understand how methods are being executed in an *execution stack*.

12. Recursion

Mathematical Recursion, Termination, Call Stack, Examples, Recursion vs. Iteration, Lindenmayer Systems

Mathematical Recursion

- Many mathematical functions can be naturally defined recursively.
- The means, the function appears in its own definition

$$n! = \begin{cases} 1, & \text{if } n \leq 1 \\ n \cdot (n-1)!, & \text{otherwise} \end{cases}$$

Recursion in Java:

$$n! = \begin{cases} 1, & \text{if } n \leq 1 \\ n \cdot (n-1)!, & \text{otherwise} \end{cases}$$

```
// POST: return value is n!
public static int fac (int n) {
  if (n <= 1)
    return 1;
  else
    return n * fac (n-1);
}
```

Infinite Recursion

- is as bad as an infinite loop. . .
- . . . but even worse: it burns time and memory

```
private static void f() {
  f(); // f() -> f() -> ... stack overflow
}
```


Recursive Functions: Termination

As with loops we need

- progress towards termination

`fac(n)`:
terminates immediately for $n \leq 1$, otherwise the function is called recusively with $<$ n .

„n is getting smaller with each call."

Recursive Functions: Evaluation

Example: `fac(4)`

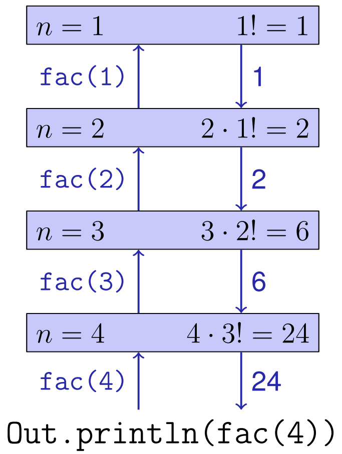
```
// POST: return value is n!
public static int fac (int n) {

  if (n <= 1) return 1;
  return n * fac(n-1); // n > 1
}
```

Initialization of the formal argument: $n = 4$
recursive call with argument $n - 1 == 3$

The Call Stack

For each method call:

- push value of the actual parameter on the stack
- work with the upper most value
- at the end of the call the upper most value is removed from the stack



Euclidean Algorithm

- finds the greatest common divisor $\gcd(a,b)$ of two natural numbers a and b
- is based on the following mathematical recursion:

$$\gcd(a,b) = \begin{cases} a, & \text{falls } b = 0 \\ \gcd(b, a \bmod b), & \text{andernfalls} \end{cases}$$

Euclidean Algorithm in Java

$$\gcd(a,b) = \begin{cases} a, & \text{falls } b = 0 \\ \gcd(b, a \bmod b), & \text{andernfalls} \end{cases}$$

```
public static int gcd (int a, int b) {
  if (b == 0)
    return a;
  else
    return gcd (b, a % b);
}
```

Termination: $a \bmod b < b$, thus b is decreased for each recursive call.

Fibonacci Numbers

$$F_n := \begin{cases} 0, & \text{falls } n = 0 \\ 1, & \text{falls } n = 1 \\ F_{n-1} + F_{n-2}, & \text{falls } n > 1 \end{cases}$$

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89 \ldots$$

Fibonacci Numbers in Java

Laufzeit

`fib(50)` takes “forever” because it computes F_{48} two times, F_{47} 3 times, F_{46} 5 times, F_{45} 8 times, F_{44} 13 times, F_{43} 21 times ... F_1 ca. 10^9 times (!)

```
public static int fib (int n) {
  if (n == 0) return 0;
  if (n == 1) return 1;
  return fib (n-1) + fib (n-2); // n > 1
}
```

Korrektheit und Terminierung sind klar.

Fast Fibonacci Numbers

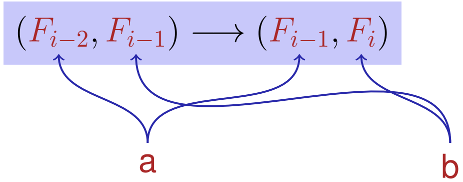
Idea:

- Compute each Fibonacci number only once, in the order $F_0, F_1, F_2, \ldots, F_n$!
- Memorize the most recent two numbers (variables `a` and `b`)!
- Compute the next number as a sum of `a` and `b`!

Fast Fibonacci Numbers in Java

```
public static int fib (int n){
  if (n == 0) return 0;
  if (n <= 2) return 1;
  int a = 1;  // F_1
  int b = 1;  // F_2
  for (int i = 3; i <= n; ++i){
    int a_old = a;  // F_i-2
    a = b;                 // F_i-1
    b += a_old;            // F_i-1 += F_i-2 -> F_i
  }
  return b;
}
```

very fast, also for `fib(50)`



Recursion and Iteration

Recursion can *always* be simulated by

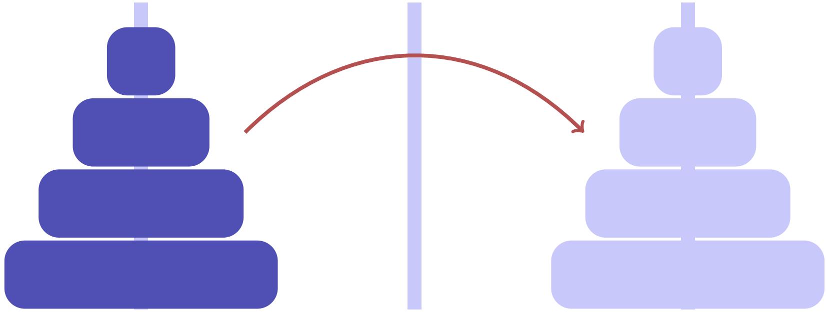
- Iteration (loops)
- explicit “call stack” (e.g. array)

Often recursive formulations are simpler, sometimes they are less efficient

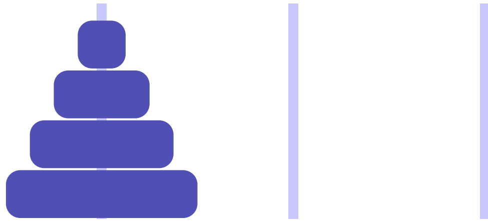
The Power of Recursion

- Some problems appear to be hard to solve without recursion. With recursion they become significantly simpler.
- Examples: *The towers of Hanoi*, The n-Queens-Problem, Sudoku-Solver, Expression Parsers, Reversing In- or Output, Searching in Trees, Divide-And-Conquer (e.g. sorting) $\rightarrow$ Informatik II,

Experiment: The Towers of Hanoi



The Towers of Hanoi – Code



Move 4 discs vom 0 to 2 with auxiliary staple 1:

```
move(4, 0, 1, 2);
```


The Towers of Hanoi – Code

```
move(4, 0, 1, 2);
        ==
```

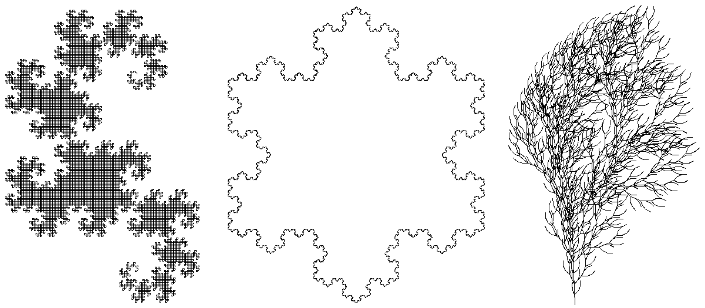
1. Move 3 discs from 0 to 1 with auxiliary staple 2:
   `move(3, 0, 2, 1);`
2. Move 1 disc from 0 to 2
   `move(1, 0, 1, 2);`
3. Move 3 discs from 1 to 2 with auxiliary staple 0
   `move(3, 1, 0, 2);`

The Towers of Hanoi – Code

```
public static void move(int n, int source, int aux, int dest){
  if (n==1){
    Out.println("move " + source + "−>" + dest);
  } else {
    move(n−1, source, dest, aux);
    move(1, source, aux, dest);
    move(n−1, aux, source, dest);
  }
}
```

Lindenmayer-Systems (L-Systems)

Fractals from Strings and Turtles



L-Systems have been invented by the Ungarian Biologist Aristid Lindenmayer (1925 – 1989) to model growth of plants.

Definition and Example

- alphabet Σ
- Σ^*: finite words over Σ
- production $P : \Sigma \to \Sigma^*$
- initial word $s_0 \in \Sigma^*$

- $\{ \mathrm{F}, +, - \}$
c	$P(c)$
F	F + F +
+	+
−	−
- F

Definition

The triple $\mathcal{L} = (\Sigma, P, s_0)$ is an L-System.

The Language Described

Wörter $w_0, w_1, w_2, \ldots \in \Sigma^*$:

$P(\mathrm{F}) = \mathrm{F} + \mathrm{F} +$

$$w_0 := s_0 \qquad w_0 := \mathrm{F}$$

$$\mathrm{F} + \mathrm{F} +$$

$$w_1 := P(w_0) \qquad w_1 := \boxed{\mathrm{F}}\boxed{+}\boxed{\mathrm{F}}\boxed{+}$$

$$w_2 := P(w_1) \qquad w_2 := \boxed{\mathrm{F} + \mathrm{F} +}\boxed{+}\boxed{\mathrm{F} + \mathrm{F} +}\boxed{+}$$

$$P(\mathrm{F})P(+)P(\mathrm{F})P(+)$$

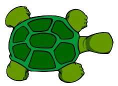
$$\vdots \qquad \vdots$$

Definition

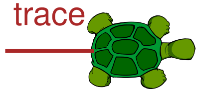
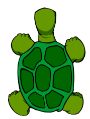
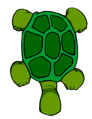
$P(c_1c_2 \ldots c_n) := P(c_1)P(c_2) \ldots P(c_n)$

Turtle Graphics

Turtle with position and direction

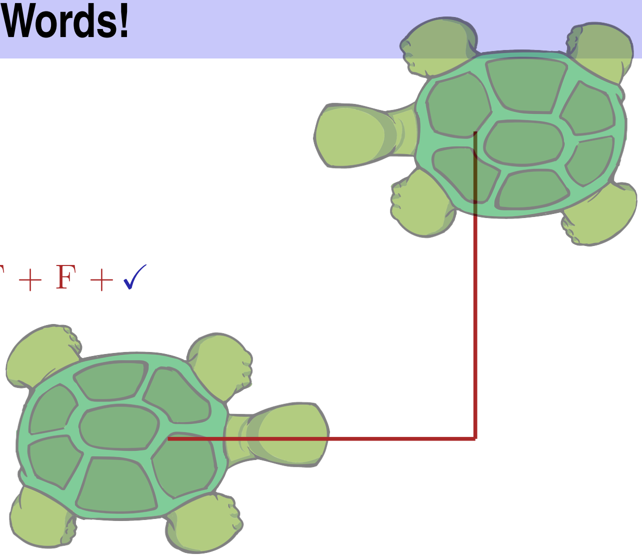


Turtle understands 3 commands:

F: move one step forwards ✓	+: rotate by 90 degrees ✓	−: rotate by −90 degrees ✓
trace 		

Draw Words!

$w_1 = \mathrm{F} + \mathrm{F} + \checkmark$



lindenmayer: Main Program

word $w_0 \in \Sigma^*$:

```
public static void main(String[] args){
    Out.print("Maximal Recursion Depth = ");
    int depth = In.readInt();
    Turtle t = new Turtle();
    produce(t,"F",depth);
    t.show();
}
```

$w = w_0 = \mathrm{F}$

lindenmayer: production

```
// POST: recursively iterate over the production of the characters
//      of a word.
//      When recursion limit is reached, the word is "drawn"
static void produce (Turtle turtle, String word, int depth){
  if (depth > 0) {
    for (int k = 0; k < word.length(); ++k){
      produce(turtle, replace(word.charAt(k)), depth−1);
    }
  } else {
    draw(turtle,word);
  }
}
```

$w = w_i \rightarrow w = w_{i+1}$

draw $w = w_n$!

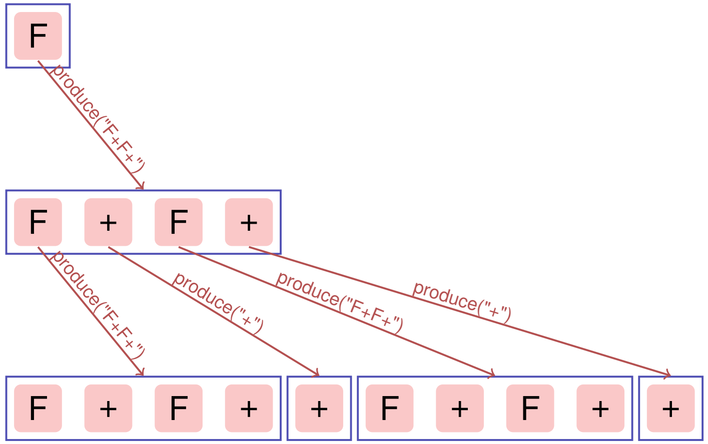
lindenmayer: replace

```
// POST: returns the production of c
static String replace (char c){
  switch (c) {
  case 'F':
    return "F+F+";
  default:
    return Character.toString(c); // trivial production c −> c
  }
}
```

lindenmayer: draw

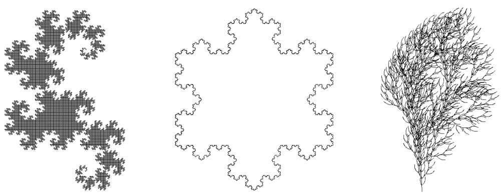
```
// POST: draws the turtle graphic interpretation of word
static void draw (Turtle turtle, String word) {
  for (int k = 0; k < word.length(); ++k){
    switch (word.charAt(k)) {
    case 'F':
      turtle.forward(1); // move one step forward
      break;
    case '+':
      turtle.left(90); // turn counterclockwise by 90 degrees
      break;
    }
  }
}
```


The Recursion



L-Systeme: Erweiterungen

- arbitrary symbols without graphical interpetation
- arbitrary angles (snowflake)
- saving and restoring the state of the turtle $\rightarrow$ plants (bush)



Challenge: we are looking forward to your contributions