Educational Objectives

- You have a good understanding how a computer represents numbers.
- You can transform integers in *binary representation* and perform computations.
- You understand how the value range of integers is chosen.
- You can describe the representation of floating-point numbers in general.
- You know the three *floating-point rules*.
- You can use booleans and *boolean expressions* in Java.

4. Number Representations

Domain of Types `int`, `float` and `double` Mixed Expressions and Conversion; Holes in the Domain;Floating Point Number Systems; IEEE Standard; Limits of Floating Point Arithmetics; Floating Point Guidelines

Binary Number Representations

Binary representation ("Bits" from $\{0, 1\}$)

$$b_n b_{n-1} \dots b_1 b_0$$

corresponds to the number $b_n \cdot 2^n + \cdots + b_1 \cdot 2 + b_0$

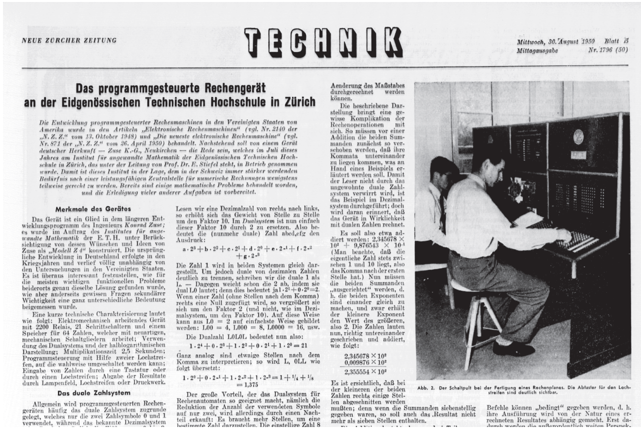
Example: `101011` corresponds to 43.

Least Significant Bit (LSB)

Most Significant Bit (MSB)

Binary Numbers: Numbers of the Computer?

Truth: Computers calculate using binary numbers.



NEUE ZÜRCHER ZEITUNG

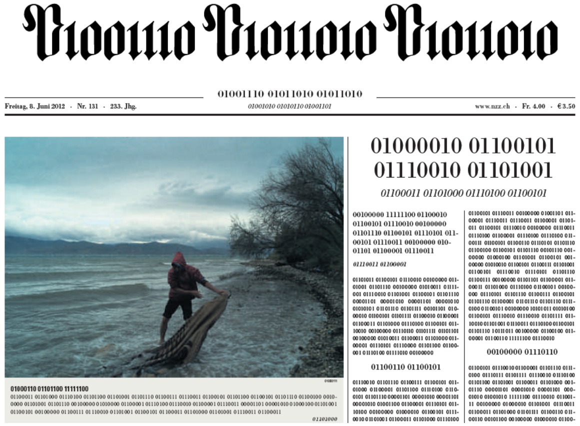
TECHNIK

Mittwoch, 30. August 1950

Das programmgesteuerte Rechengerät an der Eidgenössischen Technischen Hochschule in Zürich

Binary Numbers: Numbers of the Computer?

Stereotype: computers are talking 0/1 gibberish



Computing Tricks

- Estimate the orders of magnitude of powers of two.[2]:

$$
\begin{aligned}
2^{10} &= 1024 = 1\text{Ki} \approx 10^3.\\
2^{20} &= 1\text{Mi} \approx 10^6,\\
2^{30} &= 1\text{Gi} \approx 10^9,\\
2^{32} &= 4 \cdot (1024)^3 = 4\text{Gi} \approx 4 \cdot 10^9.\\
2^{64} &= 16\text{Ei} \approx 16 \cdot 10^{18}.
\end{aligned}
$$

[2] Decimal vs. binary units: MB - Megabyte vs. MiB - Megabibyte (etc.)
kilo (K, Ki) – mega (M, Mi) – giga (G, Gi) – tera(T, Ti) – peta(P, Pi) – exa (E, Ei)

Definition: *Domain*

For numeric types the domain defines the numeric interval a the type can cover.

Book, on page 24

Domain of Type `int`

```
public class Main {
    public static void main(String[] args) {
        Out.print("Minimum int value is ");
        Out.println(Integer.MIN_VALUE);
        Out.print("Maximum int value is ");
        Out.println(Integer.MAX_VALUE);
    }
}
```

```
Minimum int value is -2147483648.
Maximum int value is 2147483647.
```

Where do these numbers come from?

Domain of the Type `int`

Representation with 32 bits. Domain comprises the 2^{32} integers:

$$\{-2^{31}, -2^{31}+1, \ldots, -1, 0, 1, \ldots, 2^{31}-2, 2^{31}-1\}$$

Where does this partitioning come from?

Computing with Binary Numbers (4 digits)

Simple Addition

$$\begin{array}{rr} 2 & 0010 \\ +3 & +0011 \\ \hline 5 & 0101 \end{array}$$

Simple Subtraction

$$\begin{array}{rr} 5 & 0101 \\ -3 & -0011 \\ \hline 2 & 0010 \end{array}$$

Computing with Binary Numbers (4 digits)

Addition with Overflow

$$\begin{array}{rr} 7 & 0111 \\ +9 & +1001 \\ \hline 16 & (1)0000 \end{array}$$

Negative Numbers?

$$\begin{array}{rr} 5 & 0101 \\ +(-5) & ???? \\ \hline 0 & (1)0000 \end{array}$$

Computing with Binary Numbers (4 digits)

Simpler -1

$$\begin{array}{rr} 1 & 0001 \\ +(-1) & 1111 \\ \hline 0 & (1)0000 \end{array}$$

Utilize this:

$$\begin{array}{rr} 3 & 0011 \\ +? & +???? \\ \hline -1 & 1111 \end{array}$$

Computing with Binary Numbers (4 digits)

Invert!

$$\begin{array}{r} 3 \\ +(-4) \\ \hline -1 \end{array} \qquad \begin{array}{r} 0011 \\ +1100 \\ \hline 1111 \widehat{=} 2^B - 1 \end{array}$$

$$\begin{array}{r} a \\ +(-a-1) \\ \hline -1 \end{array} \qquad \begin{array}{r} a \\ \bar{a} \\ \hline 1111 \widehat{=} 2^B - 1 \end{array}$$

Computing with Binary Numbers (4 digits)

- Negation: inversion and addition of 1

$$-a \quad \widehat{=} \quad \bar{a} + 1$$

- Wrap around semantics (calculating modulo 2^B

$$-a \quad \widehat{=} \quad 2^B - a$$

Why this works

Modulo arithmetics: Compute on a circle[3]

$11 \equiv 23 \equiv -1 \equiv \ldots \mod 12$ $\quad + \quad$ $4 \equiv 16 \equiv \ldots \mod 12$ $\quad = \quad$ $3 \equiv 15 \equiv \ldots \mod 12$

[3] The arithmetics also work with decimal numbers (and for multiplication).

Negative Numbers (3 Digits)

	a	$-a$	
0	000	000	0
1	001	111	-1
2	010	110	-2
3	011	101	-3
4	100	100	-4
5	101		
6	110		
7	111		

The most significant bit decides about the sign.

Two's Complement

- Negation by bitwise negation and addition of 1

$$-2 = -[0010] = [1101] + [0001] = [1110]$$

  - Arithmetics of addition and subtraction *identical* to unsigned arithmetics

  $$3 - 2 = 3 + (-2) = [0011] + [1110] = [0001]$$

  - Intuitive "wrap-around" conversion of negative numbers.

  $$-n \to 2^B - n$$

  - Domain: $-2^{B-1} \dots 2^{B-1} - 1$

Over- and Underflow

- Arithmetic operations (`+`,`-`,`*`) can lead to numbers outside the valid domain.
- Results can be incorrect!

`power8`: $15^8 = -1732076671$

`power20`: $3^{20} = -808182895$

- There is *no error message!*

Definition: *Floating Point Numbers*

Floating point numbers represent numbers from $\mathbb{R}$ with a fixed number of significant number of digits, multiplied by a decimal power (base 10).

Book, on page 67

"Proper Calculation"

```
public class Main {

    public static void main(String[] args) {
        Out.print("Celsius: ");
        int celsius = In.readInt();
        int fahrenheit = 9 * celsius / 5 + 32;
        Out.print(celsius + " degrees Celsius are ");
        Out.println(fahrenheit + " degrees Fahrenheit");
    }
}
```

```
public class Main {

    public static void main(String[] args) {
        Out.print("Celsius: ");
        float celsius = In.readInt();
```

Types `float` and `double`

- are the fundamental types for floating point numbers
- approximate the field of real numbers $(\mathbb{R}, +, \times)$ from mathematics
- have a great domain, sufficient for many applications (`double` provides more places than `float`)
- are fast on many computers

Fixpoint numbers

- fixed number of integer places (e.g. 7)
- fixed number of decimal places (e.g. 3)

`0.0824 = 0000000.082` ← third place truncated

Nachteile

- Domain is getting *even* smaller than for integers.
- If a number can be represented depends on the position of the comma.

Floating point Numbers

- fixed number of significant places (e.g. 10)
- plus position of the comma

`82.4` = $824 \cdot 10^{-1}$

`0.0824` = $824 \cdot 10^{-4}$

- Zahl ist $\textit{Mantissa} \times 10^{\textit{Exponent}}$

Domain

Integer Types:

- Over- and Underflow relatively frequent, but ...
- the domain is contiguous (no “holes”): $\mathbb{Z}$ is “discrete”.

Floating point types:

- Overflow and Underflow seldom, but ...
- there are holes: $\mathbb{R}$ is “continuous”.

Holes in the Domain

```
public class Main {
    public static void main(String[] args) {
    Out.print("First number =? ");          input 1.1
    float n1 = In.readFloat();

    Out.print("Second number =? ");         input 1.0
    float n2 = In.readFloat();

    Out.print("Their difference =? ");      input 0.1
    float d = In.readFloat();

    Out.print("computed difference − input difference = ");
    Out.println(n1−n2−d);
    }                                       output 2.2351742E-8
}
```

What is going on here?

Floating Point Rules — Rule 1

Rule 1

Do not test rounded floating point numbers for equality.

```
for (float i = 0.1; i != 1.0; i += 0.1){
    Out.println(i);
}
```

endless loop because i never becomes exactly 1

More explanations next time!

Floating Point Rules — Rule 2

Rule 2

Do not add two numbers providing very different orders of magnitude!

$$1.000 \cdot 10^5$$
$$+1.000 \cdot 10^0$$
$$= 1.00001 \cdot 10^5$$
$$\text{“=”}\ 1.000 \cdot 10^5 \quad \text{(Rounding on 4 places)}$$

Addition of 1 does not provide any effect!

Floating Point Guidelines — Rule 3

Rule 4

Do not subtract two numbers with a very similar value.

Cancellation problems (without further explanations).

5. Logical Values

Boolean Functions; the Type `boolean`; logical and relational operators; shortcut evaluation

Our Goal

```
int a = In.readInt();
if (a % 2 == 0){
    Out.println("even");
} else {
    Out.println("odd");
}
```

Behavior depends on the value of a Boolean expression

Boolean Values in Mathematics

Boolean expressions can take on one of two values:

F or T

- F corresponds to *"wrong"*
- T corresponds to *"true"*

The Type `boolean` in Java

- represents *logical values*
- Literals `false` and `true`
- Domain {*false*, *true*}

```
boolean b = true; // Variable with value true
```

Relational Operators

```
a < b     (smaller than)
a >= b    (greater than)
a == b    (equals)
a != b    (unequal)
```

number type $\times$ number type $\rightarrow$ `boolean`

Boolean Functions in Mathematics

- Boolean function

$$f : \{F, T\}^2 \rightarrow \{F, T\}$$

- F corresponds to "false".
- T corresponds to "true".

$\text{AND}(x, y)$ $\qquad x \wedge y$

- "logical and"

$$f : \{F, T\}^2 \rightarrow \{F, T\}$$

- F corresponds to "false".
- T corresponds to "true".

x	y	$\text{AND}(x, y)$
F	F	F
F	T	F
T	F	F
T	T	T

Logischer Operator &&

```
a && b        (logical and)
```

`boolean` $\times$ `boolean` $\rightarrow$ `boolean`

```
int n = -1;
int p = 3;
boolean b = (n < 0) && (0 < p); // b = true
```

OR(x, y) $\qquad x \vee y$

- "logical or"

$$f : \{F, T\}^2 \to \{F, T\}$$

- F corresponds to "false".
- T corresponds to "true".

x	y	OR(x, y)
F	F	F
F	T	T
T	F	T
T	T	T

Logical Operator ||

`a || b` (logical or)

`boolean` $\times$ `boolean` $\to$ `boolean`

```
int n = 1;
int p = 0;
boolean b = (n < 0) || (0 < p); // b = false
```


NOT(x) $\qquad \neg x$

- "logical not"

$$f : \{F, T\} \to \{F, T\}$$

- F corresponds to "false".
- T corresponds to "true".

x	NOT(x)
F	T
T	F

Logical Operator !

`!b` (logical not)

`boolean` $\to$ `boolean`

```
int n = 1;
boolean b = !(n < 0); // b = true
```

Precedences

```
!b && a
   ⇕
(!b) && a

a && b || c && d
       ⇕
(a && b) || (c && d)

a || b   &&   c || d
         ⇕
a || (b && c) || d
```


Precedences

The unary logical operator `!`

provides a stronger binding than

binary arithmetic operators. These

bind stronger than

relational operators,

and these bind stronger than

binary logical operators.

```
7 + x < y && y != 3 * z || ! b
7 + x < y && y != 3 * z || (!b)
```


DeMorgan Rules

- `!(a && b) == (!a || !b)`
- `!(a || b) == (!a && !b)`

! (rich *and* beautiful) == (poor *or* ugly)

Application: either ... or (XOR)

`(x || y)`	`&& !(x && y)`	x or y, and not both
`(x || y)`	`&& (!x || !y)`	x or y, and one of them not
`!(!x && !y)`	`&& !(x && y)`	not none and not both
`!(!x && !y || x && y)`		not: both or none

Shortcut Evaluation

- Logical operators `&&` and `||` evaluate the *left operand first*.
- If the result is then known, the right operand will *not be* evaluated.

```
x != 0 && z / x > y
```

$\Rightarrow$ No division by 0