

9. Sorting II

Mergesort, Quicksort

9.1 Mergesort

[Ottman/Widmayer, Kap. 2.4, Cormen et al, Kap. 2.3],

Mergesort

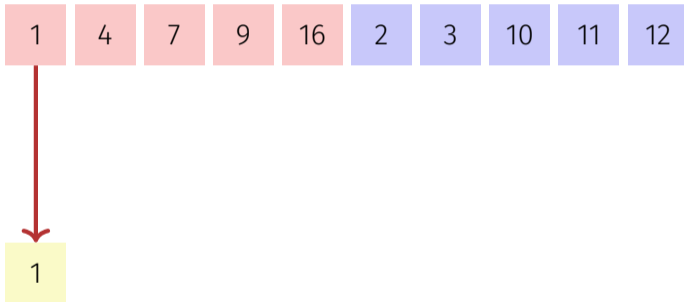
Divide and Conquer!

- Assumption: two halves of the array A are already sorted.
- Minimum of A can be evaluated with a single element comparison.
- Iteratively: merge the two presorted halves of A in $\mathcal{O}(n)$.

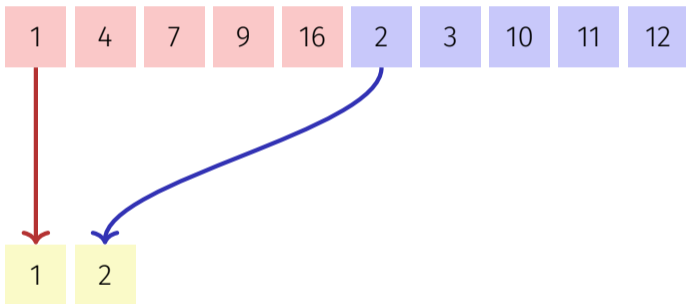
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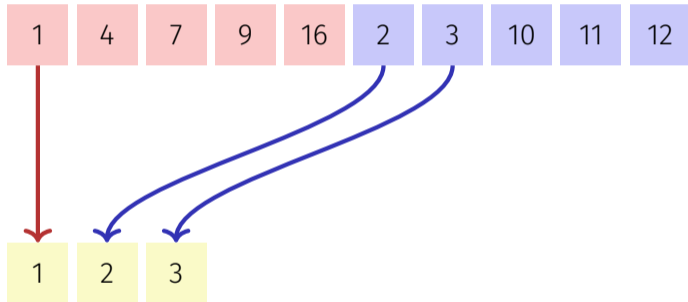
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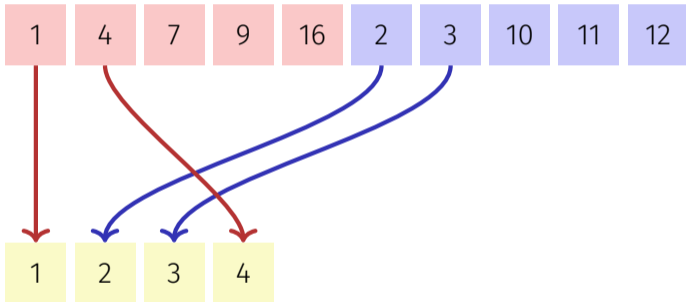
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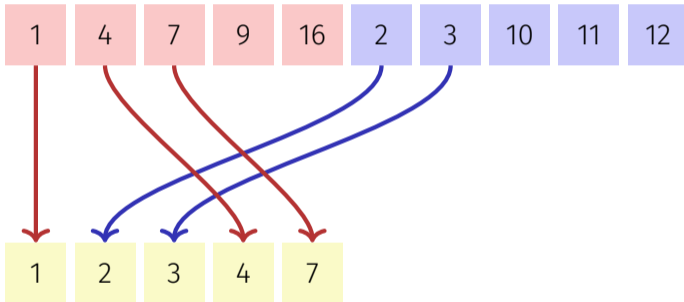
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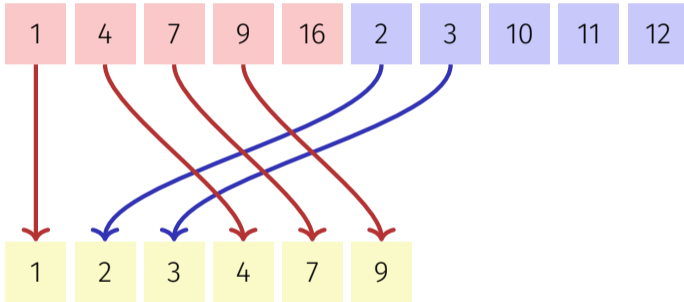
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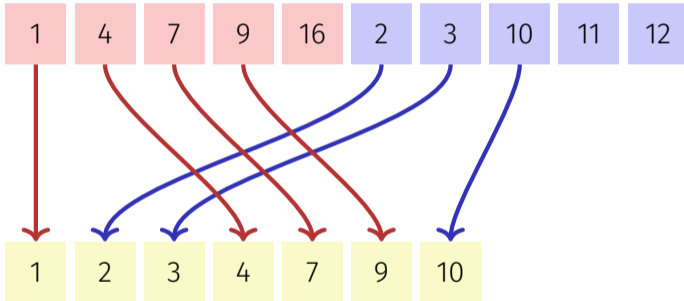
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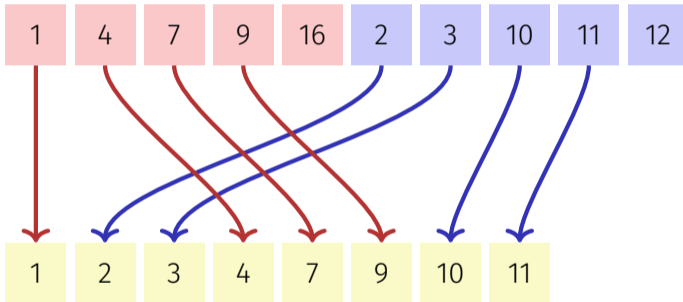
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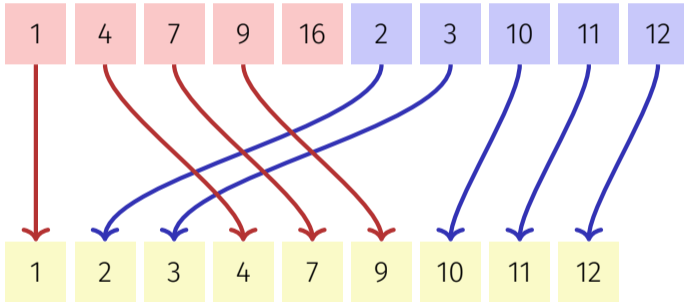
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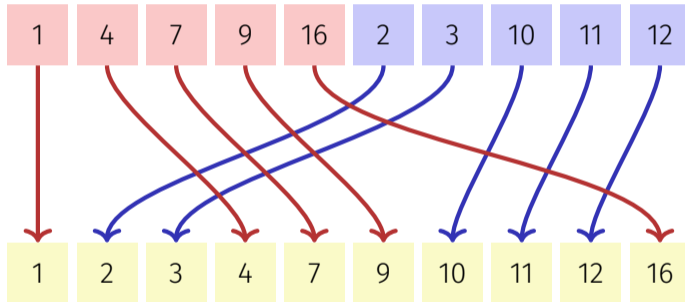
Merge



Merge



Merge



Algorithm Merge(A, l, m, r)

Input: Array A with length n , indexes $1 \leq l \leq m \leq r \leq n$.
 $A[l, \dots, m]$, $A[m + 1, \dots, r]$ sorted

Output: $A[l, \dots, r]$ sorted

```
1  $B \leftarrow$  new Array( $r - l + 1$ )
2  $i \leftarrow l$ ;  $j \leftarrow m + 1$ ;  $k \leftarrow 1$ 
3 while  $i \leq m$  and  $j \leq r$  do
4   if  $A[i] \leq A[j]$  then  $B[k] \leftarrow A[i]$ ;  $i \leftarrow i + 1$ 
5   else  $B[k] \leftarrow A[j]$ ;  $j \leftarrow j + 1$ 
6    $k \leftarrow k + 1$ ;
7 while  $i \leq m$  do  $B[k] \leftarrow A[i]$ ;  $i \leftarrow i + 1$ ;  $k \leftarrow k + 1$ 
8 while  $j \leq r$  do  $B[k] \leftarrow A[j]$ ;  $j \leftarrow j + 1$ ;  $k \leftarrow k + 1$ 
9 for  $k \leftarrow l$  to  $r$  do  $A[k] \leftarrow B[k - l + 1]$ 
```

Correctness

Hypothesis: after k iterations of the loop in line 3 $B[1, \dots, k]$ is sorted and $B[k] \leq A[i]$, if $i \leq m$ and $B[k] \leq A[j]$ if $j \leq r$.

Proof by induction:

Base case: the empty array $B[1, \dots, 0]$ is trivially sorted.

Induction step ($k \rightarrow k + 1$):

- wlog $A[i] \leq A[j]$, $i \leq m, j \leq r$.
- $B[1, \dots, k]$ is sorted by hypothesis and $B[k] \leq A[i]$.
- After $B[k + 1] \leftarrow A[i]$ $B[1, \dots, k + 1]$ is sorted.
- $B[k + 1] = A[i] \leq A[i + 1]$ (if $i + 1 \leq m$) and $B[k + 1] \leq A[j]$ if $j \leq r$.
- $k \leftarrow k + 1, i \leftarrow i + 1$: Statement holds again.

Analysis (Merge)

Lemma 12

If: array A with length n , indexes $1 \leq l < r \leq n$. $m = \lfloor (l+r)/2 \rfloor$ and $A[l, \dots, m]$, $A[m+1, \dots, r]$ sorted.

Then: in the call of $\text{Merge}(A, l, m, r)$ a number of $\Theta(r-l)$ key movements and comparisons are executed.

Proof: straightforward (Inspect the algorithm and count the operations.)

Mergesort

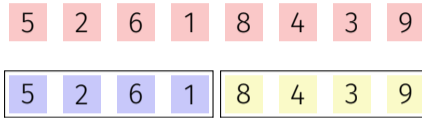
5 2 6 1 8 4 3 9

Mergesort

5 2 6 1 8 4 3 9

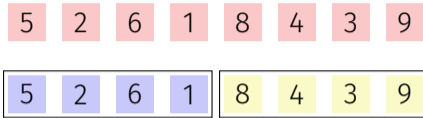
Split

Mergesort



Split

Mergesort



Split

Split

Mergesort

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Split

Split

Mergesort

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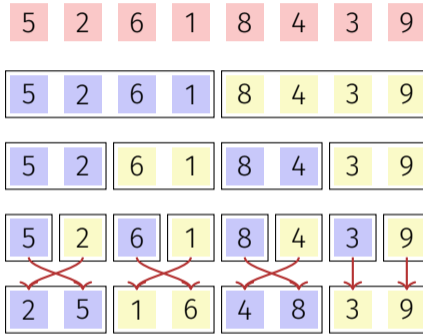
Split

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Merge

Mergesort



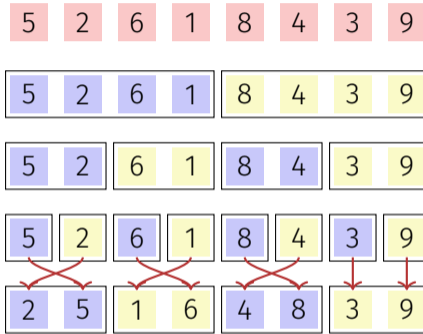
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Mergesort



Split

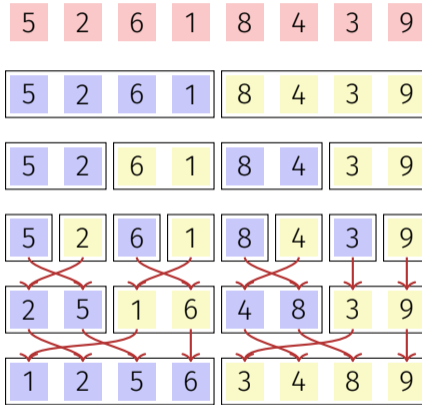
Split

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Split

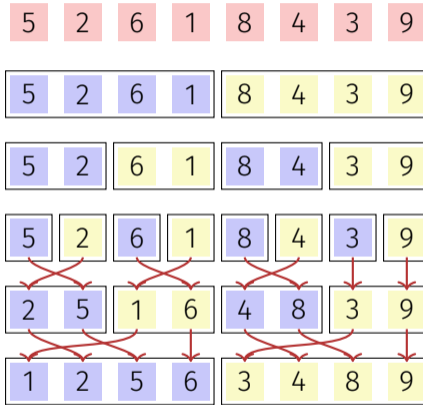
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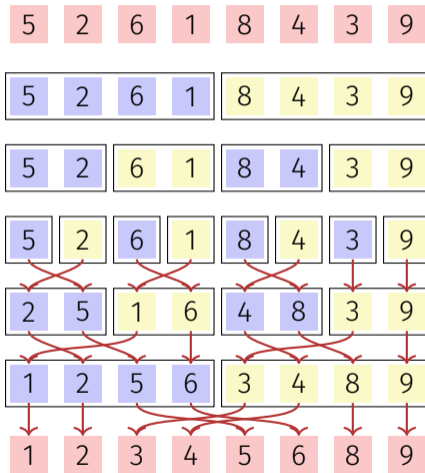
Split

Merge

Merge

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Mergesort



Split

Split

Split

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Merge

Algorithm (recursive 2-way) Mergesort(A, l, r)

Input: Array A with length n . $1 \leq l \leq r \leq n$

Output: $A[l, \dots, r]$ sorted.

if $l < r$ **then**

```
 $m \leftarrow \lfloor (l + r) / 2 \rfloor$  // middle position  
Mergesort( $A, l, m$ ) // sort lower half  
Mergesort( $A, m + 1, r$ ) // sort higher half  
Merge( $A, l, m, r$ ) // Merge subsequences
```

Recursion equation for the number of comparisons and key movements:

$$T(n) = T\left(\left\lceil \frac{n}{2} \right\rceil\right) + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \Theta(n)$$

Recursion equation for the number of comparisons and key movements:

$$T(n) = T\left(\left\lceil \frac{n}{2} \right\rceil\right) + T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + \Theta(n) \in \Theta(n \log n)$$

Algorithm StraightMergesort(A)

Avoid recursion: merge sequences of length 1, 2, 4, ... directly

Input: Array A with length n

Output: Array A sorted

$length \leftarrow 1$

while $length < n$ **do** // Iterate over lengths n

$r \leftarrow 0$

while $r + length < n$ **do** // Iterate over subsequences

$l \leftarrow r + 1$

$m \leftarrow l + length - 1$

$r \leftarrow \min(m + length, n)$

 Merge(A, l, m, r)

$length \leftarrow length \cdot 2$

Analysis

Like the recursive variant, the straight 2-way mergesort always executes a number of $\Theta(n \log n)$ key comparisons and key movements.

Natural 2-way mergesort

Observation: the variants above do not make use of any presorting and always execute $\Theta(n \log n)$ memory movements.

How can partially presorted arrays be sorted better?

Natural 2-way mergesort

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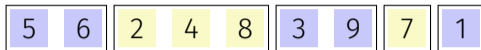
How can partially presorted arrays be sorted better?

❗ Recursive merging of previously sorted parts (*runs*) of A .

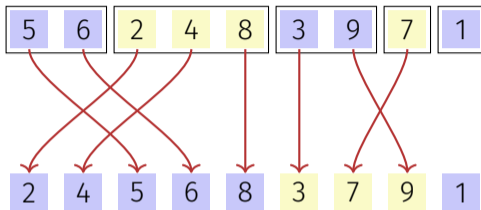
Natural 2-way mergesort

5 6 2 4 8 3 9 7 1

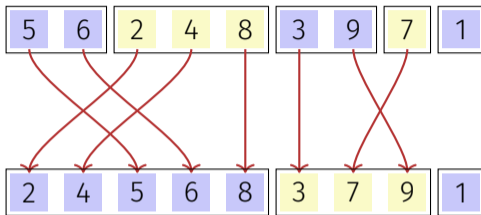
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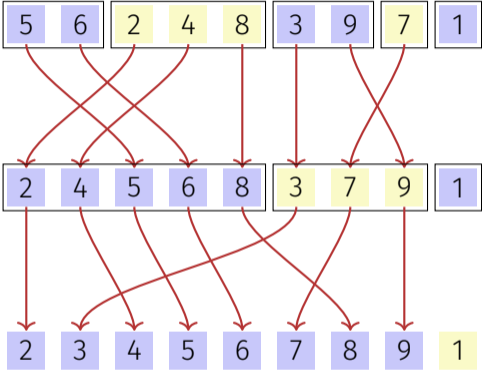
Natural 2-way mergesort



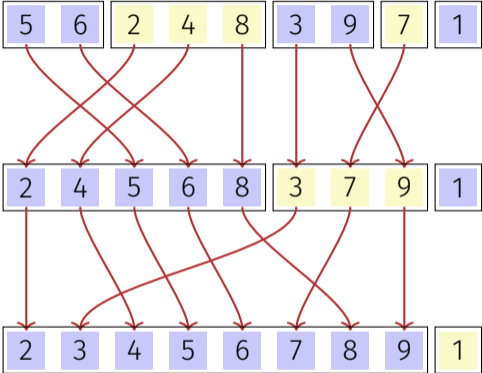
Natural 2-way mergesort



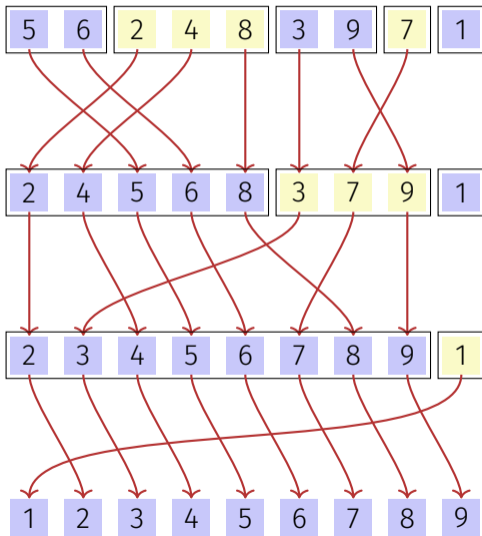
Natural 2-way mergesort



Natural 2-way mergesort



Natural 2-way mergesort



Algorithm NaturalMergesort(A)

Input: Array A with length $n > 0$

Output: Array A sorted

repeat

$r \leftarrow 0$

while $r < n$ **do**

$l \leftarrow r + 1$

$m \leftarrow l$; **while** $m < n$ **and** $A[m + 1] \geq A[m]$ **do** $m \leftarrow m + 1$

if $m < n$ **then**

$r \leftarrow m + 1$; **while** $r < n$ **and** $A[r + 1] \geq A[r]$ **do** $r \leftarrow r + 1$

 Merge(A, l, m, r);

else

$r \leftarrow n$

until $l = 1$

Is it also asymptotically better than StraightMergesort on average?

Is it also asymptotically better than StraightMergesort on average?

⚠ No. Given the assumption of pairwise distinct keys, on average there are $n/2$ positions i with $k_i > k_{i+1}$, i.e. $n/2$ runs. Only one iteration is saved on average.

Natural mergesort executes in the worst case and on average a number of $\Theta(n \log n)$ comparisons and memory movements.

9.2 Quicksort

[Ottman/Widmayer, Kap. 2.2, Cormen et al, Kap. 7]

What is the disadvantage of Mergesort?

Quicksort

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Requires additional $\Theta(n)$ storage for merging.

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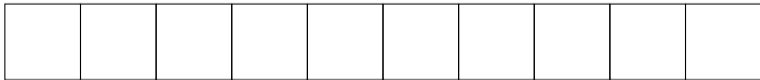
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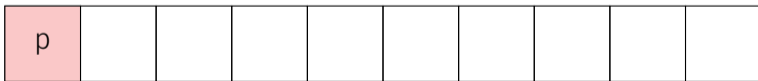
Pivot and Partition!

Use a pivot



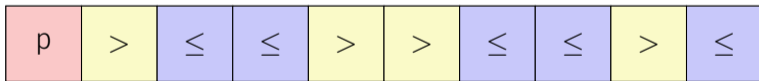
Use a pivot

1. Choose a (an arbitrary) **pivot** p



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2. Partition A in two parts, one part L with the elements with $A[i] \leq p$ and another part R with $A[i] > p$



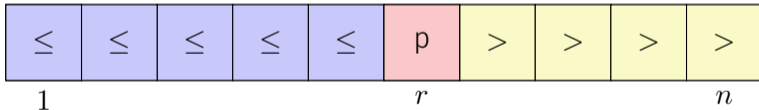
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3. Quicksort: Recursion on parts L and R



Use a pivot

1. Choose a (an arbitrary) **pivot** p
2. Partition A in two parts, one part L with the elements with $A[i] \leq p$ and another part R with $A[i] > p$
3. Quicksort: Recursion on parts L and R



Algorithm Partition(A, l, r, p)

Input: Array A , that contains the pivot p in $A[l, \dots, r]$ at least once.

Output: Array A partitioned in $[l, \dots, r]$ around p . Returns position of p .

while $l \leq r$ **do**

while $A[l] < p$ **do**

$l \leftarrow l + 1$

while $A[r] > p$ **do**

$r \leftarrow r - 1$

 swap($A[l], A[r]$)

if $A[l] = A[r]$ **then**

$l \leftarrow l + 1$

return $l-1$

Algorithm Quicksort(A, l, r)

Input: Array A with length n . $1 \leq l \leq r \leq n$.

Output: Array A , sorted in $A[l, \dots, r]$.

if $l < r$ **then**

 Choose pivot $p \in A[l, \dots, r]$

$k \leftarrow \text{Partition}(A, l, r, p)$

 Quicksort($A, l, k - 1$)

 Quicksort($A, k + 1, r$)

Quicksort (arbitrary pivot)

2 4 5 6 8 3 7 9 1

Quicksort (arbitrary pivot)

2 4 5 6 8 3 7 9 1

Quicksort (arbitrary pivot)

2 4 5 6 8 3 7 9 1

2 1 3 6 8 5 7 9 4

Quicksort (arbitrary pivot)

2 4 5 6 8 3 7 9 1

2 1 3 6 8 5 7 9 4

Quicksort (arbitrary pivot)

2 4 5 6 8 3 7 9 1

2 1 3 6 8 5 7 9 4

1 2 3 4 5 8 7 9 6

Quicksort (arbitrary pivot)

2 4 5 6 8 3 7 9 1

2 1 3 6 8 5 7 9 4

1 2 3 4 5 8 7 9 6

Quicksort (arbitrary pivot)

2 4 5 6 8 3 7 9 1

2 1 3 6 8 5 7 9 4

1 2 3 4 5 8 7 9 6

1 2 3 4 5 6 7 9 8

Quicksort (arbitrary pivot)

2 4 5 6 8 3 7 9 1

2 1 3 6 8 5 7 9 4

1 2 3 4 5 8 7 9 6

1 2 3 4 5 6 7 9 8

Quicksort (arbitrary pivot)

2 4 5 6 8 3 7 9 1

2 1 3 6 8 5 7 9 4

1 2 3 4 5 8 7 9 6

1 2 3 4 5 6 7 9 8

1 2 3 4 5 6 7 8 9

Quicksort (arbitrary pivot)

2 4 5 6 8 3 7 9 1

2 1 3 6 8 5 7 9 4

1 2 3 4 5 8 7 9 6

1 2 3 4 5 6 7 9 8

1 2 3 4 5 6 7 8 9

1 2 3 4 5 6 7 8 9

Analysis: number comparisons

Worst case.

Analysis: number comparisons

Worst case. Pivot = min or max; number comparisons:

$$T(n) = T(n - 1) + c \cdot n, T(1) = d \quad \Rightarrow \quad T(n) \in \Theta(n^2)$$

Analysis: number swaps

Result of a call to partition (pivot 3):

2 1 3 6 8 5 7 9 4

⓪ How many swaps have taken place?

Analysis: number swaps

Result of a call to partition (pivot 3):

2 1 3 6 8 5 7 9 4

- ① How many swaps have taken place?
- ② 2. The maximum number of swaps is given by the number of keys in the smaller part.

Analysis: number swaps

Thought experiment

Analysis: number swaps

Thought experiment

- Each key from the smaller part pays a coin when it is being swapped.

Analysis: number swaps

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- Each key from the smaller part pays a coin when it is being swapped.
- After a key has paid a coin the domain containing the key decreases to half its previous size.

Analysis: number swaps

Thought experiment

- Each key from the smaller part pays a coin when it is being swapped.
- After a key has paid a coin the domain containing the key decreases to half its previous size.
- Every key needs to pay at most $\log n$ coins. But there are only n keys.

Analysis: number swaps

Thought experiment

- Each key from the smaller part pays a coin when it is being swapped.
- After a key has paid a coin the domain containing the key decreases to half its previous size.
- Every key needs to pay at most $\log n$ coins. But there are only n keys.

Consequence: there are $\mathcal{O}(n \log n)$ key swaps in the worst case.

Randomized Quicksort

Despite the worst case running time of $\Theta(n^2)$, quicksort is used practically very often.

Reason: quadratic running time unlikely provided that the choice of the pivot and the pre-sorting are not very disadvantageous.

Avoidance: randomly choose pivot. Draw uniformly from $[l, r]$.

Analysis (randomized quicksort)

Expected number of compared keys with input length n :

$$T(n) = (n - 1) + \frac{1}{n} \sum_{k=1}^n (T(k - 1) + T(n - k)), \quad T(0) = T(1) = 0$$

Claim $T(n) \leq 4n \log n$.

Proof by induction:

Base case straightforward for $n = 0$ (with $0 \log 0 := 0$) and for $n = 1$.

Hypothesis: $T(n) \leq 4n \log n$ for some n .

Induction step: $(n - 1 \rightarrow n)$

Analysis (randomized quicksort)

$$\begin{aligned}T(n) &= n - 1 + \frac{2}{n} \sum_{k=0}^{n-1} T(k) \stackrel{H}{\leq} n - 1 + \frac{2}{n} \sum_{k=0}^{n-1} 4k \log k \\&= n - 1 + \sum_{k=1}^{n/2} 4k \underbrace{\log k}_{\leq \log n-1} + \sum_{k=n/2+1}^{n-1} 4k \underbrace{\log k}_{\leq \log n} \\&\leq n - 1 + \frac{8}{n} \left((\log n - 1) \sum_{k=1}^{n/2} k + \log n \sum_{k=n/2+1}^{n-1} k \right) \\&= n - 1 + \frac{8}{n} \left((\log n) \cdot \frac{n(n-1)}{2} - \frac{n}{4} \left(\frac{n}{2} + 1 \right) \right) \\&= 4n \log n - 4 \log n - 3 \leq 4n \log n\end{aligned}$$

Analysis (randomized quicksort)

Theorem 13

On average randomized quicksort requires $\mathcal{O}(n \log n)$ comparisons.

Practical Considerations

Worst case recursion depth $n - 1$ ⁹. Then also a memory consumption of $\mathcal{O}(n)$.

Can be avoided: recursion only on the smaller part. Then guaranteed $\mathcal{O}(\log n)$ worst case recursion depth and memory consumption.

⁹stack overflow possible!

Quicksort with logarithmic memory consumption

Input: Array A with length n . $1 \leq l \leq r \leq n$.

Output: Array A , sorted between l and r .

while $l < r$ **do**

 Choose pivot $p \in A[l, \dots, r]$

$k \leftarrow \text{Partition}(A, l, r, p)$

if $k - l < r - k$ **then**

 Quicksort($A[l, \dots, k - 1]$)

$l \leftarrow k + 1$

else

 Quicksort($A[k + 1, \dots, r]$)

$r \leftarrow k - 1$

The call of Quicksort($A[l, \dots, r]$) in the original algorithm has moved to iteration (tail recursion!): the if-statement became a while-statement.

Practical Considerations.

- Practically the pivot is often the median of three elements. For example: $\text{Median3}(A[l], A[r], A[\lfloor l + r/2 \rfloor])$.
- There is a variant of quicksort that requires only constant storage. Idea: store the old pivot at the position of the new pivot.
- Complex divide-and-conquer algorithms often use a trivial ($\Theta(n^2)$) algorithm as base case to deal with small problem sizes.

9.3 Appendix

Derivation of some mathematical formulas

$$\log n! \in \Theta(n \log n)$$

$$\begin{aligned} \log n! &= \sum_{i=1}^n \log i \leq \sum_{i=1}^n \log n = n \log n \\ \sum_{i=1}^n \log i &= \sum_{i=1}^{\lfloor n/2 \rfloor} \log i + \sum_{i=\lfloor n/2 \rfloor + 1}^n \log i \\ &\geq \sum_{i=2}^{\lfloor n/2 \rfloor} \log 2 + \sum_{i=\lfloor n/2 \rfloor + 1}^n \log \frac{n}{2} \\ &= \underbrace{(\lfloor n/2 \rfloor - 2 + 1)}_{>n/2-1} + \underbrace{(n - \lfloor n/2 \rfloor)}_{\geq n/2} (\log n - 1) \\ &> \frac{n}{2} \log n - 2. \end{aligned}$$

$$[n! \in o(n^n)]$$

$$\begin{aligned} n \log n &\geq \sum_{i=1}^{\lfloor n/2 \rfloor} \log 2i + \sum_{i=\lfloor n/2 \rfloor + 1}^n \log i \\ &= \sum_{i=1}^n \log i + \left\lfloor \frac{n}{2} \right\rfloor \log 2 \\ &> \sum_{i=1}^n \log i + n/2 - 1 = \log n! + n/2 - 1 \end{aligned}$$

$$\begin{aligned} n^n &= 2^{n \log_2 n} \geq 2^{\log_2 n!} \cdot 2^{n/2} \cdot 2^{-1} = n! \cdot 2^{n/2-1} \\ \Rightarrow \frac{n!}{n^n} &\leq 2^{-n/2+1} \xrightarrow{n \rightarrow \infty} 0 \Rightarrow n! \in o(n^n) = \mathcal{O}(n^n) \setminus \Omega(n^n) \end{aligned}$$

[Even $n! \in o((n/c)^n) \forall 0 < c < e$]

Konvergenz oder Divergenz von $f_n = \frac{n!}{(n/c)^n}$.

Ratio Test

$$\frac{f_{n+1}}{f_n} = \frac{(n+1)!}{\left(\frac{n+1}{c}\right)^{n+1}} \cdot \frac{\left(\frac{n}{c}\right)^n}{n!} = c \cdot \left(\frac{n}{n+1}\right)^n \rightarrow c \cdot \frac{1}{e} \lesseqgtr 1 \text{ if } c \lesseqgtr e$$

because $\left(1 + \frac{1}{n}\right)^n \rightarrow e$. Even the series $\sum_{i=1}^n f_n$ converges / diverges for $c \lesseqgtr e$.

f_n diverges for $c = e$, because (Stirling): $n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$.

[Ratio Test]

Ratio test for a sequence $(f_n)_{n \in \mathbb{N}}$: If $\frac{f_{n+1}}{f_n} \xrightarrow{n \rightarrow \infty} \lambda$, then the sequence f_n and the series $\sum_{i=1}^n f_i$

- converge, if $\lambda < 1$ and
- diverge, if $\lambda > 1$.

[Ratio Test Derivation]

Ratio test is implied by Geometric Series

$$S_n(r) := \sum_{i=0}^n r^i = \frac{1 - r^{n+1}}{1 - r}.$$

converges for $n \rightarrow \infty$ if and only if $-1 < r < 1$.

Let $0 \leq \lambda < 1$:

$$\begin{aligned} & \forall \varepsilon > 0 \exists n_0 : f_{n+1}/f_n < \lambda + \varepsilon \forall n \geq n_0 \\ \Rightarrow & \exists \varepsilon > 0, \exists n_0 : f_{n+1}/f_n \leq \mu < 1 \forall n \geq n_0 \end{aligned}$$

Thus

$$\sum_{n=n_0}^{\infty} f_n \leq f_{n_0} \cdot \sum_{n=n_0}^{\infty} \mu^{n-n_0} \text{ konvergiert.}$$

(Analogously for divergence)

L'Hospital's rule

Theorem 14

Let $f, g : \mathbb{R}^+ \rightarrow \mathbb{R}$ differentiable functions with $g'(x) \neq 0 \forall x > 0$.

If

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} g(x) = 0,$$

or

$$\lim_{x \rightarrow \infty} f(x) = \pm\infty \text{ and } \lim_{x \rightarrow \infty} g(x) = \pm\infty,$$

then

$$\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)}$$

if the limit of $f'(x)/g'(x)$ exists

L'Hospital's rule

Example

Es gilt $\log^k(n) \in o(n)$, because with $f(x) = \log^k(x)$, $g(x) = x$, we can apply L'Hospital's rule and get

$$\lim_{x \rightarrow \infty} \frac{\log^k(x)}{x} = \lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \infty} \frac{f'(x)}{g'(x)} = \lim_{x \rightarrow \infty} k \frac{\log^{k-1}(x)}{x}$$

After k iterations we get

$$\lim_{x \rightarrow \infty} \frac{\log^k(x)}{x} = \lim_{x \rightarrow \infty} k! \frac{1}{x} = 0.$$