29. Flow in Networks

Flow Network, Flow, Maximum Flow
Residual Capacity, Remainder Network, Augmenting path

Ford-Fulkerson Algorithm
Edmonds-Karp Algorithm

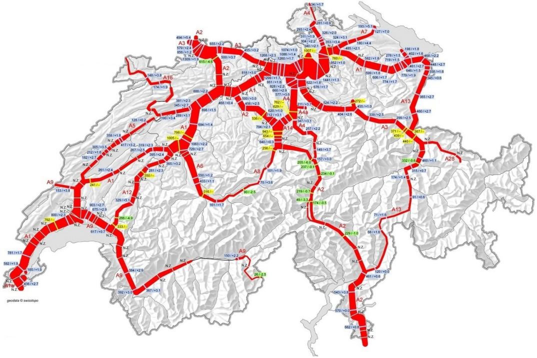
Cuts, Max-Flow Min-Cut Theorem

[Ottman/Widmayer, Kap. 9.7, 9.8.1], [Cormen et al, Kap. 26.1-26.3]

Slides redesigned by Manuela Fischer – thank you!

Maximum Traffic Flow

Given: Road Network with capacities

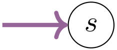


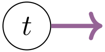
Wanted: Maximum traffic flow between Zurich and Geneva

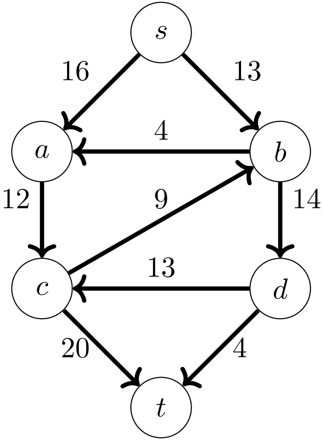
Flow Network

directed, weighted graph $G = (V, E, c)$ with capacities $c\colon\ E \to \mathbb{R}^{>0}$

- without antiparallel edges:
  $(u, v) \in E \ \Rightarrow\ (v, u) \notin E$
  

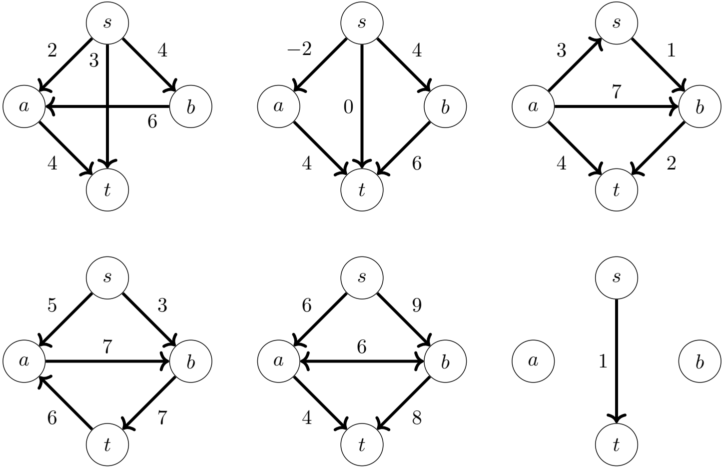
- source $s \in V$ without ingoing edges:
  $\forall v \in V\colon (v, s) \notin E$
  

- sink $t \in V$ without outgoing edges:
  $\forall v \in V\colon (t, v) \notin E$
  



Quiz Flow Network

Which of the following graphs are flow networks?

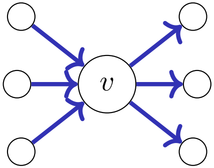


Flow in Flow Network

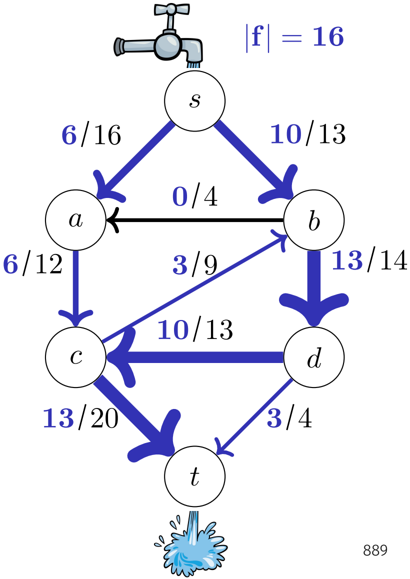
Flow is function $f\colon E \to \mathbb{R}^{\geq 0}$ such that

- **Bounded Capacity**: $\forall e \in E\colon f(e) \leq c(e)$
- **Conservation of flow**: $\forall v \in V \setminus \{s,t\}\colon$

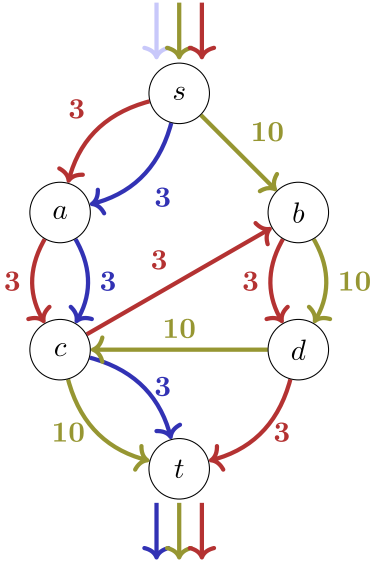
$$\underbrace{\sum_{e \in E^-(v)} f(e)}_{=:f^-(v)} = \underbrace{\sum_{e \in E^+(v)} f(e)}_{=:f^+(v)}$$

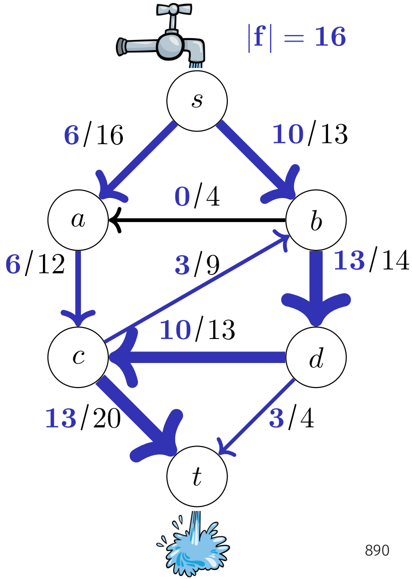


Size of flow: $|f| := f^+(s) = f^-(t)$



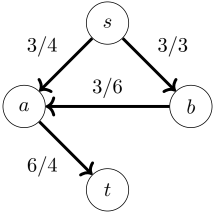
Intuition: Flow as set of paths $s \rightsquigarrow t$

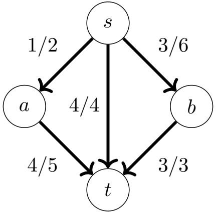


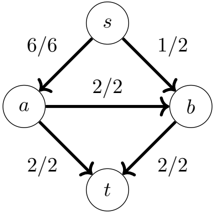


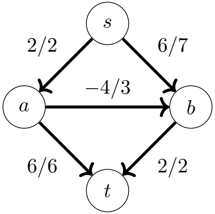
Quiz Flow

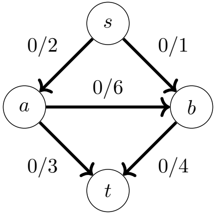
Which of the following are flows?

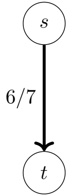






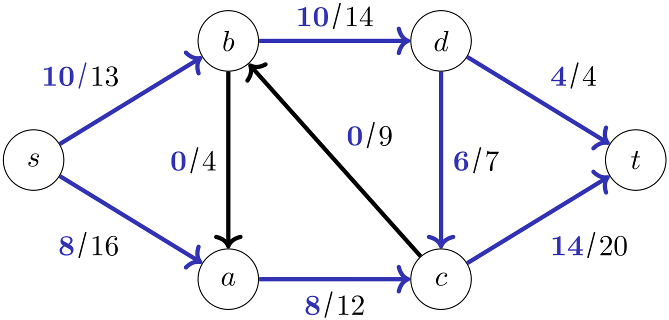






Maximal Flow

Given: Flow network: $G = (V, E, c)$, directed, positively weighted, without antiparallel edges, with source s and sink t

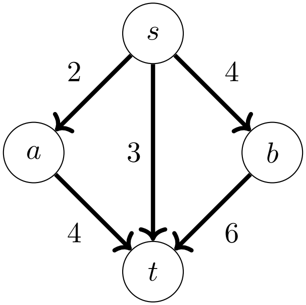


$$\mathbf{18} = |\mathbf{f}| \leq |\mathbf{f}_{\max}| = \mathbf{23}$$

Wanted: Size $|f_{\max}|$ of the maximum flow in G

Quiz Maximum Flow

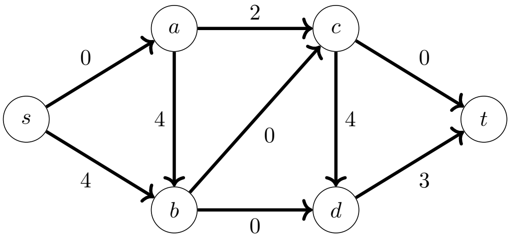
What is the maximum flow in the following flow network?



Greedy Algorithm?

Residual capacity of an edge e**:** $r(e) := c(e) - f(e)$
Residual capacity of a path P**:** $\min_{e \in P} r(e)$

Greedy: Starting with $f(e) = 0$ for all $e \in E$, as long as there exists a path $s \rightsquigarrow t$ with remaining capacity $d > 0$, increase flow along this path by d.



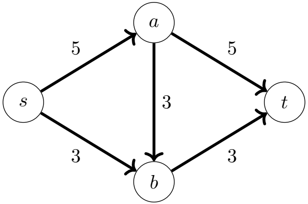
$G_f^+ := (V, E, r := c - f)$

$|f| = 8$

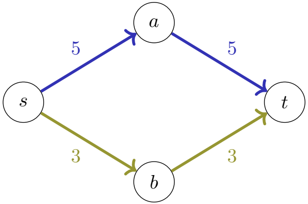
$s \to a \to b \to d \to t$: 3
$s \to a \to c \to t$: 2
$s \to b \to c \to t$: 3

but $|f_{\max}| = 10$

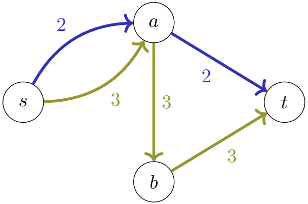
Problem with Greedy



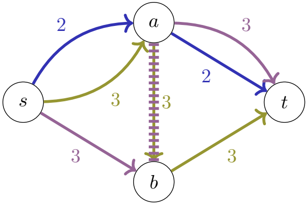
$G = (V, E, c)$



$|f_{\max}| = 8$



Greedy: $|f| = 5$



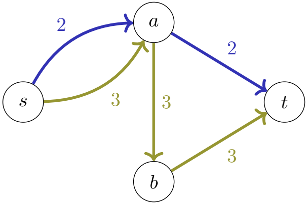
Redirection

29.1 Flow Algorithms

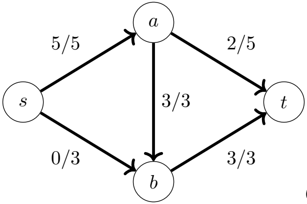
Ford-Fulkerson Algorithm

Edmonds-Karp Algorithm

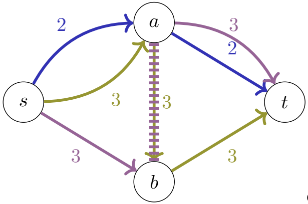
Redirection using flow decrement



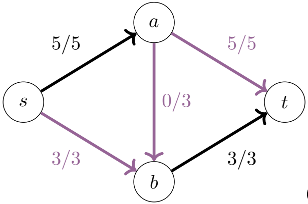
before



$$G = (V, E, f/c)$$



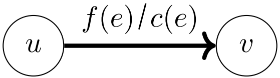
after



$$G = (V, E, f'/c)$$

$\Rightarrow$ Umleitung entspricht Verringerung des Flusses durch Kante

Idea: Flow increments and decrements

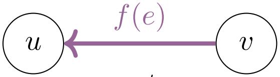


- **Increment:**
  flow through e can be increased by at most
  $r(e) := c(e) - f(e)$

$$G_f^+ := (V, E, r := c - f)$$

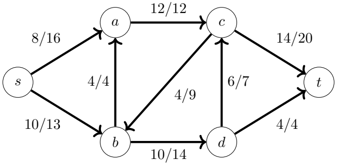
- **Decrement:**
  flow through e can be decreased by at most
  $f(e)$

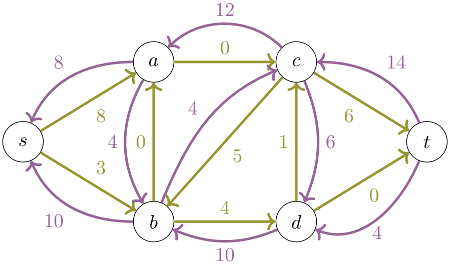
  $\Rightarrow$ flow through $\overleftarrow{e}$ can be increased by at most $f(e)$



$$G_f^- := (V, \overleftarrow{E}, f)$$

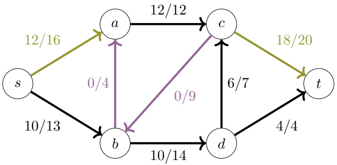
Residual Network

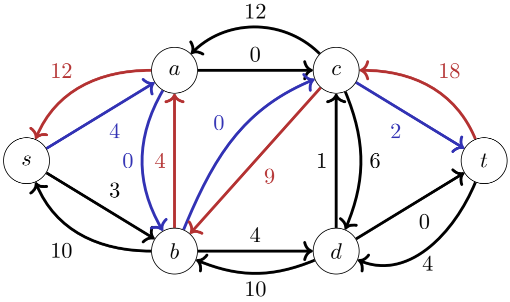




Residual network: $G_f := \mathbf{G}_\mathbf{f}^+ \cup \mathbf{G}_\mathbf{f}^- = (V, E_f, c_f)$

Ford-Fulkerson: Flow augmentation





- **Augmenting Path:** Find a path $\mathbf{P}: \mathbf{s} \rightarrow \mathbf{t}$ with residual capacity $d > 0$ in G_f
- augment flow along this path for all $e \in P$ by d:
  - decrease residual capacity $\mathbf{c_f}(\mathbf{e})$ in G_f by d; increase $\mathbf{c_f}(\overleftarrow{e})$ by d
  - increase flow through $\mathbf{e} \in \mathbf{E}$ by d; decrease through $\overleftarrow{e} \in \mathbf{E}$

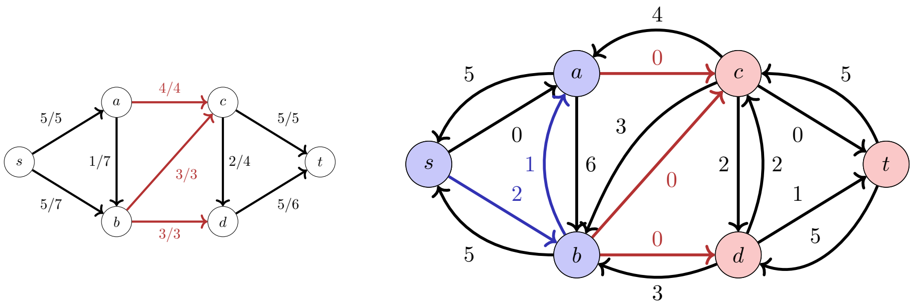
Algorithm Ford-Fulkerson(G, s, t)

Input: Flow network $G = (V, E, c)$, source s, sink t
Output: Maximal flow f

for $e \in E$ **do**
    $f(e) \leftarrow 0$

while exists positive path $P\colon s \rightsquigarrow t$ in residual network $G_f = (V, E_f, c_f)$ **do**
    $d \leftarrow \min_{e \in P} c_f(e)$
    foreach $e \in P$ **do**
        if $e \in E$ **then**
            $f(e) \leftarrow f(e) + d$
        else
            $f(\overleftarrow{e}) \leftarrow f(\overleftarrow{e}) - d$

Example Ford-Fulkerson



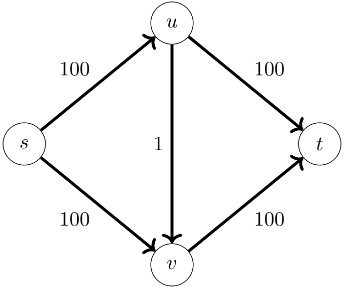
nodes reachable from s

nodes not reachable from s

all outgoing edges have residual capacity 0 in G_f

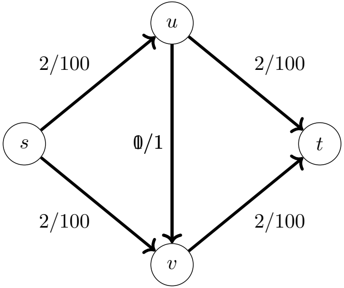
$\Rightarrow$ flow fully exhausts capacity on these edges!

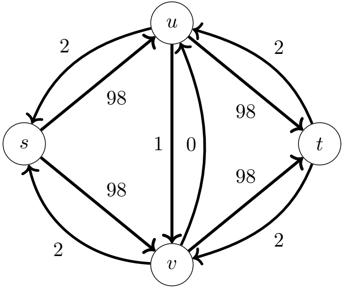
Quiz Ford-Fulkerson



How many iterations does Ford-Fulkerson need in the worst case?

Solution





After i iterations: $|f| = i$

$\Rightarrow$ in total $|f_{\max}| = 200$ iterations

Running Time Analysis of Ford-Fulkerson

Running time of each iteration: search of an augmenting path $s \rightsquigarrow t$
$\Rightarrow$ BFS or DFS: $\mathcal{O}(|V| + |E|) = \mathcal{O}(|E|)$

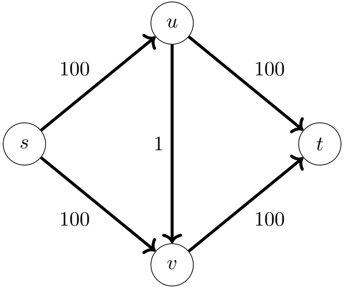
($|V| \leq |E|$, because all non-reachable nodes can be ignored.)

Number of iterations:
In every step, the size of the flow increases by $d > 0$.
integer capacities $\Rightarrow$ increment by $\geq 1 \Rightarrow$ at most $|f_{\max}|$ iterations

$\Rightarrow \mathcal{O}(|f_{\max}| \cdot |E|)$ for flow networks $G = (V, E, c)$ with $c\colon E \to \mathbb{N}^{\geq 1}$

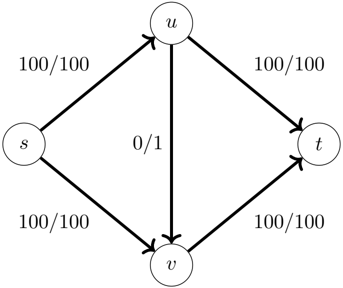
Edmonds-Karp Algorithm: (Variant of Ford-Fulkerson)
shortest augmenting path (number of edges) $\Rightarrow \mathcal{O}(|V| \cdot |E|^2)$ (without explanation)

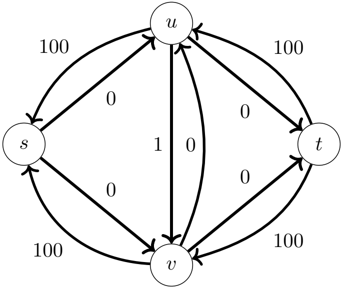
Quiz Edmonds-Karp



How many iterations does Edmonds-Karp need in the worst case?

Solution





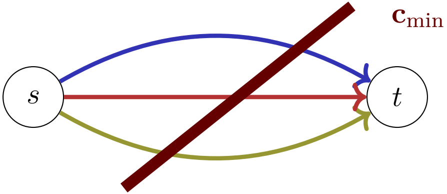
Termination after 2 iterations!

29.2 Max-Flow Min-Cut

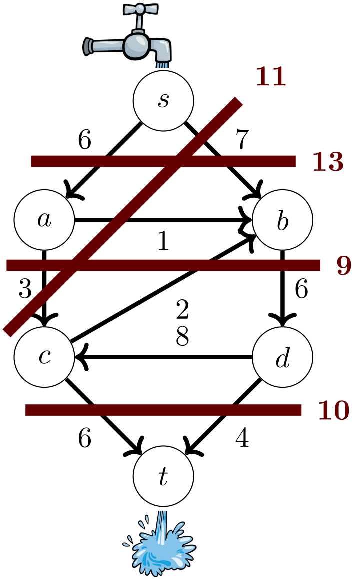
Flows and Cuts: Bottleneck Intuition

Upper bounds on size of flow:

- what can flow out of s: $c^+(s)$
- what can flow into t: $c^-(t)$
- what can flow through arbitrary cut
- what can flow through bottleneck: $c_{\min}$



$\Rightarrow$ flow $|f| \leq$ bottleneck
$\Rightarrow$ maximum flow $\leq$ bottleneck



Cut

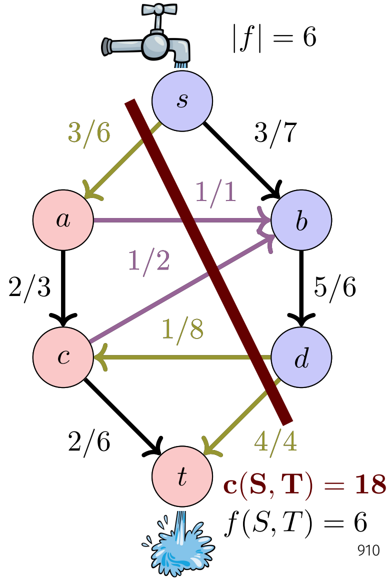
(s,t)**-Cut** of graph $G = (V, E, c)$: Partition $(\mathbf{S}, \mathbf{T})$ of V such that $s \in S, t \in T$

Size of cut:
$c(S,T) := \sum_{\mathbf{e}:\, \mathbf{S} \to \mathbf{T}} \mathbf{c}(\mathbf{e})$

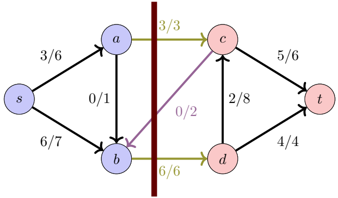
Flow through cut of flow network:
$f(S,T) := \sum_{\mathbf{e}:\, \mathbf{S} \to \mathbf{T}} \mathbf{f}(\mathbf{e}) - \sum_{\mathbf{e}:\, \mathbf{T} \to \mathbf{S}} \mathbf{f}(\mathbf{e})$

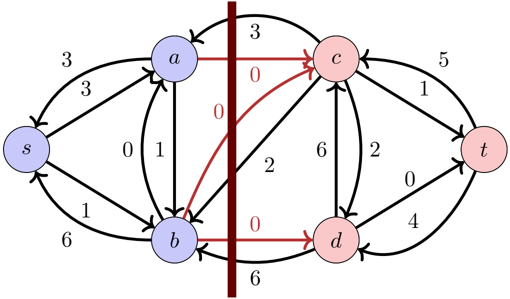
Observation:
$\forall f, S, T:\ |f| = f(S,T) \le c(S,T)$

$\Rightarrow |f_{\max}| \le c_{\min}$



Maximum Flow and Minimum Cut





after termination of Ford-Fulkerson/Edmonds-Karp:

- reachable from s, $\mathbf{T} \subseteq \mathbf{V}$ nodes not reachable from $s \Rightarrow$ **Cut** $(\mathbf{S}, \mathbf{T})$
- all outgoing edges e have remaining capacity 0 in G_f
- $f(S,T) = \sum_{\mathbf{e}:\,\mathbf{S}\to\mathbf{T}} \mathbf{f}(\mathbf{e}) - \sum_{\mathbf{e}:\,\mathbf{T}\to\mathbf{S}} \mathbf{f}(\mathbf{e}) = \sum_{\mathbf{e}:\,\mathbf{S}\to\mathbf{T}} \mathbf{c}(\mathbf{e}) = \mathbf{c}(\mathbf{S}, \mathbf{T})$
  $\Rightarrow |f_{\max}| = c_{\min}$

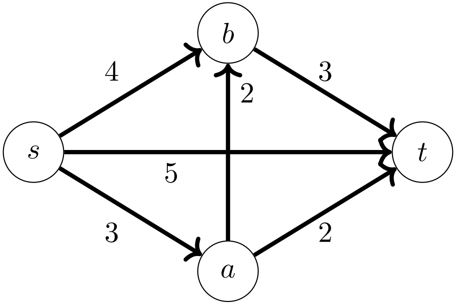
Max-Flow Min-Cut Theorem

Max-Flow Min-Cut Theorem

For a flow f in a flow network $G = (V, E, c)$ with source s and sink t, the following statements are equivalent:

1. f is a maximum flow in G
2. The residual network G_f does not provide any augmenting paths
3. $|f| = c(S, T)$ for a cut (S, T) of G.

Quiz



What is the minimum cut?
What is the maximum flow?

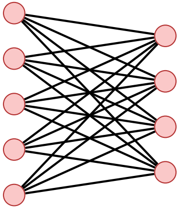
Application Examples

- Maximum Rate:
  - water in sewage system
  - cars in traffic
  - current in electrical networks
  - components on conveyors
  - information flow in communication networks
- Scheduling
- Bipartite Matching
- Image Segmentation

29.4 Maximales Bipartites Matching

Notation

A graph where V can be partitioned into disjoint sets U and W such that each $e \in E$ provides a node in U and a node in W is called **bipartite**.

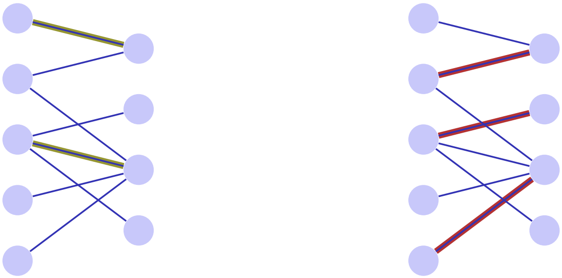


Application: maximal bipartite matching

Given: bipartite undirected graph $G = (V, E)$.

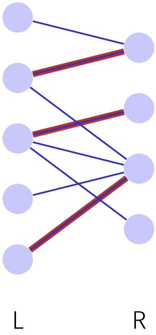
Matching M: $M \subseteq E$ such that $|\{m \in M : v \in m\}| \leq 1$ for all $v \in V$.

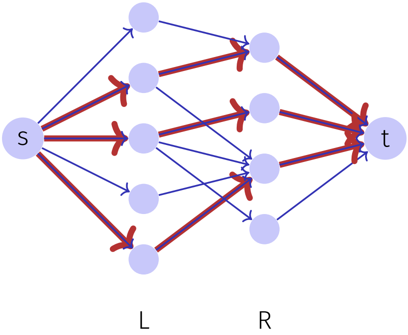
Maximal Matching M: Matching M, such that $|M| \geq |M'|$ for each matching M'.



Corresponding flow network

Construct a flow network that corresponds to the partition L, R of a bipartite graph with source s and sink t, with directed edges from s to L, from L to R and from R to t. Each edge has capacity 1.





Summary

- Definitions: flow networks, flow, cut
- Concepts: Redirection, remainder network, augmenting path
- Algorithms
  - Greedy: incorrect!
  - Ford-Fulkerson: $\mathcal{O}(|f_{\max}| \cdot |E|)$
    Greedy augmenting paths in remainder network
  - Edmonds-Karp: $\mathcal{O}(|V| \cdot |E|^2)$
    Ford-Fulkerson with shortest augmenting paths (number of edges)
- Max Flow = Min Cut

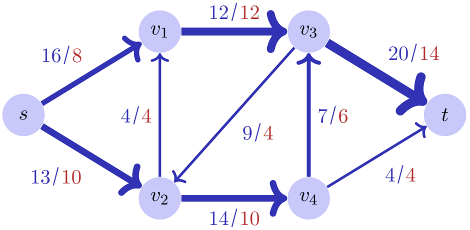
29.5 Appendix: Some Formal Things

Flow: Formulation with Skew Symmetry

A **Flow** $f : V \times V \to \mathbb{R}$ fulfills the following conditions:

- **Bounded Capacity**:
  For all $u, v \in V$: $f(u,v) \le c(u,v)$.
- **Skew Symmetry**:
  For all $u, v \in V$: $f(u,v) = -f(v,u)$.
- **Conservation of flow**:
  For all $u \in V \setminus \{s,t\}$:

$$\sum_{v \in V} f(u,v) = 0.$$



Value of the flow:
$|f| = \sum_{v \in V} f(s,v)$.
Here $|f| = 18$.

Cuts

- **Capacity** of an (s,t)—-cut: $c(S,T) = \sum_{v \in S, v' \in T} c(v,v')$
- **Minimal cut**: cut with minimal capacity.
- **Flow over the cut**: $f(S,T) = \sum_{v \in S, v' \in T} f(v,v')$

Generally: Let $U, U' \subseteq V$

$$f(U,U') := \sum_{\substack{u \in U \\ u' \in U'}} f(u,u'), \qquad f(u,U') := f(\{u\},U')$$

Then

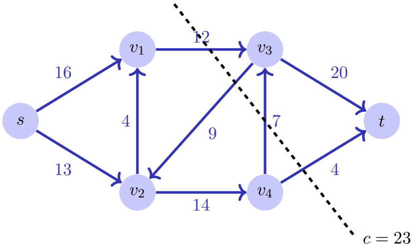
- $|f| = f(s,V)$
- $f(U,U) = 0$
- $f(U,U') = -f(U',U)$
- $f(X \cup Y, Z) = f(X,Z) + f(Y,Z)$, if $X \cap Y = \emptyset$.
- $f(R,V) = 0$ if $R \cap \{s,t\} = \emptyset$. [flow conversation!]

How large can a flow possibly be?

$$f(S,T) = f(S,V) - \underbrace{f(S,S)}_{0} = f(S,V)$$

$$= f(s,V) + f(\underbrace{S - \{s\}}_{\not\ni t, \not\ni s}, V) = |f|.$$

$$\Rightarrow |f| \le \sum_{v \in S, v' \in T} c(v, v') = c(S,T)$$



Rest Network

Rest network G_f provided by the edges with positive rest capacity:

$$\begin{aligned} G_f &:= (V, E_f, c_f) \\ c_f(u,v) &:= c(u,v) - f(u,v) \quad \forall u, v \in V \\ E_f &:= \{(u,v) \in V \times V | c_f(u,v) > 0\} \end{aligned}$$

- Increase of the flow along some edge possible, when flow can be increased along the edge,i.e. if $f(u,v) < c(u,v)$.
  Rest capacity $c_f(u,v) = c(u,v) - f(u,v) > 0$.
- Increase of flow **against the direction** of the edge possible, if flow can be reduced along the edge, i.e. if $f(u,v) > 0$.
  Rest capacity $c_f(v,u) = f(u,v) > 0$.

The increased flow is a flow

Theorem 32

Let $G = (V, E, c)$ be a flow network with source s and sink t and f a flow in G. Let G_f be the corresponding rest networks and let f' be a flow in G_f. Then $f \oplus f'$ with

$$(f \oplus f')(u, v) = f(u, v) + f'(u, v)$$

defines a flow in G with value $|f| + |f'|$.

Proof

$f \oplus f'$ defines a flow in G:

- capacity limit

$$(f \oplus f')(u,v) = f(u,v) + \underbrace{f'(u,v)}_{\leq c(u,v) - f(u,v)} \leq c(u,v)$$

- skew symmetry

$$(f \oplus f')(u,v) = -f(v,u) + -f'(v,u) = -(f \oplus f')(v,u)$$

- flow conservation $u \in V - \{s,t\}$:

$$\sum_{v \in V} (f \oplus f')(u,v) = \sum_{v \in V} f(u,v) + \sum_{v \in V} f'(u,v) = 0$$

Proof

Value of $f \oplus f'$

$$\begin{aligned} |f \oplus f'| &= (f \oplus f')(s, V) \\ &= \sum_{u \in V} f(s, u) + f'(s, u) \\ &= f(s, V) + f'(s, V) \\ &= |f| + |f'| \end{aligned}$$

■

Augmenting Paths

expansion path p: simple path from s to t in the rest network G_f.
Rest capacity $c_f(p) = \min\{c_f(u,v) : (u,v) \text{ edge in } p\}$

Theorem 33

The mapping $f_p : V \times V \to \mathbb{R}$,

$$f_p(u,v) = \begin{cases} c_f(p) & \text{if } (u,v) \text{ edge in } p \\ -c_f(p) & \text{if } (v,u) \text{ edge in } p \\ 0 & \text{otherwise} \end{cases}$$

provides a flow in G_f *with value* $|f_p| = c_f(p) > 0$.

f_p is a flow (easy to show). there is one and only one $u \in V$ with $(s,u) \in p$.
Thus $|f_p| = \sum_{v \in V} f_p(s,v) = f_p(s,u) = c_f(p)$.

Max-Flow Min-Cut Theorem

Theorem 34

Let f be a flow in a flow network $G = (V, E, c)$ with source s and sink t. The following statementsa are equivalent:

1. *f is a maximal flow in G*
2. *The rest network G_f does not provide any expansion paths*
3. *It holds that $|f| = c(S, T)$ for a cut (S, T) of G.*

Proof

- $(3) \Rightarrow (1)$:
  It holds that $|f| \leq c(S,T)$ for all cuts S,T. From $|f| = c(S,T)$ it follows that $|f|$ is maximal.
- $(1) \Rightarrow (2)$:
  f maximal Flow in G. Assumption: G_f has some expansion path $|f \oplus f_p| = |f| + |f_p| > |f|$. Contradiction.

Proof $(2) \Rightarrow (3)$

Assumption: G_f has no expansion path

Define $S = \{v \in V : \text{ there is a path } s \rightsquigarrow v \text{ in } G_f\}$.

$(S, T) := (S, V \setminus S)$ is a cut: $s \in S, t \in T$.

Let $u \in S$ and $v \in T$. Then $c_f(u, v) = 0$, also $c_f(u, v) = c(u, v) - f(u, v) = 0$.

Somit $f(u, v) = c(u, v)$.

Thus

$$|f| = f(S, T) = \sum_{u \in S} \sum_{v \in T} f(u, v) = \sum_{u \in S} \sum_{v \in T} c(u, v) = C(S, T).$$

■

Edmonds-Karp Algorithm

Theorem 35

When the Edmonds-Karp algorithm is applied to some integer valued flow network $G = (V, E)$ with source s and sink t then the number of flow increases applied by the algorithm is in $\mathcal{O}(|V| \cdot |E|)$.

$\Rightarrow$ *Overal asymptotic runtime: $\mathcal{O}(|V| \cdot |E|^2)$*

[Without proof]

Edmonds-Karp Algorithmus

Theorem 36

Wenn der Edmonds-Karp Algorithmus auf Flussnetzwerk $G = (V, E)$ *mit Quelle* s *und Senke* t *angewendet wird, dann wächst für jeden Knoten* $v \in V \setminus \{s, t\}$ *die Distanz* $\delta_f(s, v)$ *des kürzesten Pfades von* s *nach* v *im Restnetzwerk* G_f *monoton mit jeder Flusserhöhung.*

Beweis

Annahme: Distanz $\delta_f(s,v)$ wird bei Flusserhöhung $f \to f'$ kleiner für ein v: $\delta_f(s,v) < \delta_{f'}(s,v)$

Sei $p = s \rightsquigarrow u \to v$ kürzester Pfad von s nach v in $G_{f'}$, so dass $(u,v) \in E_{f'}$ und $\delta_{f'}(s,u) = \delta_{f'}(s,v) - 1$. Es gilt $\delta_{f'}(s,u) \geq \delta_f(s,u)$.

Wenn $(u,v) \in E_f$: $\delta_f s, v \leq \delta_f(s,u) + 1 \leq \delta_{f'}(s,u) + 1 = \delta_{f'}(s,v)$ Widerspruch.

Also $(u,v) \notin E_f$.

Integer number theorem

Theorem 37

If the capacities of a flow network are integers, then the maximal flow generated by the Ford-Fulkerson method provides integer numbers for each $f(u,v)$, $u, v \in V$.

[without proof]

Consequence: Ford-Fulkerson generates for a flow network that corresponds to a bipartite graph a maximal matching $M = \{(u,v) : f(u,v) = 1\}$.