22. Dynamic Programming III

Optimal Search Tree [Ottman/Widmayer, Kap. 5.7]

22.1 Optimal Search Trees

Optimal binary Search Trees

Given: n keys $k_1, k_2 \dots k_n$ (wlog $k_1 < k_2 < ... < k_n$) with weights (search probabilities[34]) $p_1, p_2, \dots, p_n$.

Wanted: optimal search tree T with key depths[35] $\mathrm{d}(\cdot)$, that minimizes the expected search costs

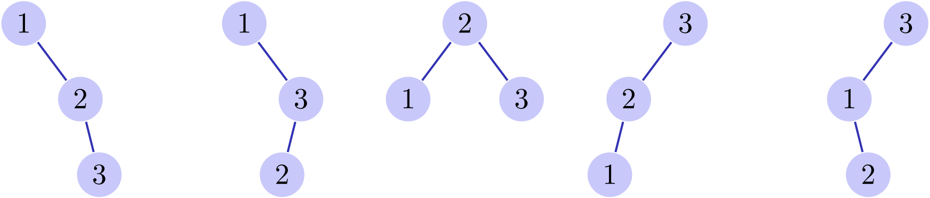
$$C(T) = \sum_{i=1}^{n} (\mathrm{d}(k_i) + 1) \cdot p_i$$

[34] It is possible to model unsuccesful search additionally, omitted for brevity here

[35] $d(k)$: Length of the path from the root to the node k

Examples

$2p_1 + p_2$ $\quad$ $p_1 + 2p_2$

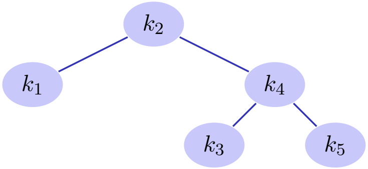


$p_1 + 2p_2 + 3p_3$ $\quad$ $p_1 + 3p_2 + 2p_2$ $\quad$ $2p_1 + p_2 + 2p_3$ $\quad$ $3p_1 + 2p_2 + p_3$ $\quad$ $2p_1 + 3p_3 + 1p_3$

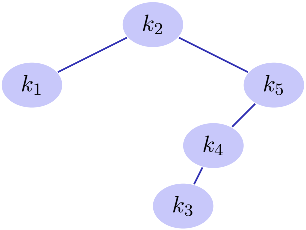
Example

Expected Frequencies

i	1	2	3	4	5
p_i	0.25	0.10	0.05	0.20	0.40

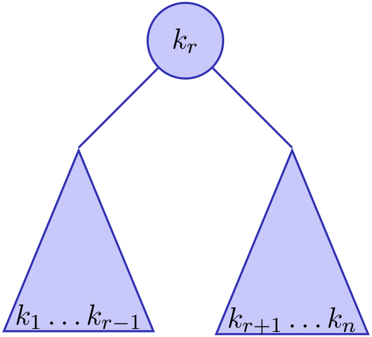


Search tree with expected costs 2.35



Search tree with expected costs 2.2

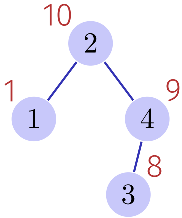
Sub-trees for Searching



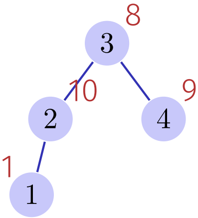
Which r to choose?

Greedy?

Scenario $p_1 = 1, p_2 = 10, p_3 = 8, p_4 = 9$



$c(T) = 54$



$c(T) = 49$

Structure of a optimal binary search tree

- Consider all subtrees with roots k_r and optimal subtrees for keys $k_i, \ldots, k_{r-1}$ and $k_{r+1}, \ldots, k_j$
- Subtrees with keys $k_i, \ldots, k_{r-1}$ and $k_{r+1}, \ldots, k_j$ must be optimal for the respective sub-problems.[36]

$$E(i,j) = \text{Costs of optimal search tree with nodes } k_i, k_{i+1}, \ldots, k_j$$

[36]The usual argument: if it was not optimal, it could be replaced by a better solution improving the overal solution.

Rekursion

With

$$p(i,j) := p_i + p_{i+1} + \cdots + p_j \qquad i \leq j$$

it holds that

$$E(i,j) = \begin{cases} 0 & \text{if } i > j \\ p(i) & \text{if } i = j \\ p(i,j) + \min\{E(i,k-1) + E(k+1,j), i \leq k \leq j\} & \text{otherwise.} \end{cases}$$

DP

0. $E(1,n)$: Costs of optimal search tree with nodes $k_1,\dots,k_n$ with search frequencies $p_1,\dots,p_n$
1. $E(i,j), 1 \le i \le j \le n$ # sub-problems $\Theta(n^2)$
2. Enumerate roots of subtree of $k_i,\dots,k_j$, # possibilities: $j-i+1$
3. Dependencies $E(i,j)$ depend on $E(i,k), E(k,j)$ $i<k<j$. Computation of the off-diagonals of E, starting with the diagonal of E
4. Solution is in $E(1,n)$, Reconstruction: store the arg-mins of the recursion in a separate table V.
5. Running time $\Theta(n^3)$. Memory $\Theta(n^2)$.

Example

i	1	2	3	4	5
p_i	0.25	0.10	0.05	0.20	0.40

E

i \ j	0	1	2	3	4	5
1	0	0.25	0.45	0.60	1.15	2.00
2		0	0.10	0.20	0.55	1.30
3			0	0.05	0.30	0.95
4				0	0.20	0.80
5					0	0.40
6						0

p

i \ j	1	2	3	4	5
1	0.25	0.35	0.40	0.60	1.00
2		0.10	0.15	0.35	0.75
3			0.05	0.25	0.65
4				0.20	0.60
5					0.40

V

i \ j	1	2	3	4	5
1	1	1	1	1	4
2		2	2	4	5
3			3	4	5
4				4	5
5					5

23. Greedy Algorithms

Fractional Knapsack Problem, Huffman Coding [Cormen et al, Kap. 16.1, 16.3]

Greedy Choice

A problem with a recursive solution can be solved with a **greedy algorithm** if it has the following properties:

- The problem has **optimal substructure**: the solution of a problem can be constructed with a combination of solutions of sub-problems.
- The problem has the **greedy choice property**: The solution to a problem can be constructed, by using a local criterion that is not depending on the solution of the sub-problems.

Examples: fractional knapsack, Huffman-Coding (below)
Counter-Example: knapsack problem, Optimal Binary Search Tree

Huffman-Codes

Goal: memory-efficient saving of a sequence of characters using a binary code with code words..

Example

File consisting of 100.000 characters from the alphabet $\{a, \ldots, f\}$.

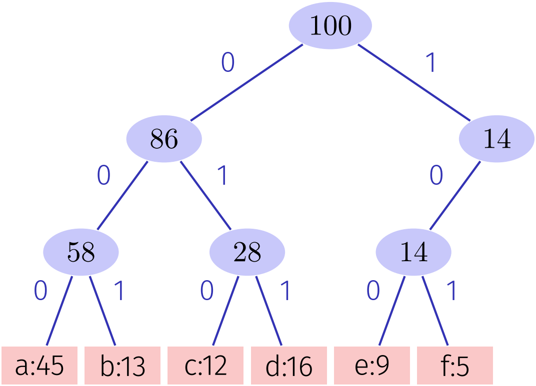
	a	b	c	d	e	f
Frequency (Thousands)	45	13	12	16	9	5
Code word with fix length	000	001	010	011	100	101
Code word variable length	0	101	100	111	1101	1100

File size (code with fix length): 300.000 bits.
File size (code with variable length): 224.000 bits.

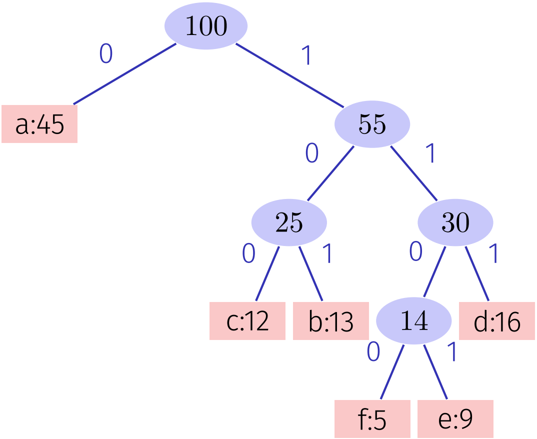
Huffman-Codes

- Consider prefix-codes: no code word can start with a different codeword.
- Prefix codes can, compared with other codes, achieve the optimal **data compression** (without proof here).
- Encoding: concatenation of the code words without stop character (difference to morsing).
  affe $\rightarrow 0 \cdot 1100 \cdot 1100 \cdot 1101 \rightarrow 0110011001101$
- Decoding simple because prefixcode
  $0110011001101 \rightarrow 0 \cdot 1100 \cdot 1100 \cdot 1101 \rightarrow$ affe

Code trees



Code words with fixed length



Code words with variable length

Properties of the Code Trees

- An optimal coding of a file is alway represented by a complete binary tree: every inner node has two children.
- Let C be the set of all code words, $f(c)$ the frequency of a codeword c and $d_T(c)$ the depth of a code word in tree T. Define the cost of a tree as

$$B(T) = \sum_{c \in C} f(c) \cdot d_T(c).$$

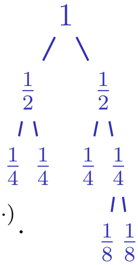
  (cost = number bits of the encoded file)

In the following a code tree is called optimal when it minimizes the costs.

Probabilitiy Distributions

The sum to be minimized

$$\sum_{c \in C} f(c) \cdot d_T(c)$$



can be written as

$$-\sum_{c \in C} f(c) \cdot \log_2 g_T(c), \text{ where } g_T(\cdot) := 2^{-d_T(\cdot)}.$$

$g_T(\cdot)$ can be understood as discrete probability distribution because it holds that $\sum_c g_T(c) = 1$. That is a property of a complete binary tree because each inner node has two child nodes.

Probabilitiy Distributions

For two discrete proability distributions f and g over C the **Gibbs inequality** holds

$$\underbrace{-\sum_{c \in C} f(c) \log f(c)}_{\text{Entropy of } f} \leq -\sum_{c \in C} f(x) \log g(c)$$

with equality if and only if $f(c) = g(c)$ for each $c \in C$.

Consequence if $f(c) \in \{2^{-k}, k \in \mathbb{N}\}$ for all $c \in C$, then the optimal code tree can be formed easily with $d_T(c) = -\log_2 f(c)$.

Shannon Fano Coding

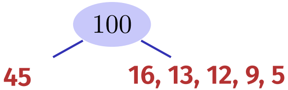
Approximative algorithm of Shannon and Fano

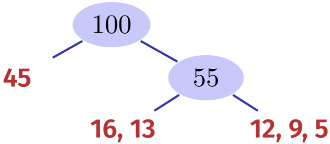
1. Sort the keys by frequency, wlog $p_1 \leq p_2 \leq ... \leq p_n$
2. Partition the keys into two sets of almost equal weight, i.e. into sets $A = \{1, \ldots, k\}$ and $B = \{k+1, \ldots, n\}$ such that $\sum_{i \in A} p_i \approx \sum_{i \in B} p_i$. Recursion until all sets contain a single element.

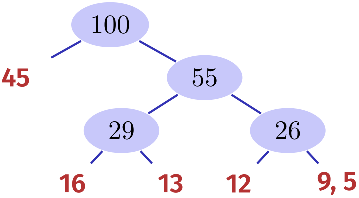
Running Time: $\Theta(n \log n)$

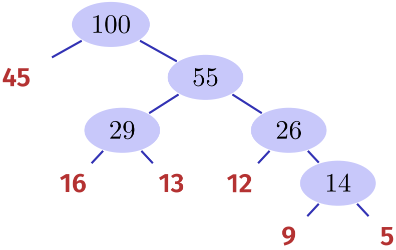
Shannon Fano Coding

45, 16, 13, 12, 9, 5





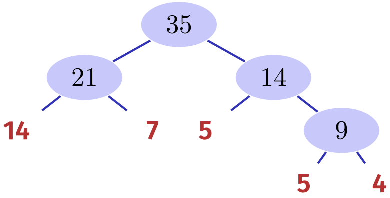




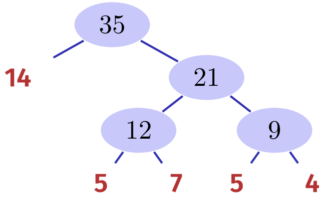
Problem

The approximate algorithm of Shannon and Fano does not always provide the optimal result

Example $\{14, 7, 5, 5, 4\}$ with lower bound (entropy) $B(T) \geq 75.35$



Shannon-Fano Coding, $B(T) = 79$

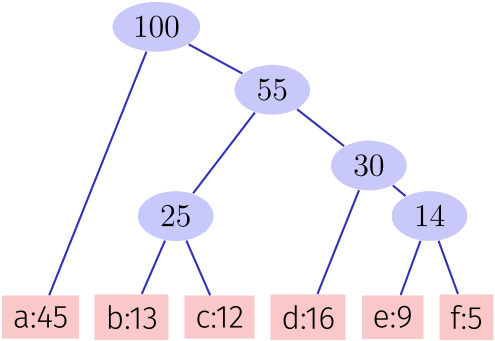


Optimal, $B(T) = 77$

Huffman's Idea

Tree construction bottom up

- Start with the set C of code words
- Replace iteriatively the two nodes with smallest frequency by a new parent node.



Algorithm Huffman(C)

Input: code words $c \in C$
Output: Root of an optimal code tree

$n \leftarrow |C|$
$Q \leftarrow C$
for $i = 1$ **to** $n - 1$ **do**
    allocate a new node z
    z.left $\leftarrow$ ExtractMin(Q) // extract word with minimal frequency.
    z.right $\leftarrow$ ExtractMin(Q)
    z.freq $\leftarrow$ z.left.freq $+$ z.right.freq
    Insert(Q, z)
return ExtractMin(Q)

Analyse

Use a heap: build Heap in $\mathcal{O}(n)$. Extract-Min in $O(\log n)$ for n Elements. Yields a runtime of $O(n \log n)$.

The greedy approach is correct

Theorem 20

Let x, y be two symbols with smallest frequencies in C and let $T'(C')$ be an optimal code tree to the alphabet $C' = C - \{x, y\} + \{z\}$ with a new symbol z with $f(z) = f(x) + f(y)$. Then the tree $T(C)$ that is constructed from $T'(C')$ by replacing the node z by an inner node with children x and y is an optimal code tree for the alphabet C.

Proof

It holds that
$f(x)\cdot d_T(x)+f(y)\cdot d_T(y) = (f(x)+f(y))\cdot(d_{T'}(z)+1) = f(z)\cdot d_{T'}(x)+f(x)+f(y)$.
Thus $B(T') = B(T) - f(x) - f(y)$.

Assumption: T is not optimal. Then there is an optimal tree T'' with $B(T'') < B(T)$. We assume that x and y are brothers in T''. Let T''' be the tree where the inner node with children x and y is replaced by z. Then it holds that $B(T''') = B(T'') - f(x) - f(y) < B(T) - f(x) - f(y) = B(T')$. Contradiction to the optimality of T'.

The assumption that x and y are brothers in T'' can be justified because a swap of elements with smallest frequency to the lowest level of the tree can at most decrease the value of B.

Recursive Problem-Solving Strategies

Brute Force Enumeration	Backtracking	Divide and Conquer	Dynamic Programming	Greedy
Recursive Enumerability	Constraint Satisfaction, Partial Validation	Optimal Substructure	Optimal Substructure, Overlapping Subproblems	Optimal Substructure, Greedy Choice Property
DFS, BFS, all Permutations, Tree Traversal	n-Queen, Sudoku, m-Coloring, SAT-Solving, naive TSP	Binary Search, Mergesort, Quicksort, Hanoi Towers, FFT	Bellman Ford, Warshall, Rod-Cutting, LAS, Editing Distance, Knapsack Problem DP	Dijkstra, Kruskal, Huffmann Coding