20. Dynamic Programming I

Memoization, Optimal Substructure, Overlapping Sub-Problems, Dependencies, General Procedure. Examples: Fibonacci, Rod Cutting, Longest Ascending Subsequence, Longest Common Subsequence, Edit Distance, Matrix Chain Multiplication (Strassen)

[Ottman/Widmayer, Kap. 1.2.3, 7.1, 7.4, Cormen et al, Kap. 15]

Fibonacci Numbers

 (again)

$$F_n := \begin{cases} n & \text{if } n < 2 \\ F_{n-1} + F_{n-2} & \text{if } n \geq 2. \end{cases}$$

Analysis: why ist the recursive algorithm so slow?

Algorithm FibonacciRecursive(n)

Input: $n \geq 0$
Output: n-th Fibonacci number

if $n < 2$ **then**
    $f \leftarrow n$
else
    $f \leftarrow$ FibonacciRecursive$(n-1)$ + FibonacciRecursive$(n-2)$
return f

Analysis

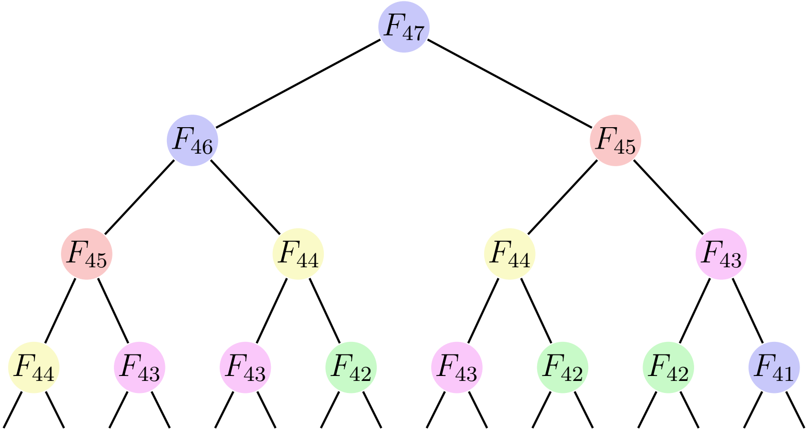
$T(n)$: Number executed operations.

- $n = 0, 1$: $T(n) = \Theta(1)$
- $n \geq 2$: $T(n) = T(n-2) + T(n-1) + c$.

$$T(n) = T(n-2) + T(n-1) + c \geq 2T(n-2) + c \geq 2^{n/2}c' = (\sqrt{2})^n c'$$

Algorithm is **exponential** in n.

Reason (visual)



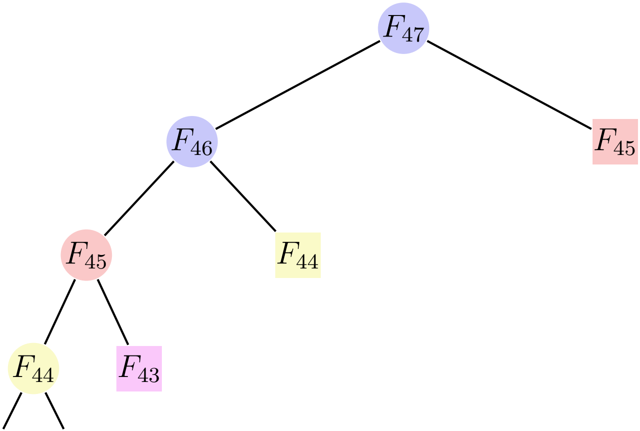
Nodes with same values are evaluated (too) often.

Memoization

Memoization (sic) saving intermediate results.

- Before a subproblem is solved, the existence of the corresponding intermediate result is checked.
- If an intermediate result exists then it is used.
- Otherwise the algorithm is executed and the result is saved accordingly.

Memoization with Fibonacci



Rectangular nodes have been computed before.

Algorithm FibonacciMemoization(n)

Input: $n \geq 0$
Output: n-th Fibonacci number

if $n \leq 2$ **then**
  $f \leftarrow 1$
else if $\exists$memo$[n]$ **then**
  $f \leftarrow$ memo$[n]$
else
  $f \leftarrow$ FibonacciMemoization$(n-1)+$ FibonacciMemoization$(n-2)$
  memo$[n] \leftarrow f$

return f

Analysis

Computational complexity:

$$T(n) = T(n-1) + c = ... = \mathcal{O}(n).$$

because after the call to $f(n-1)$, $f(n-2)$ has already been computed.

A different argument: $f(n)$ is computed exactly once recursively for each n. Runtime costs: n calls with $\Theta(1)$ costs per call $n \cdot c \in \Theta(n)$. The recursion vanishes from the running time computation.

Algorithm requires $\Theta(n)$ memory.[29]

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[29] But the naive recursive algorithm also requires $\Theta(n)$ memory implicitly.

Looking closer ...

... the algorithm computes the values of F_1, F_2, F_3,$\ldots$ in the **top-down** approach of the recursion.

Can write the algorithm **bottom-up**. This is characteristic for **dynamic programming**.

Algorithm FibonacciBottomUp(n)

Input: $n \geq 0$
Output: n-th Fibonacci number

$F[1] \leftarrow 1$
$F[2] \leftarrow 1$
for $i \leftarrow 3, \ldots, n$ **do**
  $F[i] \leftarrow F[i-1] + F[i-2]$

return $F[n]$

Dynamic Programming: Idea

- Divide a complex problem into a reasonable number of sub-problems
- The solution of the sub-problems will be used to solve the more complex problem
- Identical problems will be computed only once

Dynamic Programming Consequence

Identical problems will be computed only once

$\Rightarrow$ Results are saved

We trade speed against memory consumption

Dynamic Programming = Divide-And-Conquer ?

- In both cases the original problem can be solved (more easily) by utilizing the solutions of sub-problems. The problem provides **optimal substructure**.
- Classical Divide-And-Conquer algorithms (such as Mergesort): sub-problems are independent; their solutions are required only once in the algorithm.
- DP: sub-problems are dependent. The problem is said to have **overlapping sub-problems** that are required multiple-times in the algorithm.
- In order to avoid redundant computations, results are tabulated. For **sub-problems there must not be any circular dependencies**.

Dynamic Programming: Description

1. Use a **DP-table** with information to the subproblems.
   Dimension of the table? Semantics of the entries?
2. Computation of the **base cases**.
   Which entries do not depend on others?
3. Determine **computation order**.
   In which order can the entries be computed such that dependencies are fulfilled?
4. Read-out the **solution**
   How can the solution be read out from the table?

Runtime (typical) = number entries of the table times required operations per entry.

Dynamic Programing: Description (Fibonacci)

1. **Dimension of the table? Semantics of the entries?**

   $n \times 1$ table. nth entry contains nth Fibonacci number.

2. **Which entries do not depend on other entries?**

   Values F_1 and F_2 can be computed easily and independently.

3. **Computation order?**

   F_i with increasing i.

4. **Reconstruction of a solution?**

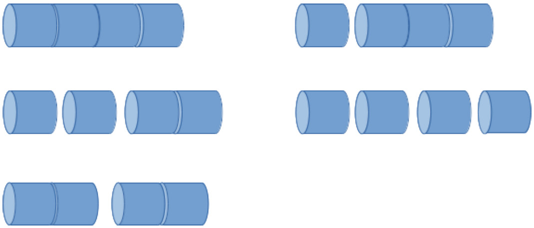
   F_n is the n-th Fibonacci number.

Rod Cutting

- Rods (metal sticks) are cut and sold.
- Rods of length $n \in \mathbb{N}$ are available. A cut does not provide any costs.
- For each length $l \in \mathbb{N}$, $l \leq n$ known is the value $v_l \in \mathbb{R}^+$
- Goal: cut the rods such (into $k \in \mathbb{N}$ pieces) that

$$\sum_{i=1}^{k} v_{l_i} \text{ is maximized subject to } \sum_{i=1}^{k} l_i = n.$$

Rod Cutting: Example



Possibilities to cut a rod of length 4 (without permutations)

Length	0	1	2	3	4
Price	0	2	3	8	9

$\Rightarrow$ Best cut: 3 + 1 with value 10.

How to Find the DP Algorithm.

0. Exact formulation of the wanted solution
1. Define sub-problems, reformulate (0.) as sub-problem
2. Recursion: relate subproblems by enumerating of local properties
3. Determine the dependencies of the sub-problems
4. Solve the problem
   Running time = #sub-problems $\times$ time/sub-problem

Structure of the problem

0. **Wanted:** r_n = maximal value of rod (cut or as a whole) with length n.
1. **sub-problems**: maximal value r_k for each $0 \leq k < n$
2. Local property: length of the first piece
   Recursion

$$
\begin{aligned}
r_k &= \max\{v_i + r_{k-i} : 0 < i \leq k\}, \quad k > 0 \\
r_0 &= 0
\end{aligned}
$$

3. **Dependency:** r_k depends (only) on values v_i, $1 \leq i \leq k$ and the optimal cuts r_i, $i < k$.
4. **Solution** in r_n. DP running time: $\Theta(n^2)$

Algorithm RodCut(v,n) (without memoization)

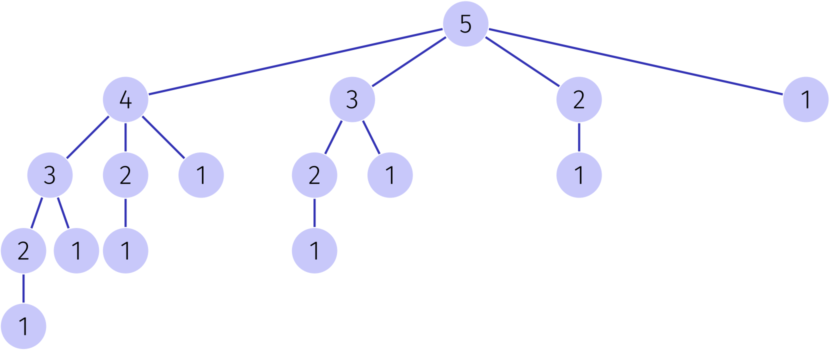
Input: $n \geq 0$, Prices v
Output: best value

$q \leftarrow 0$
if $n > 0$ **then**
    for $i \leftarrow 1, \ldots, n$ **do**
        $q \leftarrow \max\{q, v_i + \text{RodCut}(v, n-i)\}$;
return q

Running time $T(n) = \sum_{i=0}^{n-1} T(i) + c \quad \Rightarrow$[30] $\quad T(n) \in \Theta(2^n)$

[30] $T(n) = T(n-1) + \sum_{i=0}^{n-2} T(i) + c = T(n-1) + (T(n-1) - c) + c = 2T(n-1) \quad (n > 0)$

Recursion Tree



Algorithm RodCutMemoized(m, v, n)

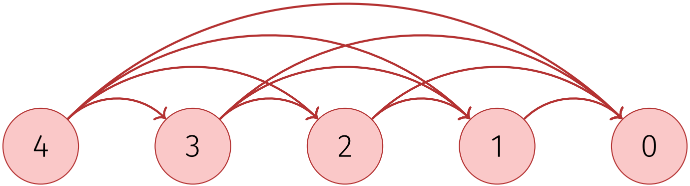
Input: $n \geq 0$, Prices v, Memoization Table m
Output: best value

$q \leftarrow 0$
if $n > 0$ **then**
  if $\exists\ m[n]$ **then**
    $q \leftarrow m[n]$
  else
    for $i \leftarrow 1, \ldots, n$ **do**
      $q \leftarrow \max\{q, v_i +$ RodCutMemoized$(m, v, n - i)\}$;
    $m[n] \leftarrow q$
return q

Running time $\sum_{i=1}^{n} i = \Theta(n^2)$

Subproblem-Graph

Describes the mutual dependencies of the subproblems



and must not contain cycles

Construction of the Optimal Cut

- During the (recursive) computation of the optimal solution for each $k \leq n$ the recursive algorithm determines the optimal length of the first rod
- Store the lenght of the first rod in a separate table of length n

Bottom-up Description with the example

1. Dimension of the table? Semantics of the entries?

   $n \times 1$ table. nth entry contains the best value of a rod of length n.

2. Which entries do not depend on other entries?

   Value r_0 is 0

3. Computation order?

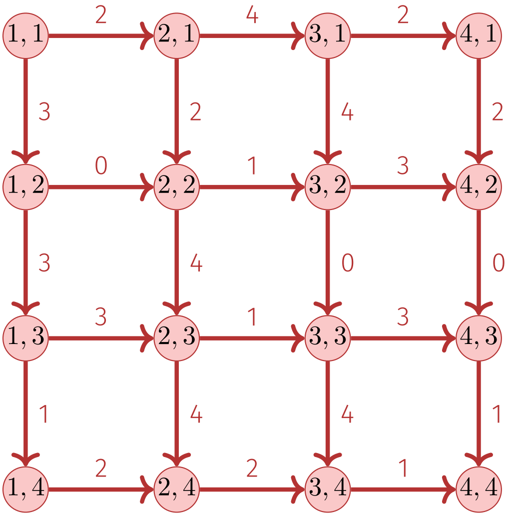
   r_i, $i = 1, \ldots, n$.

4. Reconstruction of a solution?

   r_n is the best value for the rod of length n.

Rabbit!

A rabbit sits on cite $(1, 1)$ of an $n \times n$ grid. It can only move to east or south. On each pathway there is a number of carrots. How many carrots does the rabbit collect maximally?

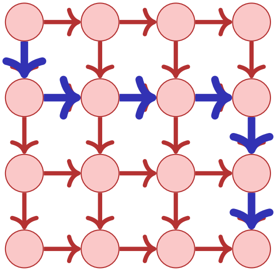


Rabbit!

Number of possible paths?

- Choice of $n-1$ ways to south out of $2n-2$ ways overal.
- $$\binom{2n-2}{n-1} \in \Omega(2^n)$$

$\Rightarrow$ No chance for a naive algorithm



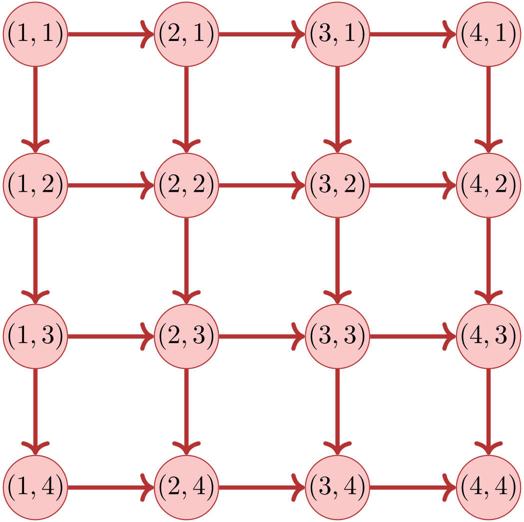
The path 100011
(1:to south, 0: to east)

Recursion

Wanted: $T_{1,1}$ **= maximal number carrots from** $(1,1)$ **to** (n,n)**.**
Let $w_{(i,j)-(i',j')}$ number of carrots on egde from (i,j) to (i',j').
Recursion (maximal number of carrots from (i,j) to (n,n))

$$T_{ij} = \begin{cases} \max\{w_{(i,j)-(i,j+1)} + T_{i,j+1}, w_{(i,j)-(i+1,j)} + T_{i+1,j}\}, & i < n, j < n \\ w_{(i,j)-(i,j+1)} + T_{i,j+1}, & i = n, j < n \\ w_{(i,j)-(i+1,j)} + T_{i+1,j}, & i < n, j = n \\ 0 & i = j = n \end{cases}$$

Graph of Subproblem Dependencies



Bottom-up Description with the example

1. **Dimension of the table? Semantics of the entries?**

   Table T with size $n \times n$. Entry at i, j provides the maximal number of carrots from (i, j) to (n, n).

2. **Which entries do not depend on other entries?**

   Value $T_{n,n}$ is 0

3. **Computation order?**

   $T_{i,j}$ with $i = n \searrow 1$ and for each i: $j = n \searrow 1$, (or vice-versa: $j = n \searrow 1$ and for each j: $i = n \searrow 1$).

4. **Reconstruction of a solution?**

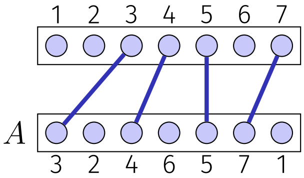
   $T_{1,1}$ provides the maximal number of carrots.

Longest Ascending Sequence (LAS)

Connect as many as possible fitting ports without lines crossing.

Formally

- Consider Sequence $A_n = (a_1, \ldots, a_n)$.
- Search for a longest increasing subsequence of A_n.
- Examples of increasing subsequences: $(3, 4, 5)$, $(2, 4, 5, 7)$, $(3, 4, 5, 7)$, $(3, 7)$.



Generalization: allow any numbers, even with duplicates (still only strictly increasing subsequences permitted). Example: $(2, 3, 3, 3, 5, 1)$ with increasing subsequence $(2, 3, 5)$.

First idea (Greedy)

Let L_i **= longest ascending subsequence of** A_i $(1 \leq i \leq n)$

Assumption: LAS L_k of A_k known. Compute L_{k+1} for A_{k+1}.

Idea

$$L_{k+1} = \begin{cases} L_k \oplus a_{k+1} & \text{if } a_k > \max(L_k) \\ L_k & \text{otherwise?} \end{cases}$$

Counterexample

$A_5 = (1, 2, 5, 3, 4)$.
$A_3 = (1, 2, 5)$ with $L_3 = A_3$ and $L_4 = A_3$.

Greedy idea fails here: we cannot directly infer L_{k+1} from L_k.

Second idea. (Prefix)

Let L_i **= longest ascending subsequence of** A_i $(1 \leq i \leq n)$

Assumption: a LAS L_j that ends in a_j is known for each $j \leq k$. Now compute LAS L_{k+1} for $k+1$.

Look at all fitting $L_{k+1} = L_j \oplus a_{k+1}$ $(j \leq k)$ and choose a longest sequence.

Example

$A_5 = (1, 2, 5, 3, 4)$.
$L_1 = (1)$, $L_2 = (1, 2)$, $L_3 = (1, 2, 5)$, $L_4 = (1, 2, 3)$, $L_5 = (1, 2, 3, 4)$.

This works with running time n^2 (and requires access to all sequences L_i.

Third approach

Let $M_{n,i}$ **= longest ascending subsequence of** A_i $(1 \le i \le n)$

Assumption: the LAS M_j for A_k, **that end with smallest element** are known for each of the lengths $1 \le j \le k$.

Consider all fitting $M_{k,j} \oplus a_{k+1}$ $(j \le k)$ and update the table of the LAS,that end with smallest possible element.

Third approach Example

Example: $A = (1, 1000, 1001, 4, 5, 2, 6, 7)$

A	LAT $M_{k,\cdot}$
1	(**1**)
$+ 1000$	$(1), (1,$ **1000**$)$
$+ 1001$	$(1), (1, 1000), (1, 1000,$ **1001**$)$
$+ 4$	$(1), (1,$ **4**$), (1, 1000, 1001)$
$+ 5$	$(1), (1, 4), (1, 4,$ **5**$)$
$+ 2$	$(1), (1,$ **2**$), (1, 4, 5)$
$+ 6$	$(1), (1, 2), (1, 4, 5), (1, 4, 5,$ **6**$)$
$+ 7$	$(1), (1, 2), (1, 4, 5), (1, 4, 5, 6), (1, 4, 5, 6,$ **7**$)$

DP Table

- Idea: save the last element of the increasing sequence $M_{k,j}$ at slot j.
- Example:
  13 12 15 11 16 14
- Problem: Table does not contain the subsequence, only the last value.
- Solution: second table with the values of the predecessors.

i	1	2	3	4	5	6
value a_i	13	12	15	11	16	14
Predecessor	$-\infty$	$-\infty$	12	$-\infty$	15	11

j	0	1	2	3	4	...
$(L_j)_j$	-∞	11	14	16	∞	

Dynamic Programming Algorithm LAS

Table dimension? Semantics?

1. Two tables $T[0,\ldots,n]$ and $V[1,\ldots,n]$. $T[j]$: last Element of the increasing subequence $M_{n,j}$
$V[j]$: Value of the predecessor of a_j.
Start with $T[0] \leftarrow -\infty$, $T[i] \leftarrow \infty \; \forall i > 1$

Computation of an entry

2. Entries in T sorted in ascending order. For each new entry a_k binary search for l, such that $T[l] < a_k < T[l+1]$. Set $T[l+1] \leftarrow a_k$. Set $V[k] = T[l]$.

Dynamic Programming algorithm LAS

3. **Computation order**

   Traverse the list anc compute $T[k]$ and $V[k]$ with ascending k

   Reconstruction of a solution?

4. Search the largest l with $T[l] < \infty$. l is the last index of the LAS. Starting at l search for the index $i < l$ such that $V[l] = a_i$, i is the predecessor of l. Repeat with $l \leftarrow i$ until $T[l] = -\infty$

Analysis

- Computation of the table:
  - Initialization: $\Theta(n)$ Operations
  - Computation of the kth entry: binary search on positions $\{1, \ldots, k\}$ plus constant number of assignments.

$$\sum_{k=1}^{n}(\log k + \mathcal{O}(1)) = \mathcal{O}(n) + \sum_{k=1}^{n}\log(k) = \Theta(n \log n).$$

- Reconstruction: traverse A from right to left: $\mathcal{O}(n)$.

Overal runtime:

$$\Theta(n \log n).$$

20.7 Editing Distance

Minimal Editing Distance

Editing distance of two sequences $A_n = (a_1, \ldots, a_n)$, $B_m = (b_1, \ldots, b_m)$.
Editing operations:

- Insertion of a character
- Deletion of a character
- Replacement of a character

Question: how many editing operations at least required in order to transform string A into string B.

TIGER $\rightarrow$ ZIGER $\rightarrow$ ZIEGER $\rightarrow$ ZIEGE

Minimal Editing Distance

Wanted: cheapest character-wise transformation $A_n \to B_m$ with costs

operation	Levenshtein	LCS[31]	general
Insert c	1	1	ins(c)
Delete c	1	1	del(c)
Replace $c \to c'$	$\mathbb{1}(c \neq c')$	$\infty \cdot \mathbb{1}(c \neq c')$	repl(c, c')

Beispiel

T	I	G	E	R	T	I	_	G	E	R	T→Z	+E	-R
Z	I	E	G	E	Z	I	E	G	E	_	Z→T	-E	+R

[31]Longest common subsequence – A special case of an editing problem

Idea

Z I E G E → T I G E R

Possibilities

1. $$c(\texttt{'ZIEG'} \rightarrow \texttt{'TIGE'}) + c(\texttt{'E'} \rightarrow \texttt{'R'})$$

   Z I E G **E** → T I G E **R**

2. $$c(\texttt{'ZIEGE'} \rightarrow \texttt{'TIGE'}) + c(\mathsf{ins}(\texttt{'R'}))$$

   Z I E G E → T I G E **+ R**

3. $$c(\texttt{'ZIEG'} \rightarrow \texttt{'TIGER'}) + c(\mathsf{del}(\texttt{'E'}))$$

   Z I E G E **- E** → T I G E R

DP

0. $E(n,m)$ = mimimum number edit operations (ED cost) $a_{1\ldots n} \to b_{1\ldots m}$
1. Subproblems $E(i,j)$ = ED of $a_{1\ldots i}$, $b_{1\ldots j}$. #SP $= n \cdot m$
2. Guess Costs$\Theta(1)$

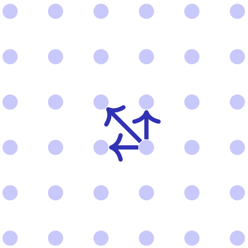
   - $a_{1..i} \to a_{1\ldots i-1}$ (delete)
   - $a_{1..i} \to a_{1\ldots i}b_j$ (insert)
   - $a_{1..i} \to a_{1\ldots i-1}b_j$ (replace)

3. Rekursion

$$E(i,j) = \min \begin{cases} \text{del}(a_i) + E(i-1,j), \\ \text{ins}(b_j) + E(i,j-1), \\ \text{repl}(a_i,b_j) + E(i-1,j-1) \end{cases}$$

DP

4. Dependencies



$\Rightarrow$ Computation from left top to bottom right. Row- or column-wise.

5. Solution in $E(n, m)$

Example (Levenshtein Distance)

$$E[i,j] \leftarrow \min\left\{E[i-1,j]+1, E[i,j-1]+1, E[i-1,j-1]+\mathbb{1}(a_i \neq b_j)\right\}$$

	∅	Z	I	E	G	E
∅	0	1	2	3	4	5
T	1	1	2	3	4	5
I	2	2	1	2	3	4
G	3	3	2	2	2	3
E	4	4	3	2	3	2
R	5	5	4	3	3	3

Editing steps: from bottom right to top left, following the recursion.

Bottom-Up DP algorithm ED

Dimension of the table? Semantics?

1. Table $E[0,\ldots,m][0,\ldots,n]$. $E[i,j]$: minimal edit distance of the strings $(a_1,\ldots,a_i)$ and $(b_1,\ldots,b_j)$

Computation of an entry

2. $E[0,i] \leftarrow i \; \forall 0 \leq i \leq m$, $E[j,0] \leftarrow i \; \forall 0 \leq j \leq n$. Computation of $E[i,j]$ otherwise via $E[i,j] = \min\{\text{del}(a_i) + E(i-1,j), \text{ins}(b_j) + E(i,j-1), \text{repl}(a_i,b_j) + E(i-1,j-1)\}$

Bottom-Up DP algorithm ED

3. **Computation order**

   Rows increasing and within columns increasing (or the other way round).

4. **Reconstruction of a solution?**

   Start with $j = m, i = n$. If $E[i,j] = \text{repl}(a_i, b_j) + E(i-1, j-1)$ then output $a_i \rightarrow b_j$ and continue with $(j,i) \leftarrow (j-1, i-1)$; otherwise, if $E[i,j] = \text{del}(a_i) + E(i-1, j)$ output $\text{del}(a_i)$ and continue with $j \leftarrow j-1$ otherwise, if $E[i,j] = \text{ins}(b_j) + E(i, j-1)$, continue with $i \leftarrow i-1$. Terminate for $i = 0$ and $j = 0$.

Analysis ED

- Number table entries: $(m + 1) \cdot (n + 1)$.
- Constant number of assignments and comparisons each. Number steps: $\mathcal{O}(mn)$
- Determination of solition: decrease i or j. Maximally $\mathcal{O}(n + m)$ steps.

Runtime overal:

$$\mathcal{O}(mn).$$

Matrix-Chain-Multiplication

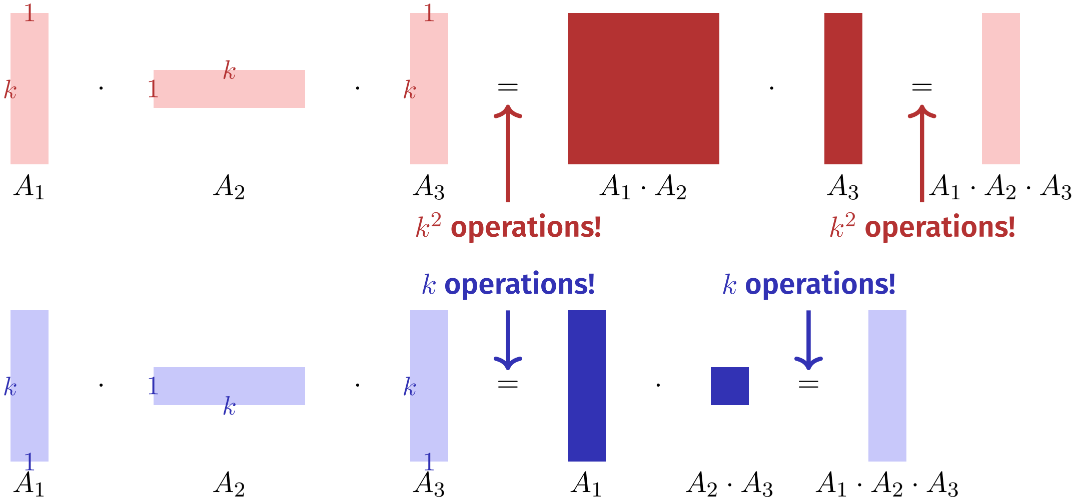
Task: Computation of the product $A_1 \cdot A_2 \cdot ... \cdot A_n$ of matrices $A_1, \ldots, A_n$.

Matrix multiplication is associative, i.e. the order of evaluation can be chosen arbitrarily

Goal: efficient computation of the product.

Assumption: multiplication of an $(r \times s)$-matrix with an $(s \times u)$-matrix provides costs $r \cdot s \cdot u$.

Does it matter?



Recursion

- Assume that the best possible computation of $(A_1 \cdot A_2 \cdots A_i)$ and $(A_{i+1} \cdot A_{i+2} \cdots A_n)$ is known for each i.
- Compute best i, done.

$n \times n$-table M. entry $M[p, q]$ provides costs of the best possible bracketing $(A_p \cdot A_{p+1} \cdots A_q)$.

$$M[p, q] \leftarrow \min_{p \le i < q} (M[p, i] + M[i + 1, q] + \text{costs of the last multiplication})$$

Computation of the DP-table

- Base cases $M[p,p] \leftarrow 0$ for all $1 \leq p \leq n$.
- Computation of $M[p,q]$ depends on $M[i,j]$ with $p \leq i \leq j \leq q$, $(i,j) \neq (p,q)$.
  In particular $M[p,q]$ depends at most from entries $M[i,j]$ with $i - j < q - p$.
  Consequence: fill the table from the diagonal.

Analysis

DP-table has n^2 entries. Computation of an entry requires considering up to $n-1$ other entries.
Overal runtime $\mathcal{O}(n^3)$.

Readout the order from M: exercise!

Digression: matrix multiplication

Consider the multiplication of two $n \times n$ matrices.
Let

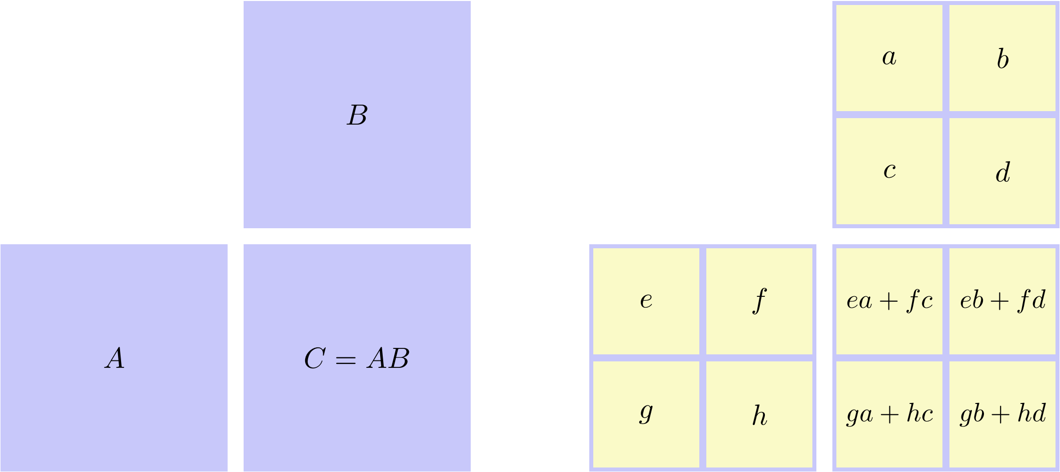
$$A = (a_{ij})_{1 \leq i,j \leq n}, B = (b_{ij})_{1 \leq i,j \leq n}, C = (c_{ij})_{1 \leq i,j \leq n},$$
$$C = A \cdot B$$

then

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

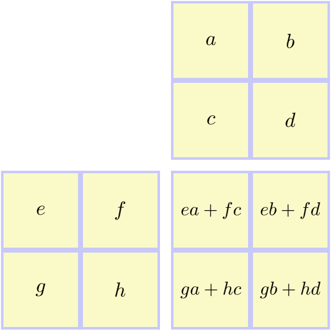
Naive algorithm requires $\Theta(n^3)$ elementary multiplications.

Divide and Conquer



Divide and Conquer

- Assumption $n = 2^k$.
- Number of elementary multiplications: $M(n) = 8M(n/2)$, $M(1) = 1$.
- yields $M(n) = 8^{\log_2 n} = n^{\log_2 8} = n^3$. No advantage ☹



Strassen's Matrix Multiplication

- Nontrivial observation by Strassen (1969): It suffices to compute the seven products $A = (e+h)\cdot(a+d)$, $B = (g+h)\cdot a$, $C = e\cdot(b-d)$, $D = h\cdot(c-a)$, $E = (e+f)\cdot d$, $F = (g-e)\cdot(a+b)$, $G = (f-h)\cdot(c+d)$. Because:
  $ea + fc = A + D - E + G$, $eb + fd = C + E$,
  $ga + hc = B + D$, $gb + hd = A - B + C + F$.
- This yields $M'(n) = 7M(n/2)$, $M'(1) = 1$. Thus $M'(n) = 7^{\log_2 n} = n^{\log_2 7} \approx n^{2.807}$.
- Fastest currently known algorithm: $\mathcal{O}(n^{2.37})$