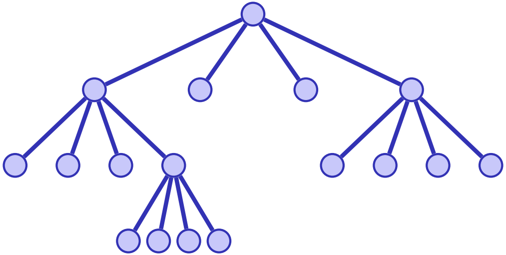
19. Quadtrees

Quadtrees, Collision Detection, Image Segmentation

Quadtree

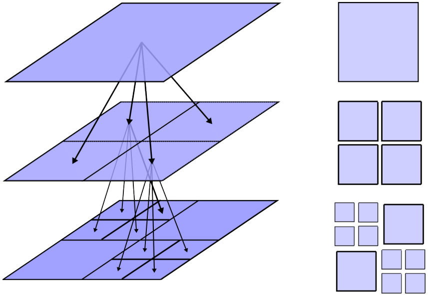
A quad tree is a tree of order 4.



... and as such it is not particularly interesting except when it is used for ...

Quadtree - Interpretation und Nutzen

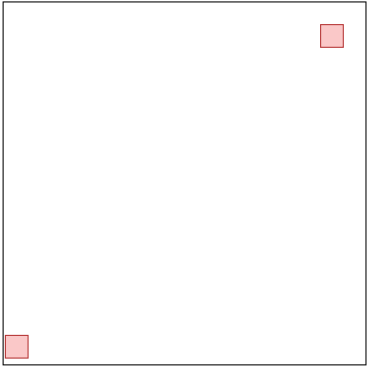
Separation of a two-dimensional range into 4 equally sized parts.



[analogously in three dimensions with an *octtree* (tree of order 8)]

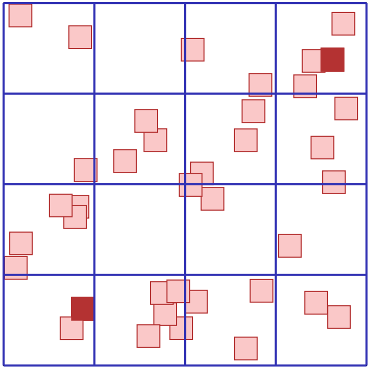
Example 1: Collision Detection

- Objects in the 2D-plane, e.g. particle simulation on the screen.
- Goal: collision detection



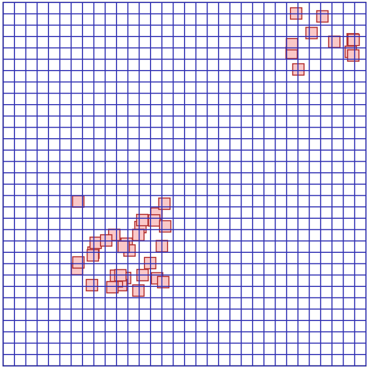
Idea

- Many objects: n^2 detections (naively)
- Improvement?
- Obviously: collision detection not required for objects far away from each other
- What is „far away"?
- Grid ($m \times m$)
- Collision detection per grid cell



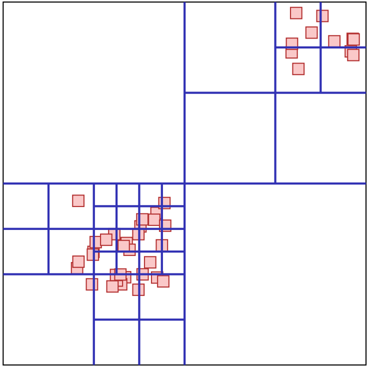
Grids

- A grid often helps, but not always
- Improvement?
- More finegrained grid?
- Too many grid cells!



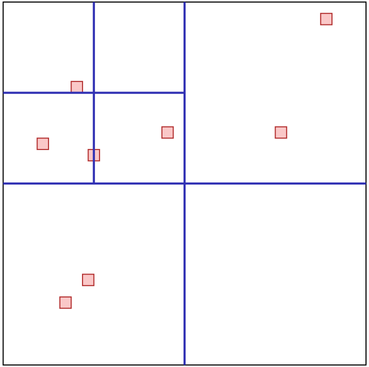
Adaptive Grids

- A grid often helps, but not always
- Improvement?
- *Adaptively* refine grid
- Quadtree!



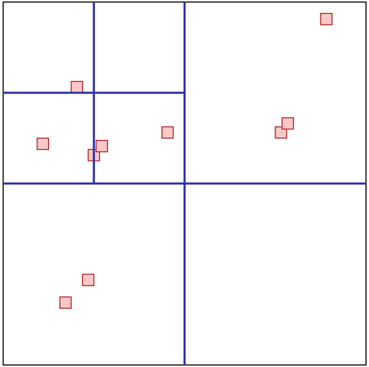
Algorithm: Insertion

- Quadtree starts with a single node
- Objects are added to the node. When a node contains too many objects, the node is split.
- Objects that are on the boundary of the quadtree remain in the higher level node.

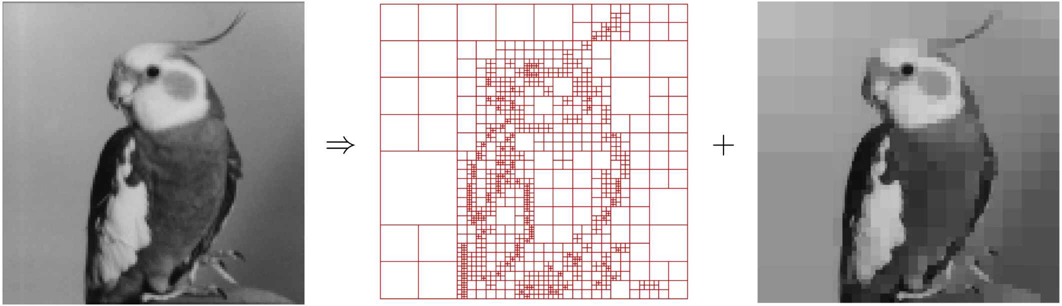


Algorithm: Collision Detection

- Run through the quadtree in a recursive way. For each node test collision with all objects contained in the same or (recursively) contained nodes.

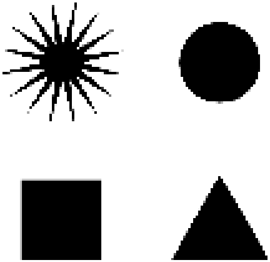


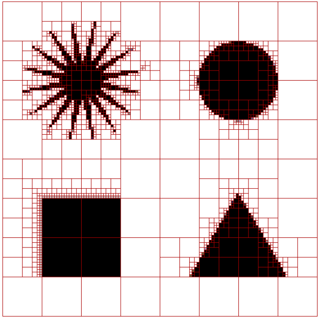
Example 2: Image Segmentation



(Possible applications: compression, denoising, edge detection)

Quadtree on Monochrome Bitmap

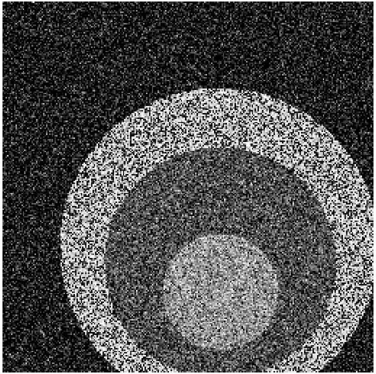


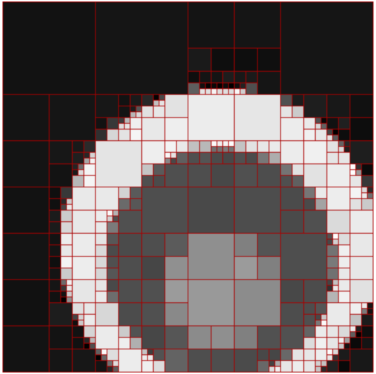


Similar procedure to generate the quadtree: split nodes recursively until each node only contains pixels of the same color.

Quadtree with Approximation

When there are more than two color values, the quadtree can get very large. ⇒ Compressed representation: *approximate* the image piecewise constant on the rectangles of a quadtree.





Piecewise Constant Approximation

(Grey-value) Image $\boldsymbol{y} \in \mathbb{R}^S$ on pixel indices S. [27]
Rectangle $r \subset S$.
Goal: determine

$$\arg\min_{v \in \mathbb{R}} \sum_{s \in r} (y_s - v)^2$$

Solution: the arithmetic mean $\mu_r = \frac{1}{|r|} \sum_{s \in r} y_s$

[27] we assume that S is a square with side length 2^k for some $k \geq 0$

Intermediate Result

The (w.r.t. mean squared error) best approximation

$$\mu_r = \frac{1}{|r|} \sum_{s \in r} y_s$$

and the corresponding error

$$\sum_{s \in r} (y_s - \mu_r)^2 =: \|\boldsymbol{y}_r - \boldsymbol{\mu}_r\|_2^2$$

can be computed quickly after a $\mathcal{O}(|S|)$ tabulation: prefix sums!

Which Quadtree?

Conflict

- **As close as possible to the data** $\Rightarrow$ small rectangles, large quadtree . Extreme case: one node per pixel. Approximation = original
- **Small amount of nodes** $\Rightarrow$ large rectangles, small quadtree Extreme case: a single rectangle. Approximation = a single grey value.

Which Quadtree?

Idea: choose between data fidelity and complexity with a regularisation parameter $\gamma \geq 0$

Choose quadtree T with leaves[28] $L(T)$ such that it minimizes the following function

$$H_\gamma(T, \boldsymbol{y}) := \gamma \cdot \underbrace{|L(T)|}_{\text{Number of Leaves}} + \underbrace{\sum_{r \in L(T)} \|y_r - \mu_r\|_2^2}_{\text{Cummulative approximation error of all leaves}} .$$

[28] here: leaf: node with null-children

Regularisation

Let T be a quadtree over a rectangle S_T and let $T_{ll}, T_{lr}, T_{ul}, T_{ur}$ be the four possible sub-trees and

$$\widehat{H}_\gamma(T, y) := \min_T \gamma \cdot |L(T)| + \sum_{r \in L(T)} \|y_r - \mu_r\|_2^2$$

Extreme cases:
$\gamma = 0 \Rightarrow$ original data;
$\gamma \to \infty \Rightarrow$ a single rectangle

Observation: Recursion

- If the (sub-)quadtree T represents only one pixel, then it cannot be split and it holds that
$$\widehat{H}_\gamma(T, \boldsymbol{y}) = \gamma$$
- Let, otherwise,

$$M_1 := \gamma + \|\boldsymbol{y}_{S_T} - \boldsymbol{\mu}_{S_T}\|_2^2$$
$$M_2 := \widehat{H}_\gamma(T_{ll}, \boldsymbol{y}) + \widehat{H}_\gamma(T_{lr}, \boldsymbol{y}) + \widehat{H}_\gamma(T_{ul}, \boldsymbol{y}) + \widehat{H}_\gamma(T_{ur}, \boldsymbol{y})$$

then
$$\widehat{H}_\gamma(T, y) = \min\{\underbrace{M_1(T, \gamma, \boldsymbol{y})}_{\text{no split}}, \underbrace{M_2(T, \gamma, \boldsymbol{y})}_{\text{split}}\}$$

Algorithmus: Minimize($\boldsymbol{y}$,r,γ)

Input: Image data $\boldsymbol{y} \in \mathbb{R}^S$, rectangle $r \subset S$, regularization $\gamma > 0$
Output: $\min_T \gamma|L(T)| + \|\boldsymbol{y} - \boldsymbol{\mu}_{L(T)}\|_2^2$

if $|r| = 0$ **then return** 0
$m \leftarrow \gamma + \sum_{s \in r} (y_s - \mu_r)^2$
if $|r| > 1$ **then**
   Split r into r_{ll},r_{lr},r_{ul},r_{ur}
   $m_1 \leftarrow \text{Minimize}(\boldsymbol{y}, r_{ll}, \gamma)$; $m_2 \leftarrow \text{Minimize}(\boldsymbol{y}, r_{lr}, \gamma)$
   $m_3 \leftarrow \text{Minimize}(\boldsymbol{y}, r_{ul}, \gamma)$; $m_4 \leftarrow \text{Minimize}(\boldsymbol{y}, r_{ur}, \gamma)$
   $m' \leftarrow m_1 + m_2 + m_3 + m_4$
else
   $m' \leftarrow \infty$
if $m' < m$ **then** $m \leftarrow m'$
return m

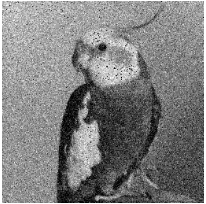
Analysis

The minimization algorithm over dyadic partitions (quadtrees) takes $\mathcal{O}(|S| \log |S|)$ steps.

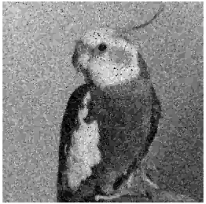
Application: Denoising (with addditional Wedgelets)

noised

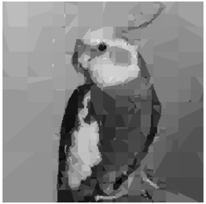
$\gamma = 0.003$



$\gamma = 0.01$

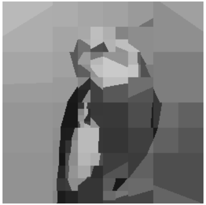


$\gamma = 0.03$



$\gamma = 0.1$

$\gamma = 0.3$

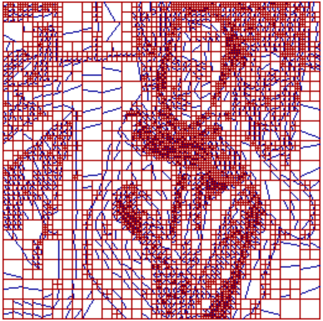


$\gamma = 1$

$\gamma = 3$

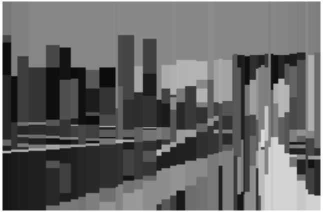
$\gamma = 10$

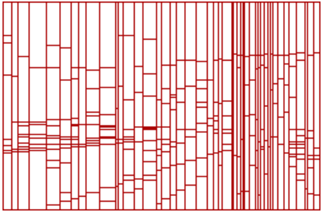
Extensions: Affine Regression + Wedgelets



Other ideas

no quadtree: hierarchical one-dimensional modell (requires dynamic programming)





19.1 Appendix

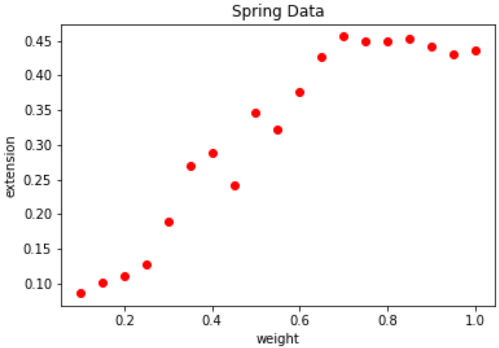
Linear Regression

The Learning Problem

Setup

- We observe N data points
- Input examples: $\boldsymbol{X} = (\boldsymbol{X}_1, \dots, \boldsymbol{X}_N)^\top$
- Output examples: $\boldsymbol{y} = (y_1, \dots, y_N)^\top$
- Assupmtion: there is an underlying truth

$$f : \mathcal{X} \to \mathcal{Y}$$



Goal: find a good approximation $h \approx g$ to make predictions $h(\boldsymbol{x})$ for new data points or to explain the data in order to find a compressed representation, for instance.

Here $\mathcal{X} = \mathbb{R}^d$. $\mathcal{Y} = \mathbb{R}$ (Regression).

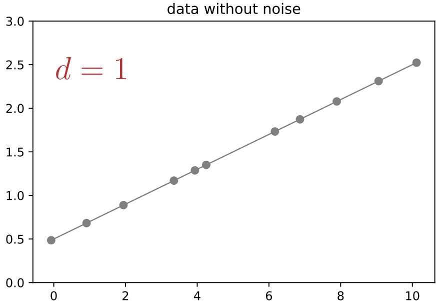
Model: Linear Regression

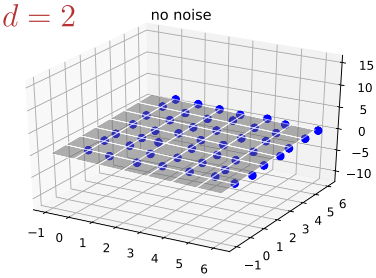
Assumption: The underlying truth can be represented as

$$h_{\boldsymbol{w}}(\boldsymbol{x}) = w_0 + w_1 x_1 + \cdots + w_d x_d = w_0 + \sum_{i=1}^{d} w_i x_i.$$

linear in $\boldsymbol{w}$!

$\Rightarrow$ We search for $\boldsymbol{w}$ (sometimes also d).





Trick for simplified notation

$$\boldsymbol{x} = (x_1, \ldots, x_d) \to (\underbrace{x_0}_{\equiv 1}, x_1, \ldots, x_d)$$

$$\begin{aligned} h_{\boldsymbol{w}}(\boldsymbol{x}) &= w_0 x_0 + w_1 x_1 + \cdots + w_d x_d \\ &= \sum_{i=0}^{d} w_i x_i \\ &= \boldsymbol{w}^\top \boldsymbol{x} \end{aligned}$$

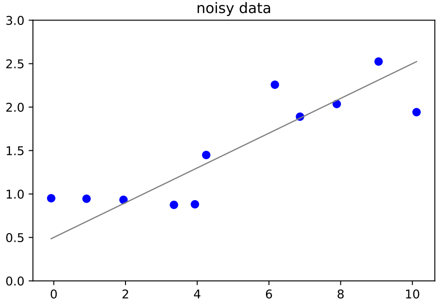
Data matrix

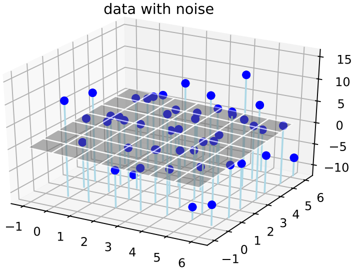
$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \\ \vdots \\ \boldsymbol{X}_n \end{bmatrix} = \begin{bmatrix} \overset{\equiv 1}{\overset{\downarrow}{X_{1,0}}} & X_{1,1} & X_{1,2} & \dots & X_{1,d} \\ X_{2,0} & X_{2,1} & X_{2,2} & \dots & X_{2,d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ X_{n,0} & X_{n,1} & X_{n,2} & \dots & X_{n,d} \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \quad \boldsymbol{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_d \end{bmatrix}$$

$$\boldsymbol{X}\boldsymbol{w} \approx \boldsymbol{y}?$$

Imprecise observations

Reality: the data are imprecise or the model is only a model.





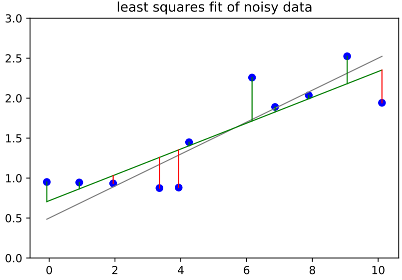
What to do?

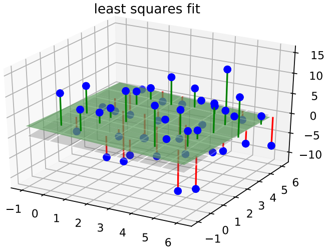
Error function

$$E(\boldsymbol{w}) = \sum_{i=1}^{N} (h_{\boldsymbol{w}}(\boldsymbol{X}_i) - y_i)^2$$

Want a $\widehat{\boldsymbol{w}}$ that minimizes E

Linarity of $h_{\boldsymbol{w}}$ in $\boldsymbol{w}$ $\Rightarrow$ solution with linear algebra.





Solution from Linear Algebra

$$\widehat{\boldsymbol{w}} = \underbrace{\left(\boldsymbol{X}^\top \boldsymbol{X}\right)^{-1} \boldsymbol{X}^\top}_{=:\boldsymbol{X}^\dagger} \boldsymbol{y}.$$

$\boldsymbol{X}^\dagger$: *Moore-Penroe Pseudo-Inverse*

Fitting Polynomials

Also works with linear regression.

$$h_{\boldsymbol{w}}(x) = w_0 + w_1 x^1 + w_2 x^2 + \cdots + w_d x^d = w_0 + \sum_{i=1}^{d} w_i x^i.$$

because $h_{\boldsymbol{w}}(x)$ remains being linear in $\boldsymbol{w}$!

$$\boldsymbol{X} = \begin{bmatrix} 1 & x_1 & (x_1)^2 & \dots & (x_1)^d \\ 1 & x_2 & (x_2)^2 & \dots & (x_2)^d \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & (x_n)^2 & \dots & (x_n)^d \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \qquad \boldsymbol{w} = \begin{bmatrix} w_0 \\ \vdots \\ w_d \end{bmatrix}$$

Example: Constant Approximation

$$\boldsymbol{X} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \qquad \boldsymbol{w} = \begin{bmatrix} w_0 \end{bmatrix}$$

$$\widehat{\boldsymbol{w}} = \left(\boldsymbol{X}^\top \boldsymbol{X}\right)^{-1} \boldsymbol{X}^\top \boldsymbol{y} = \left[\frac{1}{n} \sum y_i\right].$$

Example: Linear Approximation

$$\boldsymbol{X} = \begin{bmatrix} 1 & x_1^{(1)} & x_1^{(2)} \\ 1 & x_2^{(1)} & x_2^{(2)} \\ \vdots & & \\ 1 & x_n^{(1)} & x_n^{(2)} \end{bmatrix}, \quad \boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}, \qquad \boldsymbol{w} = \begin{bmatrix} w_0 \end{bmatrix}$$

$$\widehat{\boldsymbol{w}} = \left(\boldsymbol{X}^\top \boldsymbol{X}\right)^{-1} \boldsymbol{X}^\top \boldsymbol{y} = \begin{bmatrix} N & \sum x_i^{(1)} & \sum x_i^{(2)} \\ \sum x_i^{(1)} & \sum \left(x_i^{(1)}\right)^2 & \sum x_i^{(1)} \cdot x_i^{(2)} \\ \sum x_i^{(2)} & \sum x_i^{(1)} \cdot x_i^{(2)} & \sum \left(x_i^{(2)}\right)^2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} \sum y_i \\ \sum y_i \cdot x_i^{(1)} \\ \sum y_i \cdot x_i^{(2)} \end{bmatrix}$$