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Data Structures and Algorithms

Course at D-MATH of ETH Zurich

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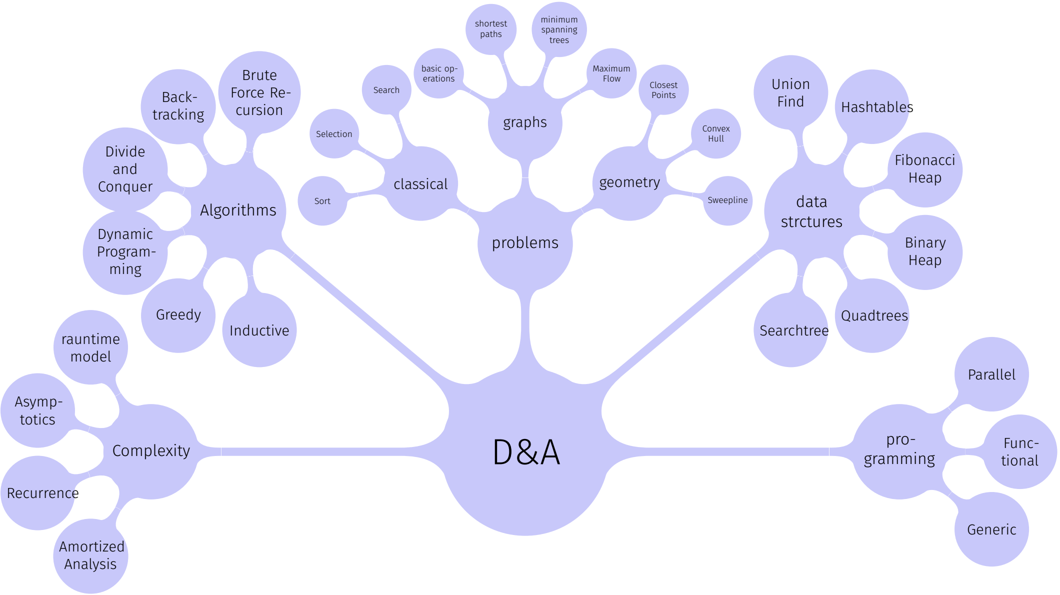
1. Introduction

Overview, Algorithms and Data Structures, Correctness, First Example

Goals of the course

- Understand the design and analysis of fundamental algorithms and data structures.
- An advanced insight into a modern programming model (with C++).
- Knowledge about chances, problems and limits of the parallel and concurrent computing.


D&A
Complexity
rauntime model
Asymp-totics
Recurrence
Amortized Analysis
Algorithms
Back-tracking
Brute Force Re-cursion
Divide and Conquer
Dynamic Program-ming
Greedy
Inductive
problems
classical
Selection
Search
Sort
graphs
basic op-erations
shortest paths
minimum spanning trees
Maximum Flow
geometry
Closest Points
Convex Hull
Sweepline
data strctures
Union Find
Hashtables
Fibonacci Heap
Binary Heap
Quadtrees
Searchtree
pro-gramming
Parallel
Func-tional
Generic

1.2 Algorithms

[Cormen et al, Kap. 1; Ottman/Widmayer, Kap. 1.1]

Algorithm

Algorithm
Well-defined procedure to compute **output** data from **input** data

Example Problem: Sorting

Input: A sequence of n numbers (comparable objects) $(a_1, a_2, \ldots, a_n)$
Output: Permutation $(a'_1, a'_2, \ldots, a'_n)$ of the sequence $(a_i)_{1 \le i \le n}$, such that $a'_1 \le a'_2 \le \cdots \le a'_n$

Possible input

$(1, 7, 3)$, $(15, 13, 12, -0.5)$, $(999, 998, 997, 996, \ldots, 2, 1)$, (1), $()$ …

Every example represents a **problem instance**

The performance (speed) of an algorithm usually depends on the problem instance. Often there are “good” and “bad” instances.

Therefore we consider algorithms sometimes **”in the average“** and most often in the **”worst case“**.

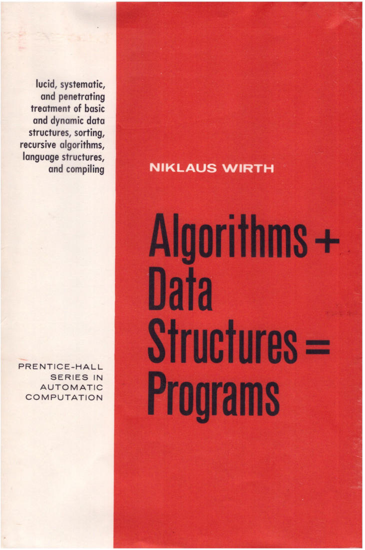
Possible solution

How many times are the lines executed each?

```
void sort(std::vector<int>& a){
  unsigned n = a.size()
  for (unsigned i = 0; i<n ; ++i){
    for (unsigned j = i+1; j<n; ++j){
      if (a[j] < a[i]){
        std::swap(a[i],a[j])
      }
    }
  }
}
```

Data Structures

- A data structure is a particular way of **organizing data** in a computer so that they can be **used efficiently** (in the algorithms operating on them).
- Programs = algorithms + data structures.



Typical Algorithm Design Steps: Example

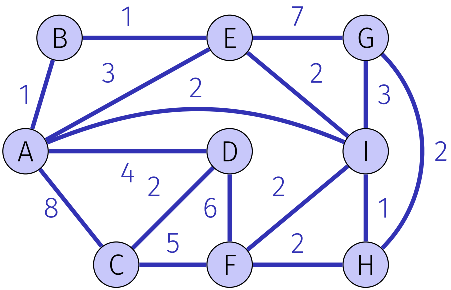
Route planning

Typical Design Steps

1. Specification of the problem: find best (shortest time) path from A to B
2. Abstraction: graph with nodes, edges and egde-weights
3. Idea (heureka!): Dijkstra
4. Data-structures and algorithms: e.g. adjacency matrix / adjacency list, min-heap, hash-table ...
5. Runtime analysis: $\mathcal{O}((n+m)\cdot\log n)$
6. Implementation: Representation choice (e.g. adjacency matrix/ adjacency list/ objects)

Difficult Problem: Travelling Salesman

Given: graph (map) with nodes (cities) and weighted edges (roads with length)
Wanted: Loop road through all cities such that each city is visited once (Hamilton-cycle) with minimal overall length.



The best known algorithm has a running time that increase exponentially with the number of nodes (cities).

Already finding a Hamilton cycle is a difficult problem in general. In contrast, the problem to find an Eulerian cycle, a cycle that uses each *edge* once, is a problem with polynomial running time.

Hard problems.

- NP-complete problems: no known efficient solution (the existence of such a solution is very improbable – but it has not yet been proven that there is none!)
- Example: travelling salesman problem

This course is *mostly* **about problems that can be solved efficiently (in polynomial time).**

Efficiency

Resources are bounded and do not come for free:

- Computing time $\rightarrow$ Efficiency
- Storage space $\rightarrow$ Efficiency

Actually, this course is nearly only about efficiency.

2. Efficiency of algorithms

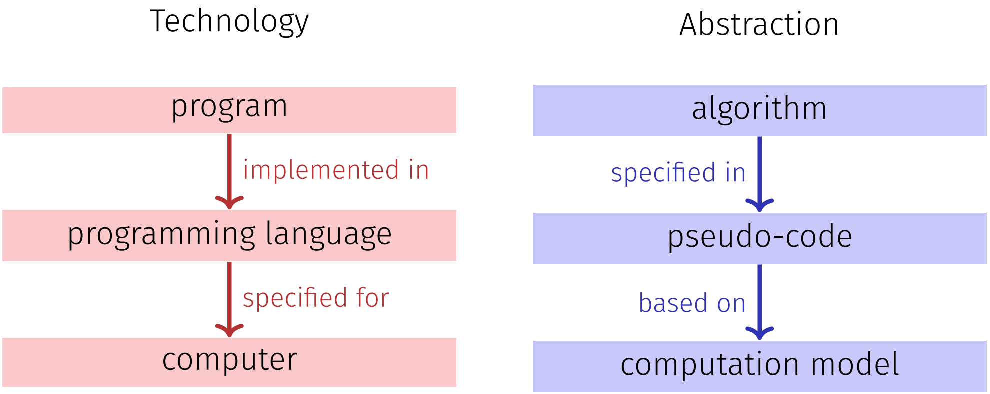
Efficiency of Algorithms, Random Access Machine Model, Function Growth, Asymptotics [Cormen et al, Kap. 2.2,3,4.2-4.4 | Ottman/Widmayer, Kap. 1.1]

Efficiency of Algorithms

Goals

- Quantify the runtime behavior of an algorithm independent of the machine.
- Compare efficiency of algorithms.
- Understand dependece on the input size.

Programs and Algorithms



Technology Model

Random Access Machine (RAM) Model

- Execution model: instructions are executed one after the other (on one processor core).
- Memory model: constant access time (big array)
- Fundamental operations: computations ($+,-,\cdot,\ldots$) comparisons, assignment / copy on machine words (registers), flow control (jumps)
- Unit cost model: fundamental operations provide a cost of 1.
- Data types: fundamental types like size-limited integer or floating point number.

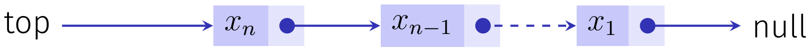
Size of the Input Data

- Typical: number of input objects (of fundamental type).
- Sometimes: number bits for a *reasonable* / *cost-effective* representation of the data.
- fundamental types fit into word of size : $w \geq \log(\text{sizeof(mem)})$ bits.

For Dynamic Data Strcutures

Pointer Machine Model

- Objects bounded in size can be dynamically allocated in constant time
- Fields (with word-size) of the objects can be accessed in constant time 1.



Asymptotic behavior

An exact running time of an algorithm can normally not be predicted even for small input data.

- We consider the asymptotic behavior of the algorithm.
- And ignore all constant factors.

An operation with cost 20 is no worse than one with cost 1
Linear growth with gradient 5 is as good as linear growth with gradient 1.

Algorithms, Programs and Execution Time

Program: concrete implementation of an algorithm.

Execution time of the program: measurable value on a concrete machine. Can be bounded from above and below.

Example 1

3GHz computer. Maximal number of operations per cycle (e.g. 8). $\Rightarrow$ lower bound.

A single operations does never take longer than a day $\Rightarrow$ upper bound.

From the perspective of the *asymptotic behavior* of the program, the bounds are unimportant.

2.2 Function growth

$\mathcal{O}$, Θ, Ω [Cormen et al, Kap. 3; Ottman/Widmayer, Kap. 1.1]

Superficially

Use the asymptotic notation to specify the execution time of algorithms.
We write $\Theta(n^2)$ and mean that the algorithm behaves for large n like n^2:
when the problem size is doubled, the execution time multiplies by four.

More precise: asymptotic upper bound

provided: a function $g : \mathbb{N} \to \mathbb{R}$.
Definition:[1]

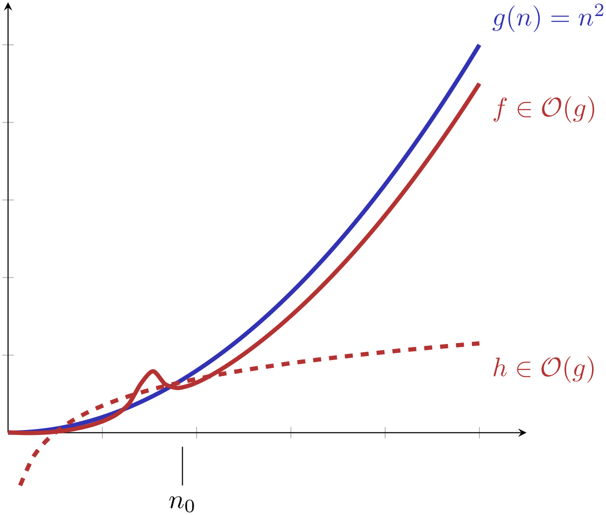
$$\begin{aligned}\mathcal{O}(g) = \{f : \mathbb{N} \to \mathbb{R}| \\ &\exists\, c > 0, \exists n_0 \in \mathbb{N} : \\ &\forall\, n \geq n_0 : 0 \leq f(n) \leq c \cdot g(n)\}\end{aligned}$$

Notation:

$$\mathcal{O}(g(n)) := \mathcal{O}(g(\cdot)) = \mathcal{O}(g).$$

[1] Ausgesprochen: Set of all functions $f : \mathbb{N} \to \mathbb{R}$ that satisfy: there is some (real valued) $c > 0$ and some $n_0 \in \mathbb{N}$ such that $0 \leq f(n) \leq n \cdot g(n)$ for all $n \geq n_0$.

Graphic

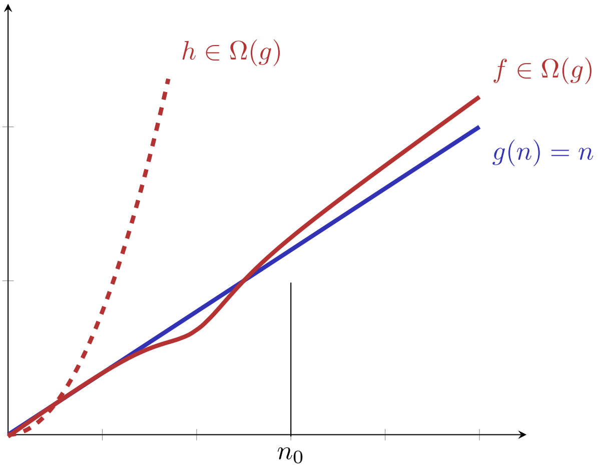


Converse: asymptotic lower bound

Given: a function $g : \mathbb{N} \to \mathbb{R}$.
Definition:

$$\begin{aligned}\Omega(g) = \{f : \mathbb{N} \to \mathbb{R}| \\ &\exists\ c > 0, \exists n_0 \in \mathbb{N} : \\ &\forall\ n \geq n_0 : 0 \leq c \cdot g(n) \leq f(n)\}\end{aligned}$$

Example



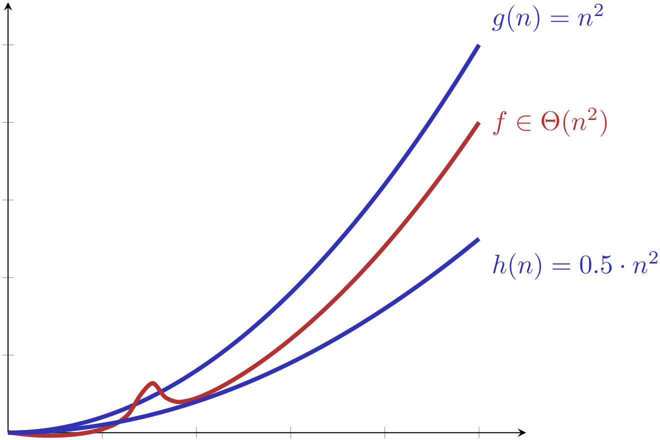
Asymptotic tight bound

Given: function $g : \mathbb{N} \to \mathbb{R}$.
Definition:

$$\Theta(g) := \Omega(g) \cap \mathcal{O}(g).$$

Simple, closed form: exercise.

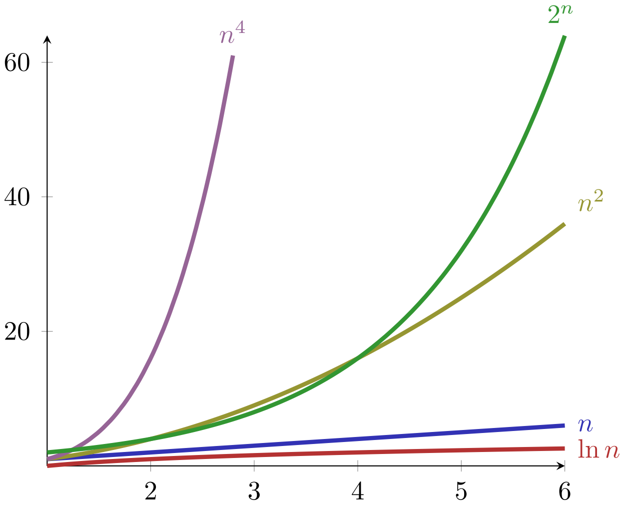
Example



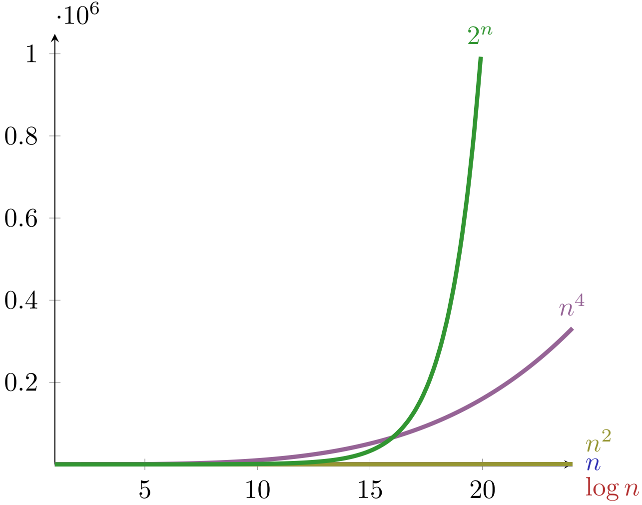
Notions of Growth

$\mathcal{O}(1)$	bounded	array access
$\mathcal{O}(\log \log n)$	double logarithmic	interpolated binary sorted sort
$\mathcal{O}(\log n)$	logarithmic	binary sorted search
$\mathcal{O}(\sqrt{n})$	like the square root	naive prime number test
$\mathcal{O}(n)$	linear	unsorted naive search
$\mathcal{O}(n \log n)$	superlinear / loglinear	good sorting algorithms
$\mathcal{O}(n^2)$	quadratic	simple sort algorithms
$\mathcal{O}(n^c)$	polynomial	matrix multiply
$\mathcal{O}(c^n)$	exponential	Travelling Salesman Dynamic Programming
$\mathcal{O}(n!)$	factorial	Travelling Salesman naively

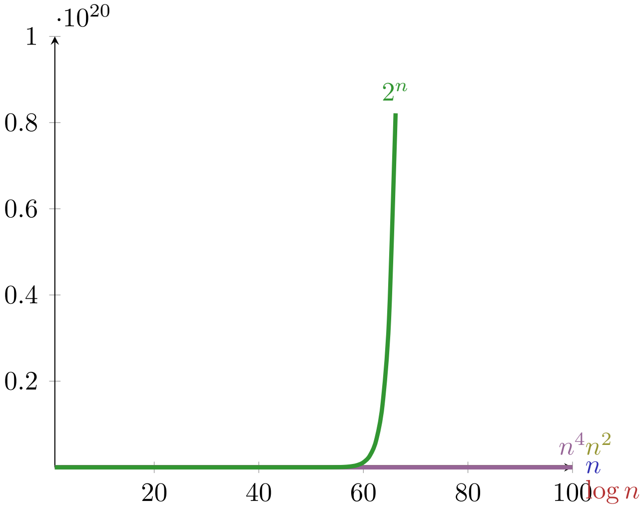
Small n



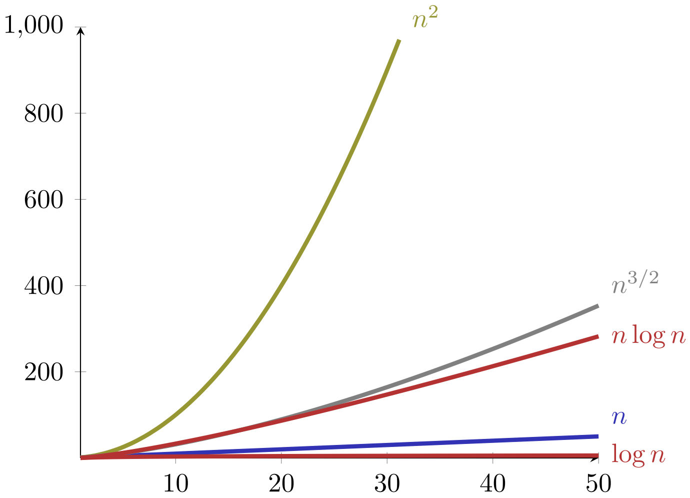
Larger n



“Large” n



Logarithms



Time Consumption

Assumption 1 Operation = $1\mu s$.

problem size	1	100	10000	10^6	10^9
$\log_2 n$	$1\mu s$	$7\mu s$	$13\mu s$	$20\mu s$	$30\mu s$
n	$1\mu s$	$100\mu s$	$1/100s$	$1s$	17 minutes
$n \log_2 n$	$1\mu s$	$700\mu s$	$13/100\mu s$	$20s$	8.5 hours
n^2	$1\mu s$	$1/100s$	1.7 minutes	11.5 days	317 centuries
2^n	$1\mu s$	10^{14} centuries	$\approx \infty$	$\approx \infty$	$\approx \infty$

Useful Tool

Theorem 2

Let $f, g : \mathbb{N} \to \mathbb{R}^+$ be two functions, then it holds that

1. $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0 \Rightarrow f \in \mathcal{O}(g)$, $\mathcal{O}(f) \subsetneq \mathcal{O}(g)$.
2. $\lim_{n\to\infty} \frac{f(n)}{g(n)} = C > 0$ *(C constant)* $\Rightarrow f \in \Theta(g)$.
3. $\frac{f(n)}{g(n)} \underset{n\to\infty}{\longrightarrow} \infty \Rightarrow g \in \mathcal{O}(f)$, $\mathcal{O}(g) \subsetneq \mathcal{O}(f)$.

About the Notation

Common casual notation

$$f = \mathcal{O}(g)$$

should be read as $f \in \mathcal{O}(g)$.
Clearly it holds that

$$f_1 = \mathcal{O}(g), f_2 = \mathcal{O}(g) \not\Rightarrow f_1 = f_2!$$

$n = \mathcal{O}(n^2), n^2 = \mathcal{O}(n^2)$ but naturally $n \neq n^2$.

We avoid this notation where it could lead to ambiguities.

Reminder: Efficiency: Arrays vs. Linked Lists

- Memory: our **`avec`** requires roughly n ints (vector size n), our **`llvec`** roughly $3n$ ints (a pointer typically requires 8 byte)
- Runtime (with **`avec`** = **`std::vector`**, **`llvec`** = **`std::list`**):

```
prepending (insert at front) [100,000x]:
    ► avec:     675 ms
    ► llvec:     10 ms
appending (insert at back) [100,000x]:
    ► avec:       2 ms
    ► llvec:      9 ms
removing first [100,000x]:
    ► avec:     675 ms
    ► llvec:      4 ms
removing last [100,000x]:
    ► avec:       0 ms
    ► llvec:      4 ms
```

```
removing randomly [10,000x]:
    ► avec:       3 ms
    ► llvec:    113 ms
inserting randomly [10,000x]:
    ► avec:      16 ms
    ► llvec:    117 ms
fully iterate sequentially (5000 elements) [5,000x]:
    ► avec:     354 ms
    ► llvec:    525 ms
```

Asymptotic Runtimes

With our new language ($\Omega, \mathcal{O}, \Theta$), we can now **state the behavior of the data structures and their algorithms more precisely**

Typical asymptotic running times (Anticipation!)

Data structure	Random Access	Insert	Next	Insert After Element	Search
`std::vector`	$\Theta(1)$	$\Theta(1)\ A$	$\Theta(1)$	$\Theta(n)$	$\Theta(n)$
`std::list`	$\Theta(n)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(n)$
`std::set`	–	$\Theta(\log n)$	$\Theta(\log n)$	–	$\Theta(\log n)$
`std::unordered_set`	–	$\Theta(1)\ P$	–	–	$\Theta(1)\ P$

A = amortized, P=expected, otherwise worst case

Complexity

Complexity of a problem P

Minimal (asymptotic) costs over all algorithms A that solve P.

Complexity of the single-digit multiplication of two numbers with n digits is $\Omega(n)$ and $\mathcal{O}(n^{\log_3 2})$ (Karatsuba Ofman).

Complexity

$$
\begin{array}{llcccc}
\text{Problem} & \text{Complexity} & \mathcal{O}(n) & \mathcal{O}(n) & \mathcal{O}(n^2) & \Omega(n \log n) \\
 & & \Uparrow & \Uparrow & \Uparrow & \Downarrow \\
\text{Algorithm} & \text{Costs}^2 & 3n-4 & \mathcal{O}(n) & \Theta(n^2) & \Omega(n \log n) \\
 & & \Downarrow & \Updownarrow & \Updownarrow & \Downarrow \\
\text{Program} & \text{Execution time} & \Theta(n) & \mathcal{O}(n) & \Theta(n^2) & \Omega(n \log n)
\end{array}
$$

[2] Number fundamental operations