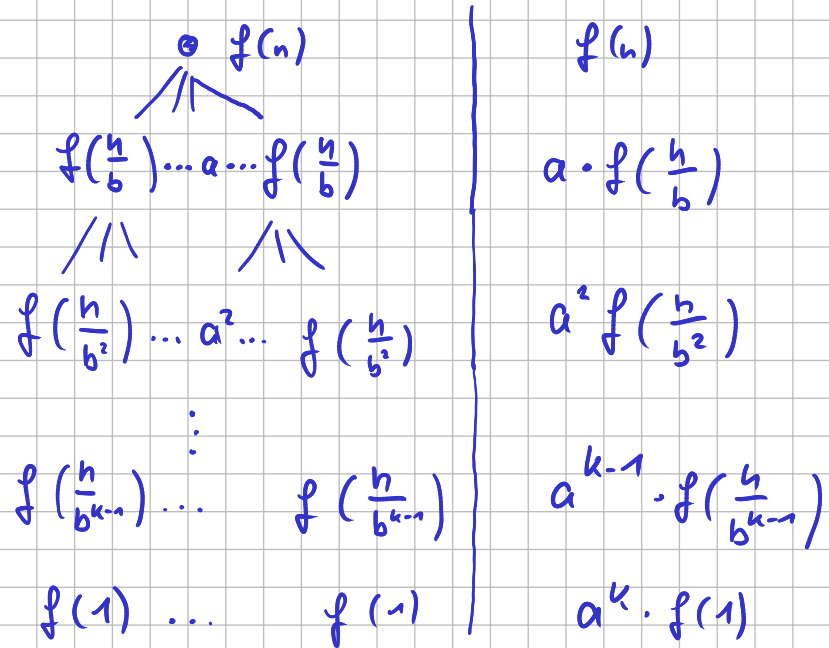


Mastertheorem

$$T(n) = \begin{cases} a T(\frac{n}{b}) + f(n) & n > 1 \\ f(1) & n = 1 \end{cases}$$

Simplification: assume $n = b^k$ ($k = \log_b n$), $k \in \mathbb{N}$

Recurrence Tree:



$$\Rightarrow T(n) = a^k f(1) + \sum_{i=0}^{k-1} a^i \cdot f\left(\frac{n}{b^i}\right) = a^k f(1) + \sum_{i=0}^{k-1} a^i \cdot f(b^{k-i})$$

① Assume $f(b^{k-i}) \cdot a^i = c \cdot a^k \Rightarrow T(n) = d \cdot a^k + c \cdot k \cdot a^k$
 $\in \Theta(\log_b n \cdot n^{\log_b a})$

$$\Downarrow$$
$$f(b^{k-i}) = c \cdot a^{k-i} \quad \forall i$$

$$\Downarrow$$
$$f(\underbrace{b^i}_m) = c \cdot a^i \quad \forall i : f(m) = c \cdot a^{\log_b m} \in \mathcal{O}(m^{\log_b a})$$

Example

Mergesort: $T(n) = 2 \cdot T(\frac{n}{2}) + cn \Rightarrow a=2, b=2$

$$f(n) = c \cdot n^{\log_2 2} = c \cdot n$$

$$\Rightarrow T(n) \in \mathcal{O}(n \log n)$$

$$\textcircled{2} \quad \text{If } f(n) \in \Theta(n^{\log_b a - \epsilon}) \quad \epsilon > 0$$

$$T(n) \geq a^k \cdot d$$

$$\begin{aligned} T(n) &\leq a^k \cdot d + c \sum_{i=0}^{k-1} (b^{k-i})^{\log_b a - \epsilon} \cdot a^i \\ &= d \cdot a^k + c \cdot b^{k(\log_b a - \epsilon)} \cdot c \cdot \sum_{i=0}^{k-1} \frac{a^i}{b^{i(\log_b a - \epsilon)}} \} = \frac{a^i}{b^{i\epsilon}} \\ &= d \cdot a^k + c \cdot a^k \cdot b^{-k\epsilon} \cdot \underbrace{\sum_{i=0}^{k-1} (b^\epsilon)^i}_{\frac{b^{k\epsilon} - 1}{b^\epsilon - 1}} \\ &\in \Theta(n^{\log_b a}) \end{aligned}$$

$$\Rightarrow T(n) \in \Theta(n^{\log_b a})$$

Example : Karatsuba : $T(n) = 3 \cdot T(\frac{n}{2}) + cn$

$$a=3, b=2 \quad f(n) \in \Theta(n^{\overbrace{\log_2 3}^{>1} - \epsilon})$$

$$\Rightarrow T(n) \in \Theta(n^{\log_2 3})$$

$$\textcircled{3} \quad f(n) \in \Theta(n^{\log_b a}) \Rightarrow T(n) \in \Theta(\log n \cdot n^{\log_b a})$$

$$\exists \epsilon > 0 : f(n) \in \Theta(n^{\log_b a - \epsilon}) \Rightarrow T(n) \in \Theta(n^{\log_b a})$$

$$f(n) \in \Omega(n^{\log_b a + \epsilon}) \quad \bullet \text{ Book}$$