# Mastertheorem

$$T(n) = \begin{cases} a\,T\left(\frac{n}{b}\right) + f(n) & n > 1 \\ f(1) & n = 1 \end{cases}$$

**Simplification:** assume $n = b^k$ $\quad (k = \log_b n)$, $k \in \mathbb{N}$

**Recurrence Tree:**

$$\begin{array}{ccc|c}
 & f(n) & & f(n) \\
f\left(\frac{n}{b}\right) & \cdots a \cdots & f\left(\frac{n}{b}\right) & a \cdot f\left(\frac{n}{b}\right) \\
f\left(\frac{n}{b^2}\right) & \cdots a^2 \cdots & f\left(\frac{n}{b^2}\right) & a^2 f\left(\frac{n}{b^2}\right) \\
 & \vdots & & \\
f\left(\frac{n}{b^{k-1}}\right) & \cdots & f\left(\frac{n}{b^{k-1}}\right) & a^{k-1} \cdot f\left(\frac{n}{b^{k-1}}\right) \\
f(1) & \cdots & f(1) & a^k \cdot f(1)
\end{array}$$

$$\Rightarrow \quad T(n) = a^k f(1) + \sum_{i=0}^{k-1} a^i \cdot f\left(\frac{n}{b^i}\right) = a^k f(1) + \sum_{i=0}^{k-1} a^i \cdot f(b^{k-i})$$

(1) Assume $f(b^{k-i}) \cdot a^i = c \cdot a^k \quad \Rightarrow T(n) = d \cdot a^k + c \cdot k \cdot a^k$

$$\in \Theta(\log_b n \cdot n^{\log_b a})$$

$$\Updownarrow$$

$$f(b^{k-i}) = c \cdot a^{k-i} \quad \forall i$$

$$\Updownarrow$$

$$f(\underbrace{b^i}_{m}) = c \cdot a^i \quad \forall i \; : \; f(m) = c \cdot a^{\log_b m}$$

$$\in \Theta(m^{\log_b a})$$

**Example**

Mergesort: $T(n) = 2 \cdot T\left(\frac{n}{2}\right) + cn \Rightarrow a = 2,\ b = 2$

$$f(n) = c \cdot n^{\log_b a} = c \cdot n$$

$$\Rightarrow T(n) \in \Theta(n \log n)$$

② If $f(n) \in \mathcal{O}(n^{\log_b a - \varepsilon})$ $\quad \varepsilon > 0$

$$T(n) \geq a^k \cdot d$$

$$T(n) \leq a^k \cdot d + c \sum_{i=0}^{k-1} (b^{k-i})^{\log_b a - \varepsilon} \cdot a^i$$

$$= d \cdot a^k + c \cdot b^{k(\log_b a - \varepsilon)} \cdot c \cdot \sum_{i=0}^{k-1} \frac{a^i}{b^{i(\log_b a - \varepsilon)}} \Big\} = a^i / b^{i\varepsilon}$$

$$= d \cdot a^k + c \cdot a^k \cdot b^{-k\varepsilon} \cdot \underbrace{\sum_{i=0}^{k-1} (b^{\varepsilon})^i}_{\frac{b^{k\varepsilon} - 1}{b^{\varepsilon} - 1}}$$

$$\in \mathcal{O}(n^{\log_b a})$$

$$\Rightarrow T(n) \in \Theta(n^{\log_b a})$$

<u>Example</u> : Karatsuba : $T(n) = 3 \cdot T(\frac{n}{2}) + cn$

$a = 3, \; b = 2 \quad f(n) \in \mathcal{O}(n^{\overbrace{\log_2 3}^{>1} - \varepsilon})$

$\Rightarrow T(n) \in \Theta(n^{\log_2 3})$

③ $f(n) \in \Theta(n^{\log_b a}) \Rightarrow T(n) \in \Theta(\log n \cdot n^{\log_b a})$

$\exists \varepsilon > 0 : f(n) \in \mathcal{O}(n^{\log_b a - \varepsilon}) \Rightarrow T(n) \in \Theta(n^{\log_b a})$

$f(n) \in \Omega(n^{\log_b a + \varepsilon})$ : Book