ETH zürich

Exercise Session 4

Data Structures and Algorithms, D-MATH, ETH Zurich

Program of today

Feedback of last exercise

Repetition theory

- Amortized Analysis
- Skip Lists

About the Bonus Task

Code-Example: Dynamic Vector

1. Feedback of last exercise

Throwing eggs

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  - Binary search. Worst case: $\log_2 n$ tries.
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  - Start from the bottom. n tries.

Throwing Eggs

Strategy using two eggs

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Strategy using two eggs

- First approach: intervals of equal length: partition n into k intervals: maximum number of trials $f(k) = k + n/k - 1$
  Minimize maximum number of trials: $f'(k) = 1 - n/k^2 = 0 \Rightarrow k = \sqrt{n}$.
  $n = 100 \Rightarrow 19$ Trials. $\Theta(\sqrt{n})$
- Second approach: take first throw trial into account by considering decreasing interval sizes. Choose smallest s such that $s + s - 1 + s - 2 + ... + 1 = s(s+1)/2 \geq 100 \Rightarrow s = 14$. Maximum number of trials: $s \in \Theta(\sqrt{n})$

Asymptotically both approaches are equally good. Practically the second way is better.

Selection algorithm

- What happens if many elements are equal?
- $99, 99, \ldots, 99$, Pivot 99, smaller partition is empty, larger $n - 1$ times 99
- May degrade runtime to n^2
- Solution?

Selection algorithm

- On equality with pivot, alternate between partitions

Selection algorithm

- On equality with pivot, alternate between partitions
- Modify algorithm to return number of elements equal to pivot

2.1 Amortized Analysis

Amortized Analysis

Three Methods

- Aggregate Analysis
- Account Method
- Potential Method

Example: simple dictionary

Supports operations insert and find. Idea:

- Collection of arrays A_i with Length 2^i
- Every array is either entirely empty or entirely full and stores items in a sorted order
- Between the arrays there is no further relationship

data $\{1, 8, 10, 18, 20, 24, 36, 48, 50, 75, 99\}$, $n = 11$

$$
\begin{aligned}
A_0&: \quad [50]\\
A_1&: \quad [8, 99]\\
A_2&: \quad \emptyset\\
A_3&: \quad [1, 10, 18, 20, 24, 36, 48, 75]
\end{aligned}
$$

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Algorithm `Find`: Run through all arrays and make a binary search each
Worst-case Runtime :

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A_3: $[1, 10, 18, 20, 24, 36, 48, 75]$

Algorithm **`Find`**: Run through all arrays and make a binary search each
Worst-case Runtime : $\Theta(\log^2 n)$,

$$\log 1 + \log 2 + \log 4 + \cdots + \log 2^k = \sum_{i=0}^{k} \log_2 2^i = \frac{k \cdot (k+1)}{2} \in \Theta(\log^2 n).$$

$(k = \lfloor \log_2 n \rfloor)$

Example: simple dictionary

Algorithm **`Insert(x)`**:

Example: simple dictionary

Algorithm **Insert(x)**:

- New array $A'_0 \leftarrow [x]$, $i \leftarrow 0$
- while $A_i \neq \emptyset$, set A'_{i+1} =Merge(A_i, A'_i), $A_i \leftarrow \emptyset$, $i \leftarrow i+1$
- Set $A_i \leftarrow A'_i$

Insert(11)

$$\begin{array}{ll} A_0\text{:} & [50] \\ A_1\text{:} & [8, 99] \\ A_2\text{:} & \emptyset \\ A_3\text{:} & [1, 10, 18, \ldots, 75] \end{array} \quad \begin{array}{ll} A'_0\text{:} & [11] \\ A'_1\text{:} & [11, 50] \\ A'_2\text{:} & [8, 11, 50, 99] \end{array} \quad \Rightarrow \quad \begin{array}{ll} A_0\text{:} & \emptyset \\ A_1\text{:} & \emptyset \\ A_2\text{:} & [8, 11, 50, 99] \\ A_3\text{:} & [1, 10, 18, \ldots, 75] \end{array}$$

Costs Insert

In the following: $n = 2^k$, $k = \log_2 n$

Assumption: creating new array A'_i with length 2^i (and, for $i > 0$ subsequent merge of A'_{i-1} and A_{i-1}) has costs $\Theta(2^i)$

In the worst case inserting an element into the data structure provides $\log_2 n$ such operations. $\Rightarrow$ **Worst-case Costs Insert**:

$$\sum_{i=0}^{k} 2^i = 2^{k+1} - 1 \in \Theta(n).$$

Aggregate Analysis

Level	Costs	Example Array
0	1	$[*]$
1	2	$[*, *]$
2	4	$[*, *, *, *]$
3	8	$\emptyset$
4	16	$[*, *, *, *, *, *, *, *, *, *, *, *, *, *, *, *]$

Observation: when you start with an empty container, an insertion sequence merges reaches level 0 each time, level 1 (with costs 2) every second time, level 2 (with costs 4) every fourth time, level 3 (with costs 8) every eighth time etc.

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Total costs: $1 \cdot \frac{n}{1} + 2 \cdot \frac{n}{2} + 4 \cdot \frac{n}{4} + \cdots + 2^k \cdot \frac{n}{2^k} = (k+1)n \in \Theta(n \log n)$.
Amortized cost per operation: $\Theta((n \log n)/n) = \Theta(\log n)$.

Account Method

- Every element i $(1 \leq i \leq n)$ pays $a_i = \log_2 n$ coins when it is inserted into the data structure.
- The element pays the allocation of the first array and every subsequent merge-step that can occur until the element has reached array A_{k+1} $(k = \lfloor \log_2 \rfloor n)$.
- The account provides enough credit to pay for all Merge operations of the n elements.

$\Rightarrow$ **Amortized costs** for insertion $\mathcal{O}(\log n)$

Potential Method

We know from the account method that **each element on the way to higher levels requires** $\log n$ **coins**, i.e. that an element on level i still needs to posess $k - i$ coins. We use the **potential**

$$\Phi_j = \sum_{0 \le i \le k : A_i \ne \emptyset} (k - i) \cdot 2^i$$

Potential Method

For the **change of the potential** $\Phi_j - \Phi_{j-1}$ we only have to consider the lower l levels that are occupied at time point $j-1$ (in analogy to the binary counter). Let l be the smallest index such that array A_l is empty.

After merging array $A_0 \dots A_{l-1}$ arrays $A_i, 0 \leq i < l$ are now empty and array A_l is now full. Therefore:

$$\Phi_j - \Phi_{j-1} = (k-l) \cdot 2^l - \sum_{i=0}^{l-1} (k-i) \cdot 2^i$$

Real costs:

$$t_j = \sum_{i=0}^{l} 2^i = 2^{l+1} - 1$$

Potential Method

$$
\begin{aligned}
\Phi_j - \Phi_{j-1} &= (k-l)\cdot 2^l - \sum_{i=0}^{l-1}(k-i)\cdot 2^i \\
&= (k-l)\cdot 2^l - k\cdot(2^l-1) + \sum_{i=0}^{l-1} i\cdot 2^i \\
&= (k-l)\cdot 2^l - k\cdot(2^l-1) + l\cdot 2^l - 2^{l+1} + 2 \\
&= k - 2^{l+1} + 2 \\
\Phi_j - \Phi_{j-1} + t_j &= k - 2^{l+1} + 2 + 2^{l+1} - 1 = k+1 \in \Theta(\log n)
\end{aligned}
$$

$\sum i \cdot \lambda^i$

Always the same trick:

$$\lambda \cdot \sum_{i=0}^{n} i \cdot \lambda^i - \sum_{i=0}^{n} i \cdot \lambda^i = \sum_{i=0}^{n} i \cdot \lambda^{i+1} - \sum_{i=0}^{n} i \cdot \lambda^i = \sum_{i=1}^{n+1} (i-1) \cdot \lambda^i - \sum_{i=0}^{n} i \cdot \lambda^i$$

$$= n \cdot \lambda^{n+1} + \sum_{i=1}^{n} (i-1) \cdot \lambda^i - i \cdot \lambda = n \cdot \lambda^{n+1} - \sum_{i=1}^{n} \lambda^i$$

$$= n \cdot \lambda^{n+1} - \frac{\lambda^{n+1} - 1}{\lambda - 1} + 1$$

$$(\lambda - 1) \cdot \sum_{i=0}^{n} i \cdot \lambda^i = n \cdot \lambda^{n+1} - \frac{\lambda^{n+1} - 1}{\lambda - 1} + 1$$

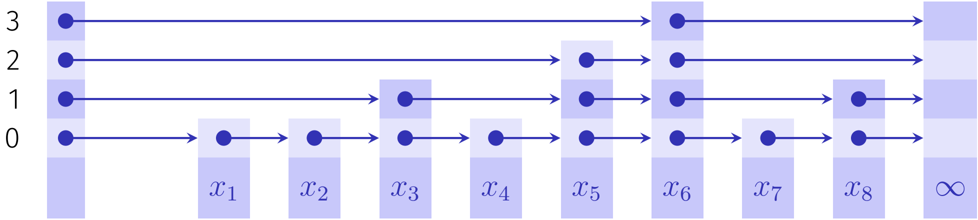
For $\lambda = 2$:

$$\sum_{i=0}^{n} i \cdot 2^i = n \cdot 2^{n+1} - 2^{n+1} + 1 + 1 = (n-1) \cdot 2^{n+1} + 2$$

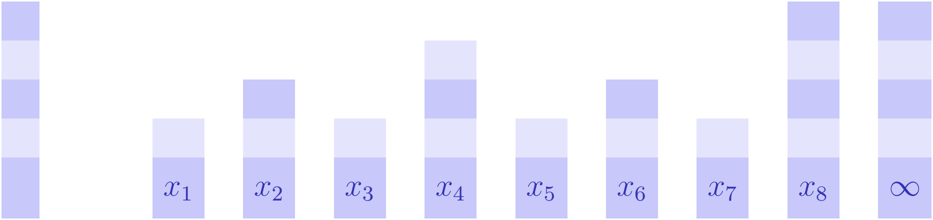
2.2 Skip Lists

Randomized Skip List

Idea: insert a key with random height H with $\mathbb{P}(H = i) = \frac{1}{2^{i+1}}$.

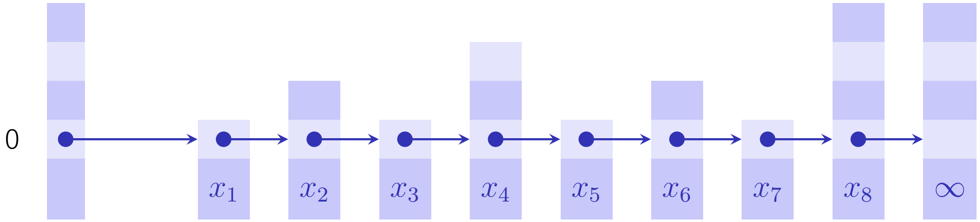


Randomized Skip List: finding element



$x_1 \leq x_2 \leq x_3 \leq \cdots \leq x_9.$

Randomized Skip List: finding element



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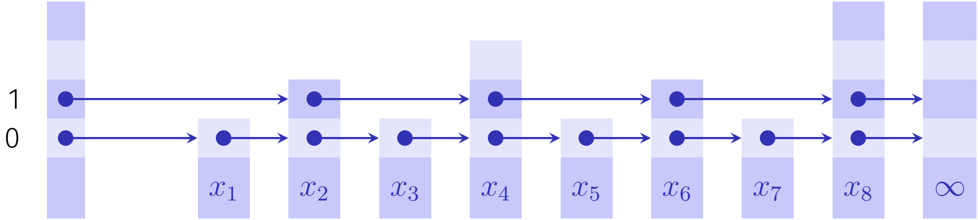

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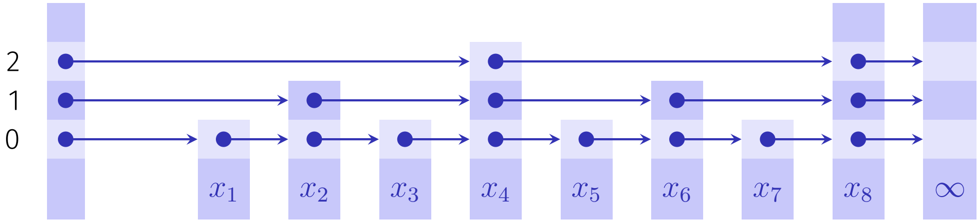

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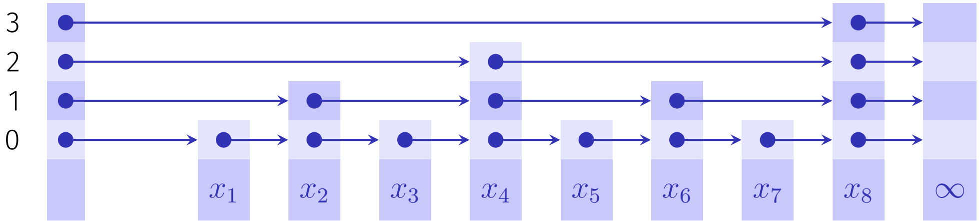

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Example: search for a key x with $x_5 < x < x_6$.

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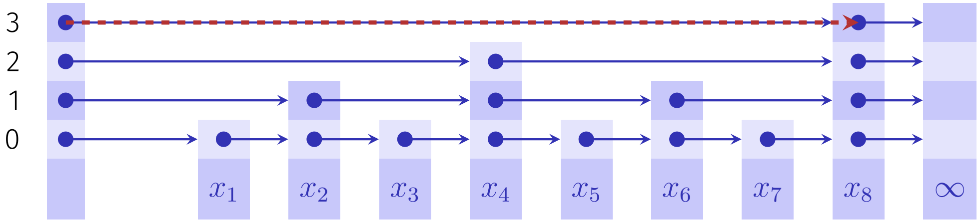

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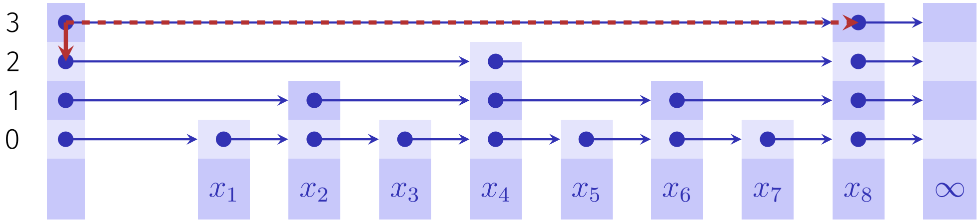

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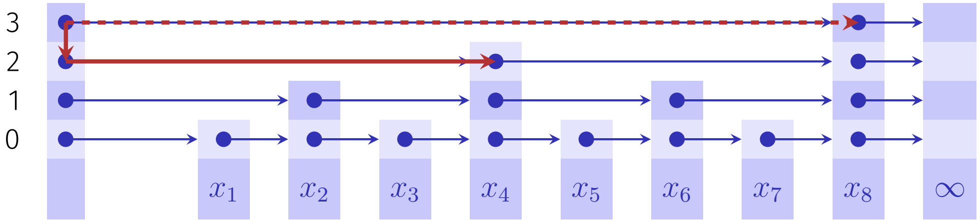

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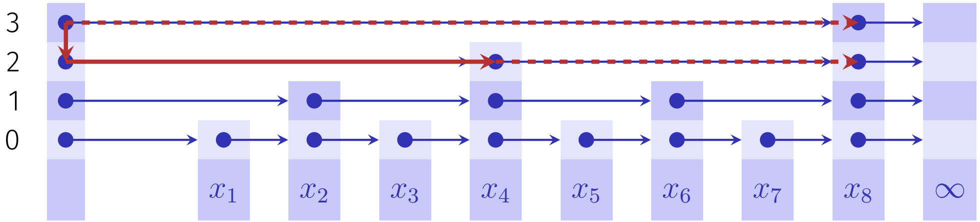

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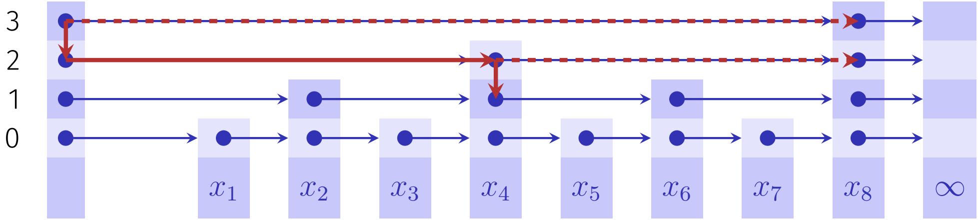

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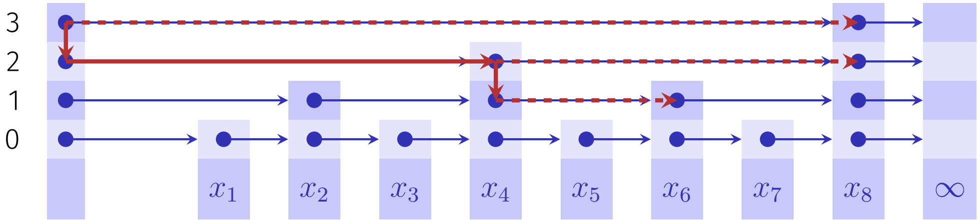

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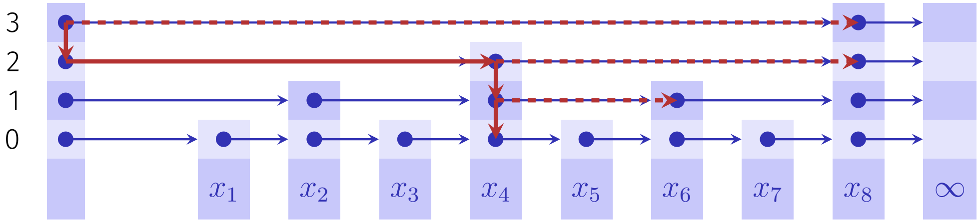

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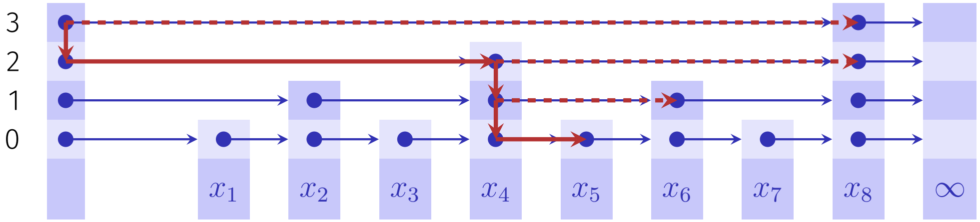

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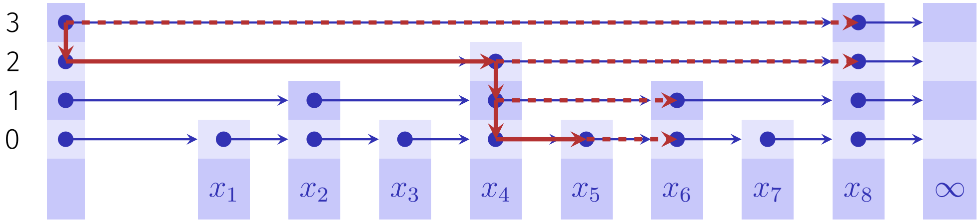

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Skip Lists Interface

```
template<typename T> class SkipList {
public:
  SkipList();
  ~SkipList();

  void insert(const T& value);
  void erase(const T& value);

  // iterator implementation ...
};
```

Partially implemented:

- A class `Node` saves an element `value` of type `T` and a `std::vector` called `forward` with pointers to successive nodes.
- First Node (without value): `head`.
- `forward[0]` points to the following element in the list.
- We use this in an already implemented iterator.

3. About the Bonus Task

Implementing `insert` and `erase`

`insert(const T& value)`

- create new node
- choose random number of levels
- for each level, find the first smaller node
- set pointers from previous nodes and new node

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- delete node if necessary

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- delete node if necessary

Warning: The same value can appear multiple times.

Recap dynamic allocated memory

Important: Every `new` needs its `delete` and only one!

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Therefore “Rule of three”:

- constructor
- copy constructor
- destructor

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Therefore “Rule of three”:

- constructor
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Being lazy “ Rule of two”:

- never copy (unsure)
- make copy constructor private (save) or deleted

4. Code-Example: Dynamic Vector

Preparation for Deque-Exercise

Questions?