ETH zürich

Exercise Session 1 – Asymptotics

Data Structures and Algorithms, D-MATH, ETH Zurich

Schedule for today

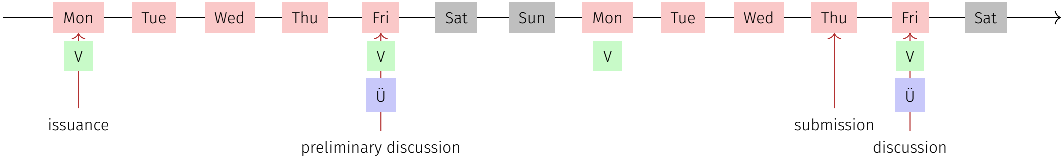
Exercise Process

Repetition Theory

- Examples (Theory)
- Aysmptotic Running Time of Program Parts

Programming Exercise

Process for the exercises



- Exercises availabe at lectures.
- Preliminary discussion in the following recitation session
- Solution of the exercise until the day before the next recitation session.
- Discussion of the exercise in the next recitation session.

2. Repetition Theory

Warm-up

- What is a problem?

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- What is an algorithm?

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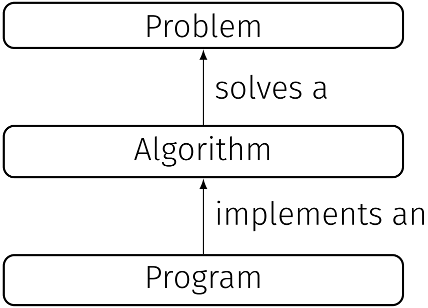
- What is a program?

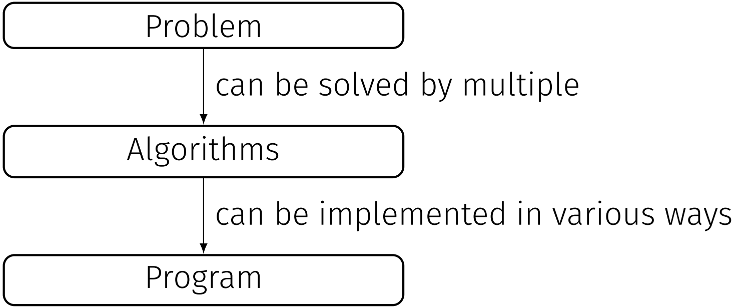
Warm-up

- What is a problem?
- What is an algorithm?
  - ➜ well-defined computing procedure to compute output data from input data.
- What is a program?
  - ➜ Concrete implementation of an algorithm

Problems, Algorithms and Programs



Warm-up



Problem
can be solved by multiple
Algorithms
can be implemented in various ways
Program

Efficiency

Program	Computing time	Measurable value on an actual machine.
Algorithm	Cost	Number of elementary operations
Problem	Complexity	Minimal (asymptotic) cost over all algorithms that solve the problem.

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Algorithm	Cost	Number of elementary operations
Problem	Complexity	Minimal (asymptotic) cost over all algorithms that solve the problem.

→ Estimation of cost or computing time depending on the input size, denoted by n.

Asymptotic behavior

- What are $\Omega(g(n))$, $\Theta(g(n))$, $\mathcal{O}(g(n))$?

Asymptotic behavior

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➜ Sets of functions!

Asymptotic behavior

- What are $\Omega(g(n))$, $\Theta(g(n))$, $\mathcal{O}(g(n))$?

➜ Sets of functions!

subset	$A \subseteq B$
proper subset	$A \subsetneq B$
intersection	$A \cap B$

Asymptotic behavior

Given: function $f : \mathbb{N} \to \mathbb{R}$.

Definition:

$$\mathcal{O}(g) = \{f : \mathbb{N} \to \mathbb{R} | \exists c > 0, n_0 \in \mathbb{N} | \forall n \geq n_0 : 0 \leq f(n) \leq c \cdot g(n)\}$$
$$\Omega(g) = \{f : \mathbb{N} \to \mathbb{R} | \exists c > 0, n_0 \in \mathbb{N} | \forall n \geq n_0 : 0 \leq c \cdot g(n) \leq f(n)\}$$
$$\Theta(g) = \mathcal{O}(g) \cap \Omega(g)$$

Intuition:
$f \in \mathcal{O}(g)$: f grows asymptotically not faster than g. Algorithm with running time f is not worse than any other algorithm with g.
$f \in \Omega(g)$: f grows asymptotically not slower than g. Algorithm with running time f is worse than any other algorithm with g.
$f \in \Theta(g)$: f grows asymptotically as fast as g. Algorithm with running time f is as good as any other algorithm with g.

Used less often

Given: function $f : \mathbb{N} \to \mathbb{R}$.
Definition:

$$\mathcal{O}(g) = \{f : \mathbb{N} \to \mathbb{R} | \exists c > 0, n_0 \in \mathbb{N} | \forall n \geq n_0 : 0 \leq f(n) \leq c \cdot g(n)\}$$
$$o(g) = \{f : \mathbb{N} \to \mathbb{R} | \forall c > 0 \; \exists n_0 \in \mathbb{N} | \forall n \geq n_0 : 0 \leq f(n) \leq c \cdot g(n)\}$$

$$\Omega(g) = \{f : \mathbb{N} \to \mathbb{R} | \exists c > 0, n_0 \in \mathbb{N} | \forall n \geq n_0 : 0 \leq c \cdot g(n) \leq f(n)\}$$
$$\omega(g) = \{f : \mathbb{N} \to \mathbb{R} | \forall c > 0 \; \exists n_0 \in \mathbb{N} | \forall n \geq n_0 : 0 \leq c \cdot g(n) \leq f(n)\}$$

$f \in o(g)$: f grows much slower than g
$f \in \omega(g)$: f grows much faster than g

Useful information for the exercise

Theorem 1

1. $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0 \Rightarrow f \in \mathcal{O}(g)$, $\mathcal{O}(f) \subsetneq \mathcal{O}(g)$.
2. $\lim_{n\to\infty} \frac{f(n)}{g(n)} = C > 0$ (C constant) $\Rightarrow f \in \Theta(g)$.
3. $\frac{f(n)}{g(n)} \underset{n\to\infty}{\rightarrow} \infty \Rightarrow g \in \mathcal{O}(f)$, $\mathcal{O}(g) \subsetneq \mathcal{O}(f)$.

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3. $\frac{f(n)}{g(n)} \underset{n\to\infty}{\to} \infty \Rightarrow g \in \mathcal{O}(f)$, $\mathcal{O}(g) \subsetneq \mathcal{O}(f)$.

Example 2

1. $\lim_{n\to\infty} \frac{n}{n^2} = 0 \Rightarrow n \in \mathcal{O}(n^2)$, $\mathcal{O}(n) \subsetneq \mathcal{O}(n^2)$.
2. $\lim_{n\to\infty} \frac{2n}{n} = 2 > 0 \Rightarrow 2n \in \Theta(n)$.
3. $\frac{n^2}{n} \underset{n\to\infty}{\to} \infty \Rightarrow n \in \mathcal{O}(n^2)$, $\mathcal{O}(n) \subsetneq \mathcal{O}(n^2)$.

Property

$$f_1 \in \mathcal{O}(g), f_2 \in \mathcal{O}(g) \Rightarrow f_1 + f_2 \in \mathcal{O}(g)$$

Examples

$$\mathcal{O}(g) = \{f : \mathbb{N} \to \mathbb{R} |\ \exists c > 0, \exists n_0 \in \mathbb{N} : \forall n \geq n_0 : 0 \leq f(n) \leq c \cdot g(n)\}$$

$f(n)$	$f \in \mathcal{O}(?)$	Example
$3n + 4$		
$2n$		
$n^2 + 100n$		
$n + \sqrt{n}$		

Examples

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$f(n)$	$f \in \mathcal{O}(?)$	Example
$3n + 4$	$\mathcal{O}(n)$	$c = 4, n_0 = 4$
$2n$		
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Examples

$$\mathcal{O}(g) = \{f : \mathbb{N} \to \mathbb{R} \mid \exists c > 0, \exists n_0 \in \mathbb{N} : \forall n \geq n_0 : 0 \leq f(n) \leq c \cdot g(n)\}$$

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$2n$	$\mathcal{O}(n)$	$c = 2, n_0 = 0$
$n^2 + 100n$	$\mathcal{O}(n^2)$	$c = 2, n_0 = 100$
$n + \sqrt{n}$	$\mathcal{O}(n)$	$c = 2, n_0 = 1$

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- $\mathcal{O}(n) \subseteq \mathcal{O}(n^2)$ is correct
- $\Theta(n) \subseteq \Theta(n^2)$ is wrong $n \notin \Omega(n^2) \supset \Theta(n^2)$

Quiz

$1 \in \mathcal{O}(15)$?

Quiz

$1 \in \mathcal{O}(15)$? ✓ better $1 \in \mathcal{O}(1)$

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$2n + 1 \in \Theta(n)$?

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$1 \in \mathcal{O}(15)$?	✓ better $1 \in \mathcal{O}(1)$
$2n + 1 \in \Theta(n)$?	✓

Quiz

$1 \in \mathcal{O}(15)$?	✓ better $1 \in \mathcal{O}(1)$
$2n + 1 \in \Theta(n)$?	✓
$\sqrt{n} \in \mathcal{O}(n)$?	

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$2n + 1 \in \Theta(n)$?	✓
$\sqrt{n} \in \mathcal{O}(n)$?	✓
$\sqrt{n} \in \Omega(n)$?	✗
$n \in \Omega(\sqrt{n})$?	

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$1 \in \mathcal{O}(15)$?	✓ better $1 \in \mathcal{O}(1)$
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$\sqrt{n} \in \Omega(n)$?	✗
$n \in \Omega(\sqrt{n})$?	✓

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$\sqrt{n} \in \Omega(n)$?	✗
$n \in \Omega(\sqrt{n})$?	✓
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$n \in \Omega(\sqrt{n})$?	✓
$\sqrt{n} \notin \Theta(n)$?	✓
$\mathcal{O}(\sqrt{n}) \subset \mathcal{O}(n)$?	

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$n \in \Omega(\sqrt{n})$?	✓
$\sqrt{n} \notin \Theta(n)$?	✓
$\mathcal{O}(\sqrt{n}) \subset \mathcal{O}(n)$?	✓
$2^n \notin \mathcal{O}(\exp(n))$?	

Quiz

$1 \in \mathcal{O}(15)$?	✓ better $1 \in \mathcal{O}(1)$
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$\mathcal{O}(\sqrt{n}) \subset \mathcal{O}(n)$?	✓
$2^n \notin \mathcal{O}(\exp(n))$?	✗

A good strategy?

… Then I simply buy a new machine!

A good strategy?

… Then I simply buy a new machine! If today I can solve a problem of size n, then with a 10 or 100 times faster machine I can solve …

Komplexität	(speed $\times 10$)	(speed $\times 100$)
$\log_2 n$		
n		
n^2		
2^n		

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n	$n \rightarrow 10 \cdot n$	$n \rightarrow 100 \cdot n$
n^2	$n \rightarrow 3.16 \cdot n$	$n \rightarrow 10 \cdot n$
2^n		

A good strategy?

… Then I simply buy a new machine! If today I can solve a problem of size n, then with a 10 or 100 times faster machine I can solve … [1]

Komplexität	(speed $\times 10$)	(speed $\times 100$)
$\log_2 n$	$n \to n^{10}$	$n \to n^{100}$
n	$n \to 10 \cdot n$	$n \to 100 \cdot n$
n^2	$n \to 3.16 \cdot n$	$n \to 10 \cdot n$
2^n	$n \to n + 3.32$	$n \to n + 6.64$

[1] To see this, you set $f(n') = c \cdot f(n)$ ($c = 10$ or $c = 100$) and solve for n'

Asymptotic Running Times with Θ

```
void run(int n){
  for (int i = 1; i<n; ++i)
  op();
}
```

How often is **`op()`** called as a function of n?

Asymptotic Running Times with Θ

```
void run(int n){
  for (int i = 1; i<n; ++i)
    for (int j = 1; j<n; ++j)
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    }
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How often is `op()` called?

Asymptotic Running Times with Θ

```
void run(int n){
  for (int i = 1; i<n; ++i){
    op();
    for (int j = 1; j<i*i; ++j)
      op();
    }
}
```

How often is `op()` called?

Asymptotic Running Times with Θ

```
void run(int n){
  for(int i = 1; i <= n; ++i)
    for(int j = 1; j*j <= n; ++j)
      for(int k = n; k >= 2; --k)
        op();
}
```

How often is `op()` called as a function of n?

Asymptotic Running Times with Θ

```
int f(int n){
    i=1;
    while (i <= n*n*n){
        i = i*2;
    }
    return i;
}
```

How often is **op()** called as a function of n?

3. Programming Exercise

Preparing remarks for the homework (Prefix Sum in 2D)

Sum in Subarray (naive algorithm)

Input: A sequence of n numbers $(a_0, a_1, \ldots, a_{n-1})$ and a sub-interval $I = [x_0, x_1]$

Output: $\sum_{i=x_0}^{x_1} a_i$.

$\mathcal{S} \leftarrow 0$
for $i \in \{x_0, \ldots, x_1\}$ **do**
  $\mathcal{S} \leftarrow \mathcal{S} + a_i$
return $\mathcal{S}$

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Idea of the exercise

- Use the prefix sum to compute the sum of arbitrary sub-intervals with constant running time
- **Generalize** to two dimensions.

4. Appendix

Some formulas with derivation

Sums

$$\sum_{i=0}^{n} i = ?$$

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Intuition

$$1 + \ldots + 100 = (1 + 100) + (2 + 99) + (3 + 98) + \ldots + (50 + 51)$$

Sums

$$\sum_{i=0}^{n} i = \frac{n \cdot (n+1)}{2}$$

Why?

Intuition

$$1 + \ldots + 100 = (1 + 100) + (2 + 99) + (3 + 98) + \ldots + (50 + 51)$$

More formally?

Sums

$$\sum_{i=0}^{n}(n-i) = ?$$

Sums

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Sums

$$\sum_{i=0}^{n}(n-i) = \sum_{i=0}^{n} i$$

$$\begin{aligned}\Rightarrow 2 \cdot \sum_{i=0}^{n} i &= \sum_{i=0}^{n} i + \sum_{i=0}^{n}(n-i) \\ &= \sum_{i=0}^{n}(i + (n-i)) = \sum_{i=0}^{n} n = (n+1) \cdot n\end{aligned}$$

Sums

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Sums

$$\sum_{i=0}^{n} i^2 = ?$$

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This you do not need to know by heart. But you should know that it is a polynome of third degree.

[Sums]

How do you derive something like this?

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How do you derive something like this? Interesting Trick: On the one hand

$$\sum_{i=0}^{n} i^3 - \sum_{i=1}^{n} (i-1)^3 = \sum_{i=0}^{n} i^3 - \sum_{i=0}^{n-1} i^3 = n^3,$$

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on the other hand

$$\begin{aligned}\sum_{i=0}^{n} i^3 - \sum_{i=1}^{n} (i-1)^3 &= \sum_{i=1}^{n} i^3 - \sum_{i=1}^{n} (i-1)^3 \\ &= \sum_{i=1}^{n} i^3 - (i-1)^3 = \sum_{i=1}^{n} 3 \cdot i^2 - 3 \cdot i + 1\end{aligned}$$

Exponents and Logarithms

$$\log_a y = x \Leftrightarrow a^x = y \quad (a > 0, y > 0)$$

$$a^x \cdot a^y = ?$$

$$\frac{a^x}{a^y} = ?$$

$$a^{x \cdot y} = ?$$

$$\log_b x = ?$$

$$\log_a (x \cdot y) = ?$$

$$\log_a \frac{x}{y} = ?$$

$$\log_a x^y = ?$$

$$\log_a n! = ?$$

$$a^{\log_b x} = ?$$

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$$a^x \cdot a^y = a^{x+y} \qquad \log_a(x \cdot y) = ?$$

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$$\log_a n! = \sum_{i=1}^{n} \log i$$

$$\log_b x = ? \qquad a^{\log_b x} = ?$$

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$$\log_b x = \log_b a \cdot \log_a x$$
$$a^{\log_b x} = x^{\log_b a}$$

To see the last line, replace $x \to a^{\log_a x}$

Comparisons

$$\frac{n^2}{2^n} \underset{n \to \infty}{\longrightarrow} ?$$

Comparisons

$$\frac{n^2}{2^n} \underset{n \to \infty}{\longrightarrow} 0$$

Comparisons

$$\frac{n^{10000}}{2^n} \underset{n\to\infty}{\longrightarrow} ?$$

Comparisons

$$\frac{n^{10000}}{2^n} \xrightarrow[n\to\infty]{} 0$$

Comparisons

$d > 1, c > 0$

$$\frac{n^c}{d^n} \xrightarrow[n \to \infty]{} ?$$

Comparisons

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Comparisons

$d > 1, c > 0$

$$\frac{n^c}{d^n} \xrightarrow[n\to\infty]{} 0$$

because

$$\frac{n^c}{d^n} = \frac{2^{\log_2 n^c}}{2^{\log_2 d^n}} = 2^{c \cdot \log_2 n - n \log_2 d}$$

Comparisons

$$\frac{n}{\log n} \xrightarrow[n\to\infty]{} ?$$

Comparisons

$$\frac{n}{\log n} \xrightarrow[n \to \infty]{} \infty$$

Comparisons

$$\frac{n \log n}{\sqrt{n}} \xrightarrow[n\to\infty]{} ?$$

Comparisons

$$\frac{n \log n}{\sqrt{n}} \xrightarrow[n\to\infty]{} \infty$$

Comparisons

$$\frac{\log_2 n^2}{\sqrt{n}} \underset{n\to\infty}{\longrightarrow} ?$$

Comparisons

$$\frac{\log_2 n^2}{\sqrt{n}} \xrightarrow[n \to \infty]{} 0$$

Comparisons

$$\frac{\log_2 n^2}{\sqrt{n}} \xrightarrow[n\to\infty]{} 0$$

$$\begin{aligned}
\log_2 n^2 &= 2\log_2 n \\
\sqrt{n} &= n^{1/2} = 2^{\log_2 n^{1/2}} = \left(\sqrt{2}\right)^{\log_2 n} \\
\frac{\log n^2}{\sqrt{n}} &= 2\frac{\log_2 n}{\left(\sqrt{2}\right)^{\log_2 n}}
\end{aligned}$$

which behaves because of $\log_2 n \to \infty$ for $n \to \infty$ like $2\frac{n}{\left(\sqrt{2}\right)^n}$