9. Sorting III

Lower bounds for the comparison based sorting, radix- and bucket-sort

9.1 Lower bounds for comparison based sorting

[Ottman/Widmayer, Kap. 2.8, Cormen et al, Kap. 8.1]

Lower bound for sorting

Up to here: worst case sorting takes $\Omega(n \log n)$ steps.

Is there a better way? No:

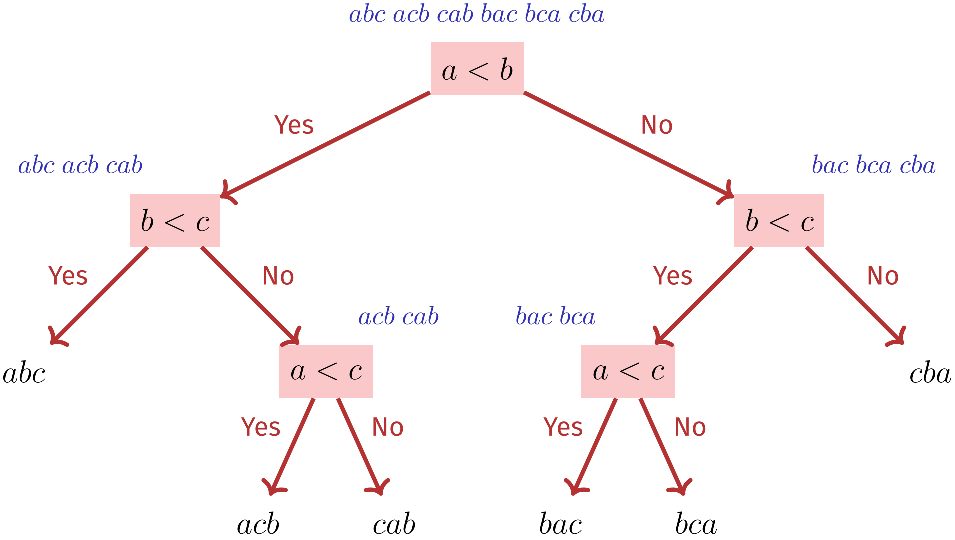
Theorem 14

Sorting procedures that are based on comparison require in the worst case and on average at least $\Omega(n \log n)$ *key comparisons.*

Comparison based sorting

- An algorithm must identify the correct one of $n!$ permutations of an array $(A_i)_{i=1,\dots,n}$.
- At the beginning the algorithm knows nothing about the array structure.
- We consider the knowledge gain of the algorithm in the form of a decision tree:
  - Nodes contain the remaining possibilities.
  - Edges contain the decisions.

Decision tree



Decision tree

A binary tree with L leaves provides $K = L - 1$ inner nodes.[10]

The height of a binary tree with L leaves is at least $\log_2 L$. $\Rightarrow$ The height of the decision tree $h \geq \log n! \in \Omega(n \log n)$.

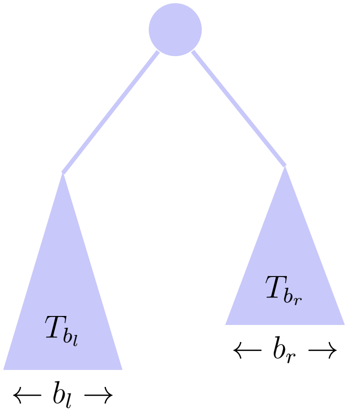
Thus the length of the longest path in the decision tree $\in \Omega(n \log n)$.

Remaining to show: mean length $M(n)$ of a path $M(n) \in \Omega(n \log n)$.

---

[10] Proof: start with emtpy tree ($K = 0, L = 1$). Each added node replaces a leaf by two leaves, i.e.} $K \to K + 1 \Rightarrow L \to L + 1$.

Average lower bound



- Decision tree T_n with n leaves, average height of a leaf $m(T_n)$
- Assumption $m(T_n) \geq \log n$ not for all n.
- Choose smallest b with $m(T_b) < \log b \Rightarrow b \geq 2$
- $b_l + b_r = b$ with $b_l > 0$ and $b_r > 0 \Rightarrow$ $b_l < b, b_r < b \Rightarrow m(T_{b_l}) \geq \log b_l$ und $m(T_{b_r}) \geq \log b_r$

Average lower bound

Average height of a leaf:

$$\begin{aligned} m(T_b) &= \frac{b_l}{b}(m(T_{b_l})+1) + \frac{b_r}{b}(m(T_{b_r})+1) \\ &\geq \frac{1}{b}(b_l(\log b_l + 1) + b_r(\log b_r + 1)) = \frac{1}{b}(b_l \log 2b_l + b_r \log 2b_r) \\ &\geq \frac{1}{b}(b \log b) = \log b. \end{aligned}$$

Contradiction. ■

The last inequality holds because $f(x) = x \log x$ is convex ($f''(x) = 1/x > 0$) and for a convex function it holds that $f((x+y)/2) \leq 1/2 f(x) + 1/2 f(y)$ ($x = 2b_l$, $y = 2b_r$).[11] Enter $x = 2b_l$, $y = 2b_r$, and $b_l + b_r = b$.

---

[11] generally $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$ for $0 \leq \lambda \leq 1$.

9.2 Radixsort and Bucketsort

Radixsort, Bucketsort [Ottman/Widmayer, Kap. 2.5, Cormen et al, Kap. 8.3]

Radix Sort

Sorting based on comparison: comparable keys ($<$ or $>$, often $=$). No further assumptions.

Different idea: use more information about the keys.

Assumptions

Assumption: keys representable as words from an alphabet containing m elements.

Examples

$m = 10$	decimal numbers	$183 = 183_{10}$
$m = 2$	dual numbers	101_2
$m = 16$	hexadecimal numbers	$A0_{16}$
$m = 26$	words	`"INFORMATIK"`

m is called the radix of the representation.

Assumptions

- keys = m-adic numbers with same length.
- Procedure z for the extraction of digit k in $\mathcal{O}(1)$ steps.

Example

$$z_{10}(0, 85) = 5$$
$$z_{10}(1, 85) = 8$$
$$z_{10}(2, 85) = 0$$

Radix-Exchange-Sort

Keys with radix 2.
Observation: if for some $k \geq 0$:

$$z_2(i, x) = z_2(i, y) \text{ for all } i > k$$

and

$$z_2(k, x) < z_2(k, y),$$

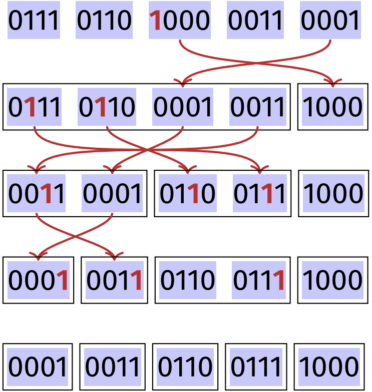
then it holds that $x < y$.

Radix-Exchange-Sort

Idea:

- Start with a maximal k.
- Binary partition the data sets with $z_2(k,\cdot) = 0$ vs. $z_2(k,\cdot) = 1$ like with quicksort.
- $k \leftarrow k - 1$.

Radix-Exchange-Sort



0111 0110 **1**000 0011 0001

0**1**11 0**1**10 0001 0011 1000

00**1**1 0001 01**1**0 01**1**1 1000

000**1** 001**1** 0110 011**1** 1000

0001 0011 0110 0111 1000

Algorithm RadixExchangeSort(A, l, r, b)

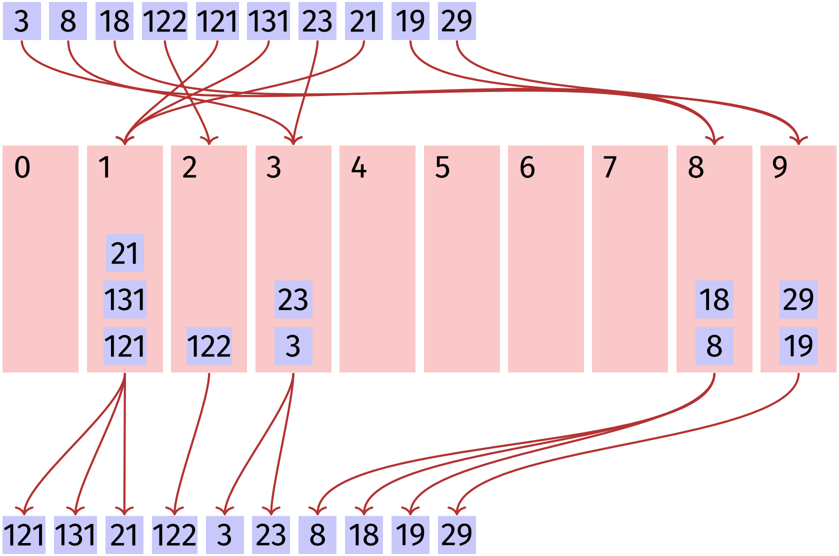
```
Input:  Array $A$ with length $n$, left and right bounds $1 \leq l \leq r \leq n$, bit position $b$
Output: Array $A$, sorted in the domain $[l, r]$ by bits $[0, \dots, b]$ .
if $l < r$ and $b \geq 0$ then
    $i \leftarrow l - 1$
    $j \leftarrow r + 1$
    repeat
        repeat $i \leftarrow i + 1$ until $z_2(b, A[i]) = 1$ or $i \geq j$
        repeat $j \leftarrow j - 1$ until $z_2(b, A[j]) = 0$ or $i \geq j$
        if $i < j$ then swap($A[i], A[j]$)
    until $i \geq j$
    RadixExchangeSort($A, l, i - 1, b - 1$)
    RadixExchangeSort($A, i, r, b - 1$)
```

Analysis

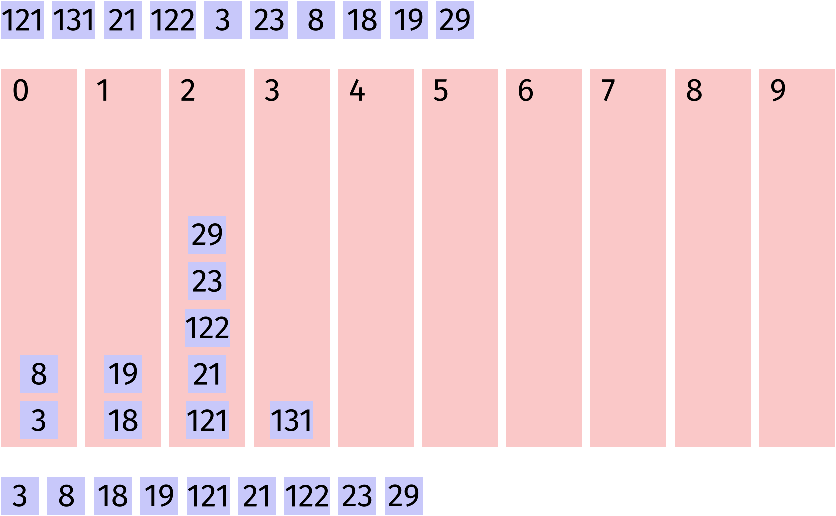
RadixExchangeSort provides recursion with maximal recursion depth = maximal number of digits p.

Worst case run time $\mathcal{O}(p \cdot n)$.

Bucket Sort


3
8
18
122
121
131
23
21
19
29
0
1
2
3
4
5
6
7
8
9
21
131
121
122
23
3
18
8
29
19
121
131
21
122
3
23
8
18
19
29

Bucket Sort



121 131 21 122 3 23 8 18 19 29

0 1 2 3 4 5 6 7 8 9

0: 3, 8

1: 18, 19

2: 121, 21, 122, 23, 29

3: 131

3 8 18 19 121 21 122 23 29

Bucket Sort

3 8 18 19 121 21 122 23 29

0	1	2	3	4	5	6	7	8	9
29									
23									
21									
19									
18	131								
8	122								
3	121								

3 8 18 19 21 23 29 121 122 131 🙂

implementation details

Bucket size varies greatly. Possibilities

- Linked list or dynamic array for each digit.
- One array of length n. compute offsets for each digit in the first iteration.

Assumptions: Input length n , Number bits / integer: k , Number Buckets: 2^b
Asymptotic running time $\mathcal{O}(\frac{k}{b} \cdot (n + 2^b)$.
For Example: $k = 32$, $2^b = 256$: $\frac{k}{b} \cdot (n + 2^b) = 4n + 1024$.

Bucket Sort – Different Assumption

Hypothesis: uniformly distributed data e.g. from $[0,1)$

```
**Input**: Array $A$ with length $n$, $A_i \in [0,1)$, constant $M \in \mathbb{N}^+$
**Output**: Sorted array
$k \leftarrow \lceil n/M \rceil$
$B \leftarrow$ new array of $k$ empty lists
for $i \leftarrow 1$ to $n$ do
    $B[\lfloor A_i \cdot k \rfloor]$.append$(A[i])$
for $i \leftarrow 1$ to $k$ do
    sort $B[i]$ // e.g. insertion sort, running time $\mathcal{O}(M^2)$
return $B[0] \circ B[1] \circ \cdots \circ B[k]$ // concatenated
```

Expected asymptotic running time $\mathcal{O}(n)$ (Proof in Cormen et al, Kap. 8.4)