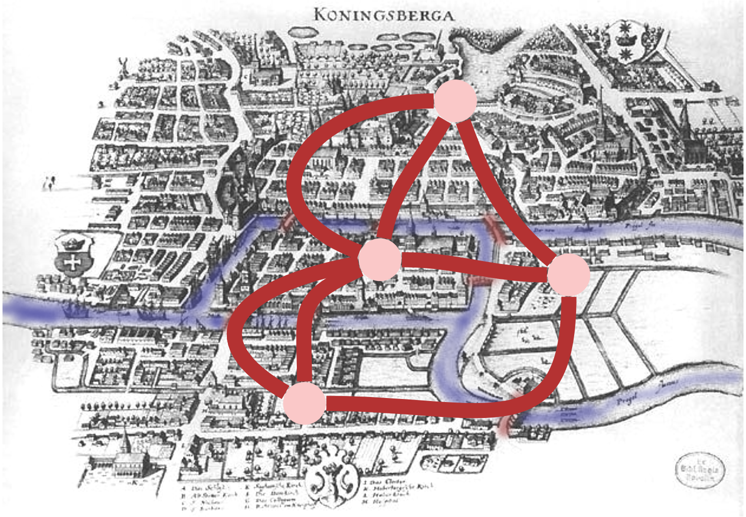
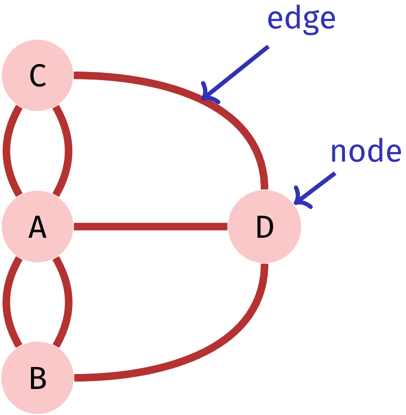
24. Graphs

Notation, Representation, Graph Traversal (DFS, BFS), Topological Sorting , Reflexive transitive closure, Connected components [Ottman/Widmayer, Kap. 9.1 - 9.4,Cormen et al, Kap. 22]

Königsberg 1736



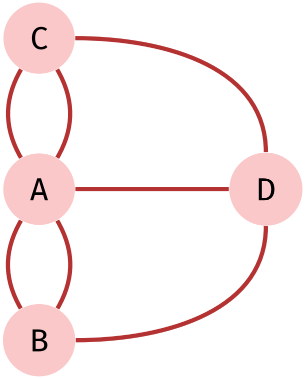
[Multi]Graph



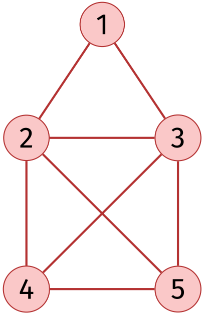
Cycles

- Is there a cycle through the town (the graph) that uses each bridge (each edge) exactly once?
- Euler (1736): no.
- Such a *cycle* is called *Eulerian path*.
- Eulerian path ⇔ each node provides an even number of edges (each node is of an *even degree*).

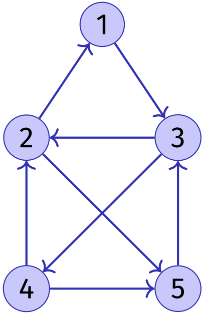
  ‘⇒” is straightforward, “⇐” ist a bit more difficult but still elementary.



Notation



undirected



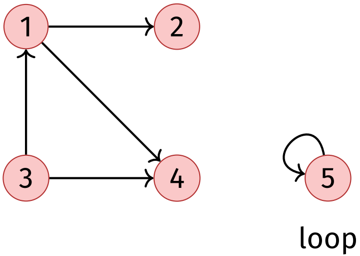
directed

$$
\begin{aligned}
V =& \{1, 2, 3, 4, 5\} \\
E =& \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \\
& \{2, 5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}\}
\end{aligned}
$$

$$
\begin{aligned}
V =& \{1, 2, 3, 4, 5\} \\
E =& \{(1, 3), (2, 1), (2, 5), (3, 2), \\
& (3, 4), (4, 2), (4, 5), (5, 3)\}
\end{aligned}
$$

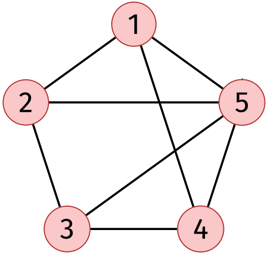
Notation

A directed graph consists of a set $V = \{v_1, \ldots, v_n\}$ of nodes (*Vertices) and a set* $E \subseteq V \times V$ *of* Edges. The same edges may not be contained more than once.



Notation

An undirected graph consists of a set $V = \{v_1, \ldots, v_n\}$ of nodes a and a set $E \subseteq \{\{u, v\} | u, v \in V\}$ of edges. Edges may not be contained more than once.[37]

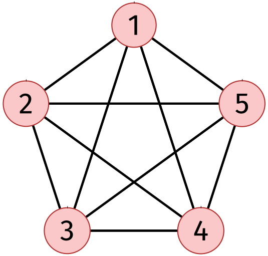


undirected graph

[37] As opposed to the introductory example – it is then called multi-graph.

Notation

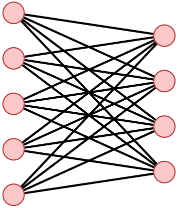
An undirected graph $G = (V, E)$ without loops where E comprises all edges between pairwise different nodes is called complete.



a complete undirected graph

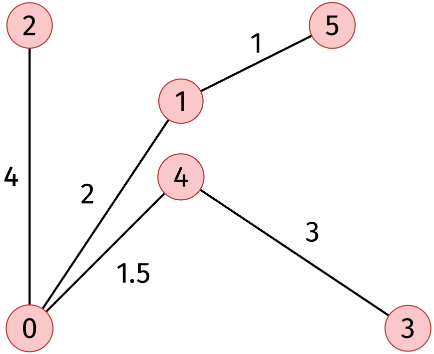
Notation

A graph where V can be partitioned into disjoint sets U and W such that each $e \in E$ provides a node in U and a node in W is called bipartite.



Notation

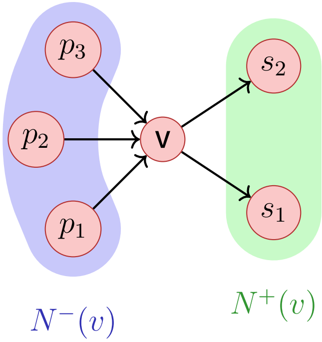
A weighted graph $G = (V, E, c)$ is a graph $G = (V, E)$ with an edge weight function $c : E \to \mathbb{R}$. $c(e)$ is called weight of the edge e.



Notation

For directed graphs $G = (V, E)$

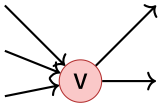
- $w \in V$ is called adjacent to $v \in V$, if $(v, w) \in E$
- Predecessors of $v \in V$: $N^-(v) := \{u \in V | (u, v) \in E\}$.
  Successors: $N^+(v) := \{u \in V | (v, u) \in E\}$

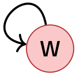


Notation

For directed graphs $G = (V, E)$

- **In-Degree:** $\deg^-(v) = |N^-(v)|$,
  Out-Degree: $\deg^+(v) = |N^+(v)|$



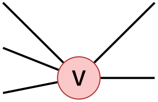


$\deg^-(v) = 3$, $\deg^+(v) = 2$ $\qquad$ $\deg^-(w) = 1$, $\deg^+(w) = 1$

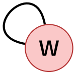
Notation

For undirected graphs $G = (V, E)$:

- $w \in V$ is called adjacent to $v \in V$, if $\{v, w\} \in E$
- Neighbourhood of $v \in V$: $N(v) = \{w \in V | \{v, w\} \in E\}$
- Degree of v: $\deg(v) = |N(v)|$ with a special case for the loops: increase the degree by 2.



$\deg(v) = 5$



$\deg(w) = 2$

Node Degrees $\leftrightarrow$ Number of Edges

For each graph $G = (V, E)$ it holds

1. $\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|$, for G directed
2. $\sum_{v \in V} \deg(v) = 2|E|$, for G undirected.

Paths

- Path: a sequence of nodes $\langle v_1, \ldots, v_{k+1}\rangle$ such that for each $i \in \{1 \ldots k\}$ there is an edge from v_i to v_{i+1} .
- Length of a path: number of contained edges k.
- Weight of a path (in weighted graphs): $\sum_{i=1}^{k} c((v_i, v_{i+1}))$ (bzw. $\sum_{i=1}^{k} c(\{v_i, v_{i+1}\})$)
- Simple path: path without repeating vertices

Connectedness

- An undirected graph is called connected, if for each pair $v, w \in V$ there is a connecting path.
- A directed graph is called strongly connected, if for each pair $v, w \in V$ there is a connecting path.
- A directed graph is called weakly connected, if the corresponding undirected graph is connected.

Simple Observations

- generally: $0 \leq |E| \in \mathcal{O}(|V|^2)$
- connected graph: $|E| \in \Omega(|V|)$
- complete graph: $|E| = \frac{|V| \cdot (|V|-1)}{2}$ (undirected)
- Maximally $|E| = |V|^2$ (directed),$|E| = \frac{|V| \cdot (|V|+1)}{2}$ (undirected)

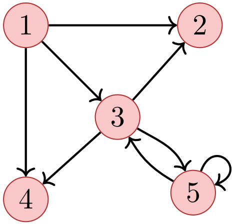
Cycles

- Cycle: path $\langle v_1, \ldots, v_{k+1} \rangle$ with $v_1 = v_{k+1}$
- Simple cycle: Cycle with pairwise different $v_1, \ldots, v_k$, that does not use an edge more than once.
- Acyclic: graph without any cycles.

Conclusion: undirected graphs cannot contain cycles with length 2 (loops have length 1)

Representation using a Matrix

Graph $G = (V, E)$ with nodes $v_1 \dots, v_n$ stored as adjacency matrix $A_G = (a_{ij})_{1 \le i,j \le n}$ with entries from $\{0, 1\}$. $a_{ij} = 1$ if and only if edge from v_i to v_j.

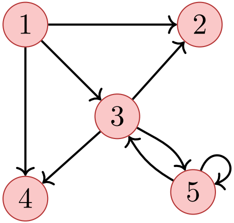


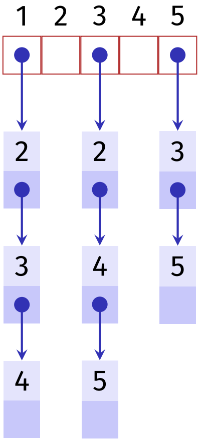
$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Memory consumption $\Theta(|V|^2)$. A_G is symmetric, if G undirected.

Representation with a List

Many graphs $G = (V, E)$ with nodes $v_1, \dots, v_n$ provide much less than n^2 edges. Representation with adjacency list: Array $A[1], \dots, A[n]$, A_i comprises a linked list of nodes in $N^+(v_i)$.



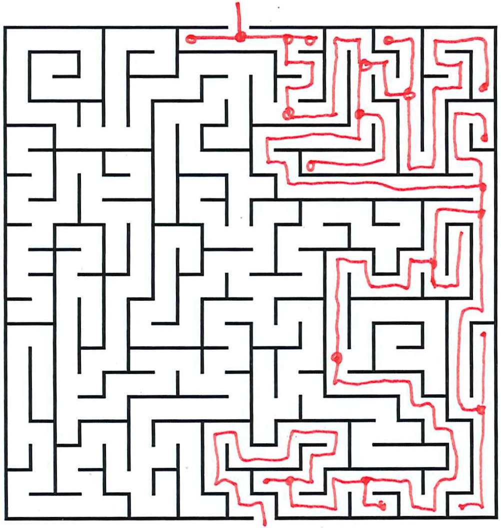


Memory Consumption $\Theta(|V| + |E|)$.

Runtimes of simple Operations

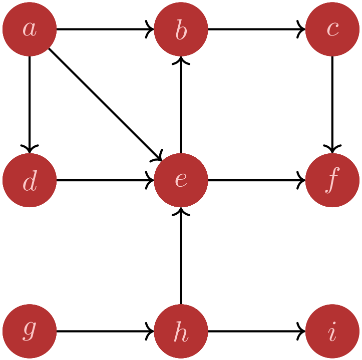
Operation	Matrix	List
Find neighbours/successors of $v \in V$	$\Theta(n)$	$\Theta(\deg^+ v)$
find $v \in V$ without neighbour/successor	$\Theta(n^2)$	$\Theta(n)$
$(v, u) \in E$?	$\Theta(1)$	$\Theta(\deg^+ v)$
Insert edge	$\Theta(1)$	$\Theta(1)$
Delete edge (v, u)	$\Theta(1)$	$\Theta(\deg^+ v)$

Depth First Search



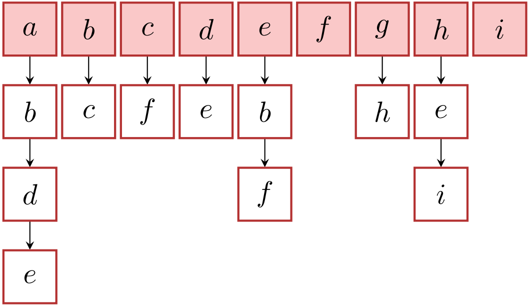
Graph Traversal: Depth First Search

Follow the path into its depth until nothing is left to visit.



Order $a, b, c, f, d, e, g, h, i$

adjacency list



Colors

Conceptual coloring of nodes

- **white:** node has not been discovered yet.
- **grey:** node has been discovered and is marked for traversal / being processed.
- **black:** node was discovered and entirely processed.

Algorithm Depth First visit DFS-Visit(G, v)

Input: graph $G = (V, E)$, Knoten v.

$v.color \leftarrow$ grey
foreach $w \in N^+(v)$ **do**
    if $w.color =$ white **then**
        DFS-Visit(G, w)

$v.color \leftarrow$ black

Depth First Search starting from node v. Running time (without recursion): $\Theta(\deg^+ v)$

Algorithm Depth First visit DFS-Visit(G)

Input: graph $G = (V, E)$

foreach $v \in V$ **do**
    $v.color \leftarrow$ white

foreach $v \in V$ **do**
    if $v.color =$ white **then**
        DFS-Visit(G,v)

Depth First Search for all nodes of a graph. Running time:
$\Theta(|V| + \sum_{v \in V}(\deg^+(v) + 1)) = \Theta(|V| + |E|)$.

Iterative DFS-Visit(G, v)

Input: graph $G = (V, E)$, $v \in V$ with $v.color =$ white

```
Stack S ← ∅
v.color ← grey; S.push(v)                      // invariant: grey nodes always on stack
while S ≠ ∅ do
    w ← nextWhiteSuccessor(v)                  // code: next slide
    if w ≠ null then
        w.color ← grey; S.push(w)
        v ← w                                  // work on w. parent remains on the stack
    else
        v.color ← black                        // no grey successors, v becomes black
        if S ≠ ∅ then
            v ← S.pop()                        // visit/revisit next node
            if v.color = grey then  S.push(v)
```

Memory Consumption Stack $\Theta(|V|)$

nextWhiteSuccessor(v)

Input: node $v \in V$
Output: Successor node u of v with $u.color =$ white, null otherwise

foreach $u \in N^+(v)$ **do**
    if $u.color =$ white **then**
        return u

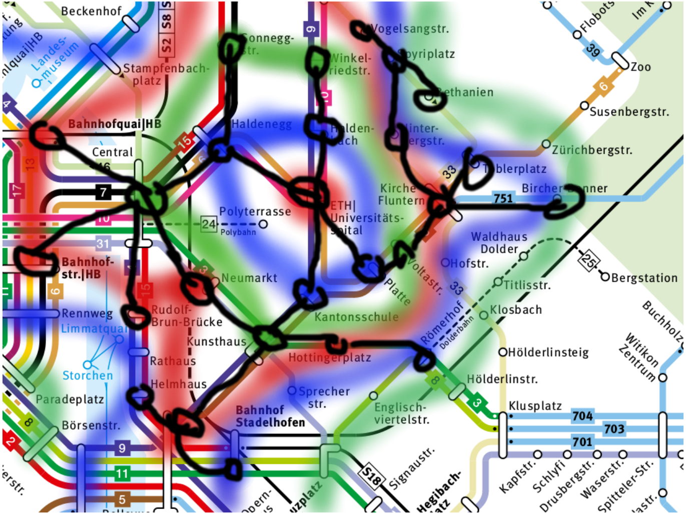
return null

Interpretation of the Colors

When traversing the graph, a tree (or Forest) is built. When nodes are discovered there are three cases

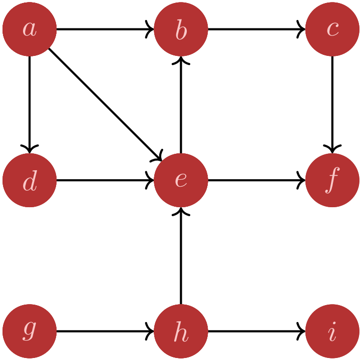
- White node: new tree edge
- Grey node: cycle (“back-edge”)
- Black node: forward- / cross edge

Breadth First Search



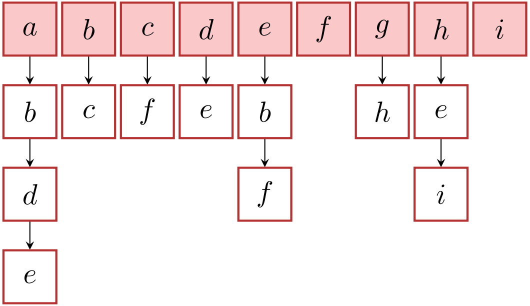
Graph Traversal: Breadth First Search

Follow the path in breadth and only then descend into depth.



Order $a, b, d, e, c, f, g, h, i$

Adjazenzliste



(Iterative) BFS-Visit(G, v)

Input: graph $G = (V, E)$

```
Queue Q ← ∅
v.color ← grey
enqueue(Q, v)
while Q ≠ ∅ do
    w ← dequeue(Q)
    foreach c ∈ N⁺(w) do
        if c.color = white then
            c.color ← grey
            enqueue(Q, c)
    w.color ← black
```

Algorithm requires extra space of $\mathcal{O}(|V|)$.

Main program BFS-Visit(G)

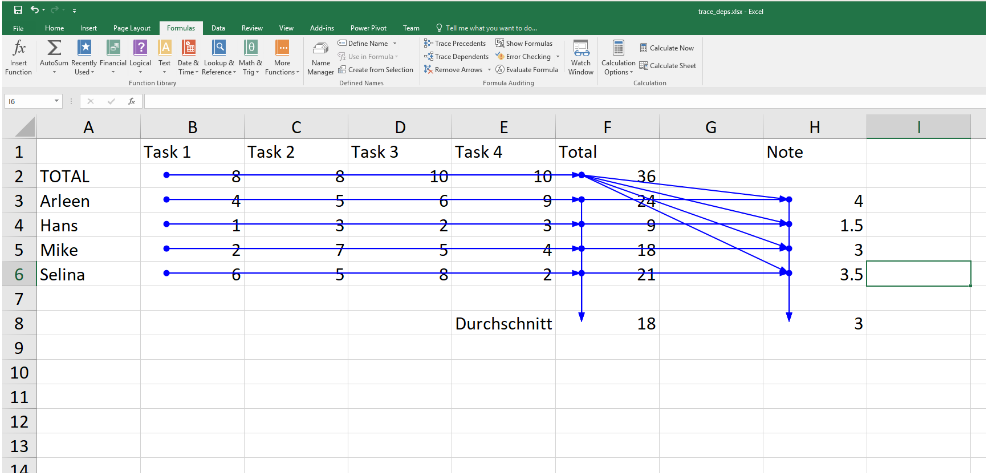
Input: graph $G = (V, E)$

foreach $v \in V$ **do**
  $v.color \leftarrow$ white

foreach $v \in V$ **do**
  if $v.color =$ white **then**
    BFS-Visit(G,v)

Breadth First Search for all nodes of a graph. Running time: $\Theta(|V| + |E|)$.

Topological Sorting



	A	B	C	D	E	F	G	H	I
1		Task 1	Task 2	Task 3	Task 4	Total		Note	
2	TOTAL	8	8	10	10	36			
3	Arleen	4	5	6	9	24		4	
4	Hans	1	3	2	3	9		1.5	
5	Mike	2	7	5	4	18		3	
6	Selina	6	5	8	2	21		3.5	
7									
8					Durchschnitt	18		3	

Evaluation Order?

Topological Sorting

Topological Sorting of an acyclic directed graph $G = (V, E)$:
Bijective mapping

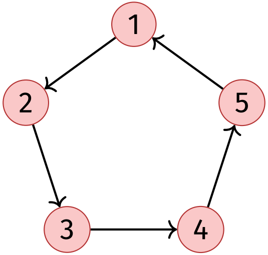
$$\mathrm{ord} : V \to \{1, \ldots, |V|\}$$

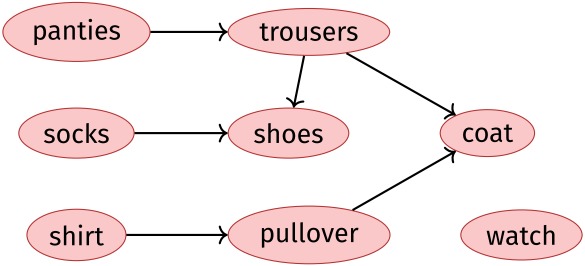
such that

$$\mathrm{ord}(v) < \mathrm{ord}(w) \;\forall\; (v, w) \in E.$$

Identify i with Element $v_i := \mathrm{ord}^1(i)$. Topological sorting $\widehat{=} \langle v_1, \ldots, v_{|V|} \rangle$.

(Counter-)Examples





Cyclic graph: cannot be sorted topologically.

A possible toplogical sorting of the graph:
shirt, pullover, panties, watch, trousers, coat, socks, shoes

Observation

Theorem 20

A directed graph $G = (V, E)$ permits a topological sorting if and only if it is acyclic.

Proof "$\Rightarrow$"

If G contains a cycle it cannot permit a topological sorting, because in a cycle $\langle v_{i_1}, \ldots, v_{i_m} \rangle$ it would hold that $v_{i_1} < \cdots < v_{i_m} < v_{i_1}$.

Proof "$\Leftarrow$"

- Base case ($n = 1$): Graph with a single node without loop can be sorted topologically, set$\operatorname{ord}(v_1) = 1$.
- Hypothesis: Graph with n nodes can be sorted topologically
- Step ($n \to n + 1$):

  1. G contains a node v_q with in-degree $\deg^-(v_q) = 0$. Otherwise iteratively follow edges backwards – after at most $n + 1$ iterations a node would be revisited. Contradiction to the cycle-freeness.
  2. Graph without node v_q and without its edges can be topologically sorted by the hypothesis. Now use this sorting and set $\operatorname{ord}(v_i) \leftarrow \operatorname{ord}(v_i) + 1$ for all $i \neq q$ and set $\operatorname{ord}(v_q) \leftarrow 1$.

Algorithm Topological-Sort(G)

Input: graph $G = (V, E)$.
Output: Topological sorting ord

Stack $S \leftarrow \emptyset$
foreach $v \in V$ **do** $A[v] \leftarrow 0$
foreach $(v, w) \in E$ **do** $A[w] \leftarrow A[w] + 1$ // Compute in-degrees
foreach $v \in V$ with $A[v] = 0$ **do** push(S, v) // Memorize nodes with in-degree 0
$i \leftarrow 1$
while $S \neq \emptyset$ **do**
  $v \leftarrow$ pop(S); $\mathrm{ord}[v] \leftarrow i$; $i \leftarrow i + 1$ // Choose node with in-degree 0
  foreach $(v, w) \in E$ **do** // Decrease in-degree of successors
    $A[w] \leftarrow A[w] - 1$
    if $A[w] = 0$ **then** push(S, w)

if $i = |V| + 1$ **then return** ord **else return** “Cycle Detected”

Algorithm Correctness

Theorem 21

Let $G = (V, E)$ be a directed acyclic graph. Algorithm **TopologicalSort**(G) *computes a topological sorting* ord *for* G *with runtime* $\Theta(|V| + |E|)$.

Proof: follows from previous theorem:

1. Decreasing the in-degree corresponds with node removal.
2. In the algorithm it holds for each node v with $A[v] = 0$ that either the node has in-degree 0 or that previously all predecessors have been assigned a value $\mathrm{ord}[u] \leftarrow i$ and thus $\mathrm{ord}[v] > \mathrm{ord}[u]$ for all predecessors u of v. Nodes are put to the stack only once.
3. Runtime: inspection of the algorithm (with some arguments like with graph traversal)

Algorithm Correctness

Theorem 22

Let $G = (V, E)$ be a directed graph containing a cycle. Algorithm TopologicalSort terminates within $\Theta(|V| + |E|)$ steps and detects a cycle.

Proof: let $\langle v_{i_1}, \ldots, v_{i_k} \rangle$ be a cycle in G. In each step of the algorithm remains $A[v_{i_j}] \geq 1$ for all $j = 1, \ldots, k$. Thus k nodes are never pushed on the stack und therefore at the end it holds that $i \leq V + 1 - k$.

The runtime of the second part of the algorithm can become shorter. But the computation of the in-degree costs already $\Theta(|V| + |E|)$.

Alternative: Algorithm DFS-Topsort(G, v)

Input: graph $G = (V, E)$, node v, node list L.

if $v.color =$ grey **then**
  stop (Cycle)

if $v.color =$ black **then**
  return

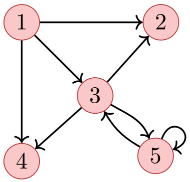
$v.color \leftarrow$ grey
foreach $w \in N^+(v)$ **do**
  DFS-Topsort(G, w)

$v.color \leftarrow$ black
Add v to head of L

Call this algorithm for each node that has not yet been visited. Asymptotic Running Time $\Theta(|V| + |E|)$.

Adjacency Matrix Product

$$B := A_G^2 = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 2 \end{pmatrix}$$



Interpretation

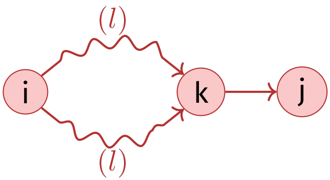
Theorem 23

Let $G = (V, E)$ *be a graph and* $k \in \mathbb{N}$. *Then the element* $a_{i,j}^{(k)}$ *of the matrix* $(a_{i,j}^{(k)})_{1 \leq i,j \leq n} = (A_G)^k$ *provides the number of paths with length* k *from* v_i *to* v_j .

Proof

By Induction.

Base case: straightforward for $k = 1$. $a_{i,j} = a_{i,j}^{(1)}$.
Hypothesis: claim is true for all $k \leq l$
Step ($l \to l+1$):



$$a_{i,j}^{(l+1)} = \sum_{k=1}^{n} a_{i,k}^{(l)} \cdot a_{k,j}$$

$a_{k,j} = 1$ iff egde k to j, 0 otherwise. Sum counts the number paths of length l from node v_i to all nodes v_k that provide a direct direction to node v_j, i.e. all paths with length $l + 1$.

Relation

Given a finite set V

(Binary) **Relation** R on V: Subset of the cartesian product $V \times V = \{(a, b) | a \in V, b \in V\}$

Relation $R \subseteq V \times V$ is called

- reflexive, if $(v, v) \in R$ for all $v \in V$
- symmetric, if $(v, w) \in R \Rightarrow (w, v) \in R$
- transitive, if $(v, x) \in R$, $(x, w) \in R \Rightarrow (v, w) \in R$

The (Reflexive) Transitive Closure R^* of R is the smallest extension $R \subseteq R^* \subseteq V \times V$ such that R^* is reflexive and transitive.

Graphs and Relations

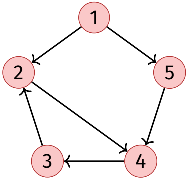
Graph $G = (V, E)$
adjacencies $A_G \mathrel{\hat{=}}$ Relation $E \subseteq V \times V$ over V

- reflexive $\Leftrightarrow a_{i,i} = 1$ for all $i = 1, \ldots, n$. (loops)
- symmetric $\Leftrightarrow a_{i,j} = a_{j,i}$ for all $i, j = 1, \ldots, n$ (undirected)
- transitive $\Leftrightarrow (u, v) \in E$, $(v, w) \in E \Rightarrow (u, w) \in E$. (reachability)

Reflexive Transitive Closure

Reflexive transitive closure of G $\Leftrightarrow$ Reachability relation E^*: $(v, w) \in E^*$ iff $\exists$ path from node v to w.

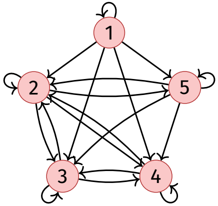
$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$



$G = (V, E)$

$$\Rightarrow$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$



$G^* = (V, E^*)$

Algorithm $A \cdot A$

Input: (Adjacency-)Matrix $A = (a_{ij})_{i,j=1\dots n}$
Output: Matrix Product $B = (b_{ij})_{i,j=1\dots n} = A \cdot A$

```
B ← 0
for r ← 1 to n do
    for c ← 1 to n do
        for k ← 1 to n do
            b_rc ← b_rc + a_rk · a_kc        // Number of Paths
return B
```

Counts number of paths of length 2

Algorithm $A \otimes A$

Input: Adjacency-Matrix $A = (a_{ij})_{i,j=1 \dots n}$
Output: Modified Matrix Product $B = (b_{ij})_{i,j=1 \dots n} = A \otimes A$

```
B ← A                                          // Keep paths
for r ← 1 to n do
    for c ← 1 to n do
        for k ← 1 to n do
            b_rc ← max{b_rc, a_rk · a_kc}      // Path: yes/no
return B
```

Computes which paths of length 1 and 2 exist

Computation of the Reflexive Transitive Closure

Goal: computation of $B = (b_{ij})_{1 \le i,j \le n}$ with $b_{ij} = 1 \Leftrightarrow (v_i, v_j) \in E^*$ First idea:

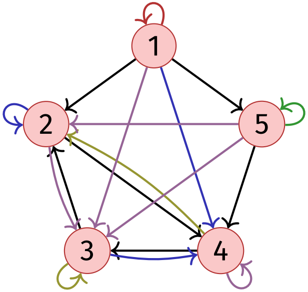
- Start with $B \leftarrow A$ and set $b_{ii} = 1$ for each i (Reflexivity.).
- Compute

$$B_n = \bigotimes_{i=1}^{n} B$$

with powers of 2 $B_2 := B \otimes B$, $B_4 := B_2 \otimes B_2$, $B_8 = B_4 \otimes B_4$ ...
$\Rightarrow$ running time $n^3 \lceil \log_2 n \rceil$

Improvement: Algorithm of Warshall (1962)

Inductive procedure: all paths known over nodes from $\{v_i : i < k\}$. Add node v_k.



$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Algorithm TransitiveClosure(A_G)

Input: Adjacency matrix $A_G = (a_{ij})_{i,j=1\dots n}$
Output: Reflexive transitive closure $B = (b_{ij})_{i,j=1\dots n}$ of G

$B \leftarrow A_G$
for $k \leftarrow 1$ **to** n **do**
  $b_{kk} \leftarrow 1$ // Reflexivity
  for $r \leftarrow 1$ **to** n **do**
    for $c \leftarrow 1$ **to** n **do**
      $b_{rc} \leftarrow \max\{b_{rc}, b_{rk} \cdot b_{kc}\}$ // All paths via v_k

return B

Runtime $\Theta(n^3)$.

Correctness of the Algorithm (Induction)

Invariant (k): all paths via nodes with maximal index $< k$ considered.

- **Base case ($k = 1$)**: All directed paths (all edges) in A_G considered.
- **Hypothesis**: invariant (k) fulfilled.
- **Step** ($k \to k + 1$): For each path from v_i to v_j via nodes with maximal index k: by the hypothesis $b_{ik} = 1$ and $b_{kj} = 1$. Therefore in the k-th iteration: $b_{ij} \leftarrow 1$.