23. Greedy Algorithms

Fractional Knapsack Problem, Huffman Coding [Cormen et al, Kap. 16.1, 16.3]

Greedy Choice

A problem with a recursive solution can be solved with a **greedy algorithm** if it has the following properties:

- The problem has **optimal substructure**: the solution of a problem can be constructed with a combination of solutions of sub-problems.
- The problem has the **greedy choice property**: The solution to a problem can be constructed, by using a local criterion that is not depending on the solution of the sub-problems.

Examples: fractional knapsack, Huffman-Coding (below)
Counter-Example: knapsack problem, Optimal Binary Search Tree

The Fractional Knapsack Problem

set of $n \in \mathbb{N}$ items $\{1, \ldots, n\}$ Each item i has value $v_i \in \mathbb{N}$ and weight $w_i \in \mathbb{N}$. The maximum weight is given as $W \in \mathbb{N}$. Input is denoted as $E = (v_i, w_i)_{i=1,\ldots,n}$.

Wanted: Fractions $0 \leq q_i \leq 1$ ($1 \leq i \leq n$) that maximise the sum $\sum_{i=1}^{n} q_i \cdot v_i$ under $\sum_{i=1}^{n} q_i \cdot w_i \leq W$.

Greedy heuristics

Sort the items decreasingly by value per weight v_i/w_i.

Assumption $v_i/w_i \geq v_{i+1}/w_{i+1}$

Let $j = \max\{0 \leq k \leq n : \sum_{i=1}^{k} w_i \leq W\}$. Set

- $q_i = 1$ for all $1 \leq i \leq j$.
- $q_{j+1} = \frac{W - \sum_{i=1}^{j} w_i}{w_{j+1}}$.
- $q_i = 0$ for all $i > j + 1$.

That is fast: $\Theta(n \log n)$ for sorting and $\Theta(n)$ for the computation of the q_i.

Correctness

Assumption: optimal solution (r_i) ($1 \leq i \leq n$).
The knapsack is full: $\sum_i r_i \cdot w_i = \sum_i q_i \cdot w_i = W$.
Consider k: smallest i with $r_i \neq q_i$ Definition of greedy: $q_k > r_k$. Let $x = q_k - r_k > 0$.
Construct a new solution (r'_i): $r'_i = r_i \forall i < k$. $r'_k = q_k$. Remove weight $\sum_{i=k+1}^{n} \delta_i = x \cdot w_k$ from items $k+1$ to n. This works because $\sum_{i=k}^{n} r_i \cdot w_i = \sum_{i=k}^{n} q_i \cdot w_i$.

Correctness

$$
\begin{aligned}
\sum_{i=k}^{n} r'_i v_i &= r_k v_k + x w_k \frac{v_k}{w_k} + \sum_{i=k+1}^{n} (r_i w_i - \delta_i) \frac{v_i}{w_i} \\
&\geq r_k v_k + x w_k \frac{v_k}{w_k} + \sum_{i=k+1}^{n} r_i w_i \frac{v_i}{w_i} - \delta_i \frac{v_k}{w_k} \\
&= r_k v_k + x w_k \frac{v_k}{w_k} - x w_k \frac{v_k}{w_k} + \sum_{i=k+1}^{n} r_i w_i \frac{v_i}{w_i} = \sum_{i=k}^{n} r_i v_i .
\end{aligned}
$$

Thus (r'_i) is also optimal. Iterative application of this idea generates the solution (q_i).

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File consisting of 100.000 characters from the alphabet $\{a, \ldots, f\}$.

	a	b	c	d	e	f
Frequency (Thousands)	45	13	12	16	9	5
Code word with fix length	000	001	010	011	100	101
Code word variable length	0	101	100	111	1101	1100

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File size (code with fix length): 300.000 bits.
File size (code with variable length): 224.000 bits.

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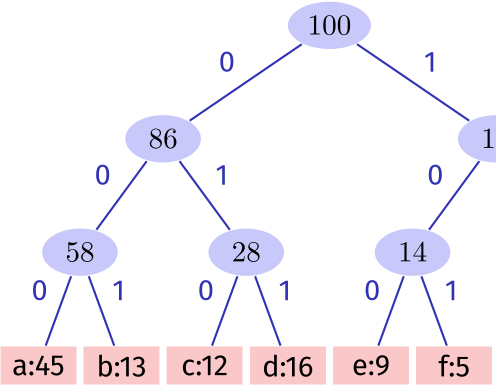
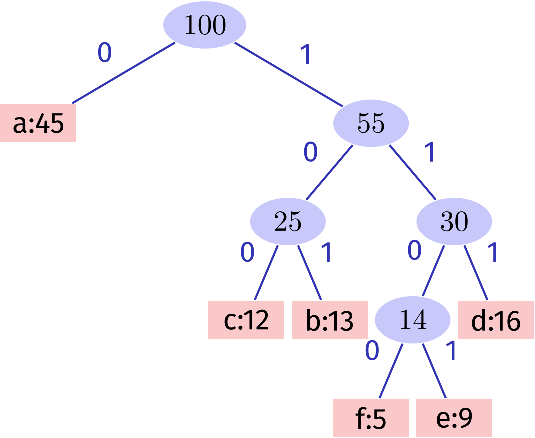
- Decoding simple because prefixcode
  $0110011001101 \to 0 \cdot 1100 \cdot 1100 \cdot 1101 \to affe$

Code trees



Code words with fixed length



Code words with variable length

Properties of the Code Trees

- An optimal coding of a file is alway represented by a complete binary tree: every inner node has two children.

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- Let C be the set of all code words, $f(c)$ the frequency of a codeword c and $d_T(c)$ the depth of a code word in tree T. Define the cost of a tree as

$$B(T) = \sum_{c \in C} f(c) \cdot d_T(c).$$

(cost = number bits of the encoded file)

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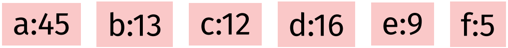
In the following a code tree is called optimal when it minimizes the costs.

Algorithm Idea

Tree construction bottom up

- Start with the set C of code words
- Replace iteriatively the two nodes with smallest frequency by a new parent node.



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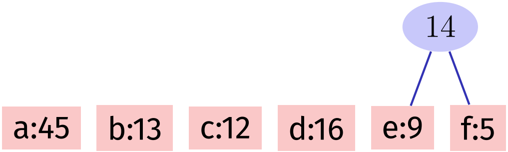

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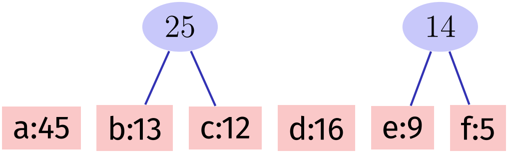

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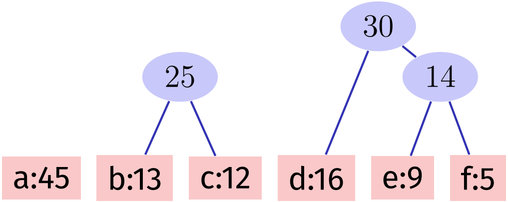

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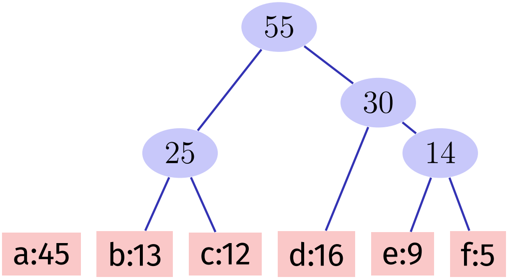

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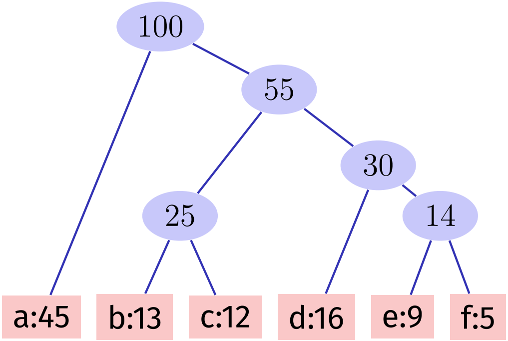

Algorithm Huffman(C)

Input: code words $c \in C$
Output: Root of an optimal code tree

```
n ← |C|
Q ← C
for i = 1 to n − 1 do
    allocate a new node z
    z.left ← ExtractMin(Q)        // extract word with minimal frequency.
    z.right ← ExtractMin(Q)
    z.freq ← z.left.freq + z.right.freq
    Insert(Q, z)
return ExtractMin(Q)
```

Analyse

Use a heap: build Heap in $\mathcal{O}(n)$. Extract-Min in $O(\log n)$ for n Elements.
Yields a runtime of $O(n \log n)$.

The greedy approach is correct

Theorem 19

Let x, y be two symbols with smallest frequencies in C and let $T'(C')$ be an optimal code tree to the alphabet $C' = C - \{x, y\} + \{z\}$ with a new symbol z with $f(z) = f(x) + f(y)$. Then the tree $T(C)$ that is constructed from $T'(C')$ by replacing the node z by an inner node with children x and y is an optimal code tree for the alphabet C.

Proof

It holds that
$f(x) \cdot d_T(x) + f(y) \cdot d_T(y) = (f(x) + f(y)) \cdot (d_{T'}(z) + 1) = f(z) \cdot d_{T'}(x) + f(x) + f(y)$.
Thus $B(T') = B(T) - f(x) - f(y)$.

Assumption: T is not optimal. Then there is an optimal tree T'' with $B(T'') < B(T)$. We assume that x and y are brothers in T''. Let T''' be the tree where the inner node with children x and y is replaced by z. Then it holds that $B(T''') = B(T'') - f(x) - f(y) < B(T) - f(x) - f(y) = B(T')$.
Contradiction to the optimality of T'.

The assumption that x and y are brothers in T'' can be justified because a swap of elements with smallest frequency to the lowest level of the tree can at most decrease the value of B.