Datenstrukturen und Algorithmen

Exercise 7

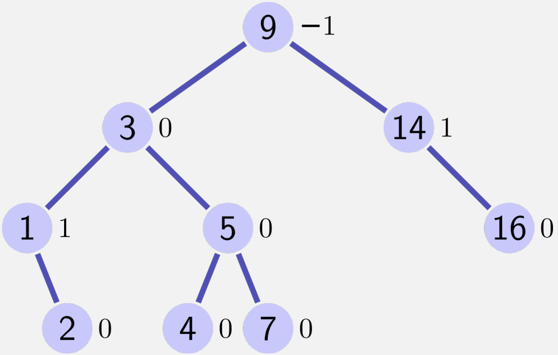
FS 2021

Program of today

1. Feedback of last exercise(s)
2. Repetition theory
   - Quadtrees

AVL insertion

- Given an AVL tree, is there an order that produces the same tree and does not cause any rotations



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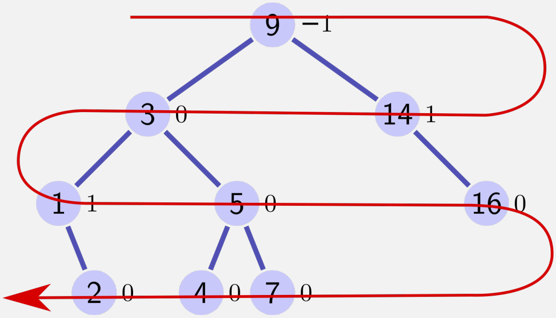

AVL insertion - sketch of proof

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- By induction over the height of the tree.
- Hypothesis: Keys at height h and lower are placed in the same place and do not cause rotation.
- Step: Show that the traversal is the same as in the original tree, yields same position. Use AVL property of T to show that there will not be a height difference bigger than 1, and therefore no rotation.

2. Repetition theory

Minimization of a functional for signal segmentation

$\mathcal{P}$ Partition $\gamma \geq 0$ regularization parameter
$f_{\mathcal{P}}$ approxmation z image $=$ 'data'

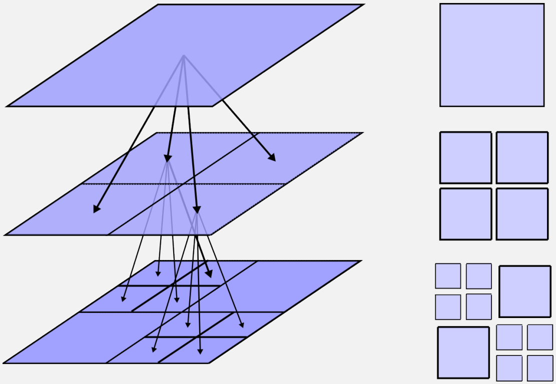
Goal: Efficient mimization of the functional

$$H_{\gamma,z} : \mathfrak{S} \to \mathbb{R}, \quad (\mathcal{P}, f_{\mathcal{P}}) \mapsto \gamma \cdot |\mathcal{P}| + \|z - f_{\mathcal{P}}\|_2^2.$$

Result $(\hat{\mathcal{P}}, \hat{f}_{\hat{\mathcal{P}}}) \in \operatorname{argmin}_{(\mathcal{P}, f_{\mathcal{P}})} H_{\gamma,z}$ can be interpreted as *optimal compromise between regularity and fidelity to data.*

Minimization of a Functional using Quadtrees

Separation of a two-dimensional range into 4 equally seized parts.

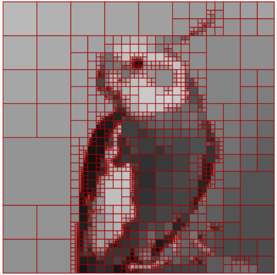


Algorithmus: Minimize(z,r,γ)

Input: Image data $z \in \mathbb{R}^S$, rectangle $r \subset S$, regularization $\gamma > 0$
Output: $\min_T \gamma|L(T)| + \|z - \mu_{L(T)}\|_2^2$

if $|r| = 0$ **then return** 0
$m \leftarrow \gamma + \sum_{s \in r} (z_s - \mu_r)^2$
if $|r| > 1$ **then**
  Split r into r_{ll},r_{lr},r_{ul},r_{ur}
  $m_1 \leftarrow \mathrm{Minimize}(z, r_{ll}, \gamma)$; $m_2 \leftarrow \mathrm{Minimize}(z, r_{lr}, \gamma)$
  $m_3 \leftarrow \mathrm{Minimize}(z, r_{ul}, \gamma)$; $m_4 \leftarrow \mathrm{Minimize}(z, r_{ur}, \gamma)$
  $m' \leftarrow m_1 + m_2 + m_3 + m_4$
else
  $m' \leftarrow \infty$
if $m' < m$ **then** $m \leftarrow m'$
return m

Minimization of a Functional using Quadtrees



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Longest ascending Sequence in matrix

Given $n \times m$ matrix A:

9	27	42	41	48
35	39	8	3	5
12	49	2	38	4
15	47	29	28	6
19	1	25	33	10

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Wanted longest ascending sequence:

$$4, 6, 28, 29, 47, 49$$

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- What is the meaning of each entry?
  - In $T[x][y]$ is the length of the longest ascending sequence that ends in $A[x][y]$
  - In $S[x][y]$ are the coordinates of the predecessor in ascending sequence (if exists)

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- Arbitrary order, if entry is already computed skip it otherwise compute for smaller neighbor recursively.

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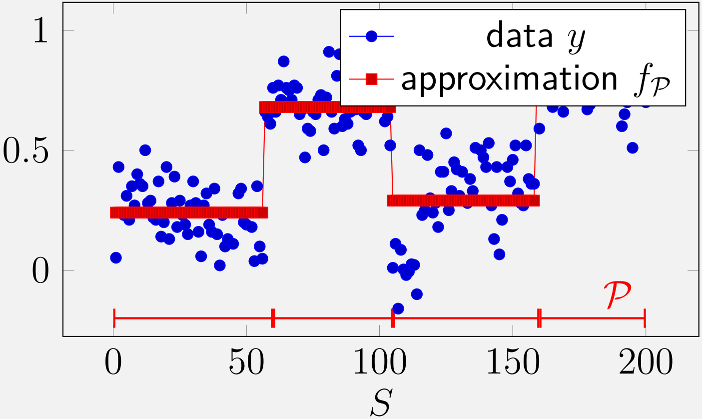
  - Consider all entries to find one with a longest sequence. From there, we can reconstruct the solution by following the corresponding predecessors.

Piecewise Constant Approximation



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- $\mathcal{P}$: (set of intervals I_i, such that $\cup_i I_i = S$).
- **Goal:** find the partition $\hat{\mathcal{P}}$ such that $H_{\gamma,y}(\hat{\mathcal{P}})$ is minimal
- Utilize: efficient computation of the mean using prefix sums (exercise 1): $\mu_I = \frac{1}{|I|} \sum_{i \in I} y_i$

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- **Goal:** find the partition $\hat{\mathcal{P}}$ such that $H_{\gamma,y}(\hat{\mathcal{P}})$ is minimal
- **Dynamic programming**: definition of the table, computation of an entry, calculation order, extracting solution
- Utilize*: $H_{\gamma,y}(\mathcal{P} \cup \{[l,r)\}) = H_{\gamma,y}(\mathcal{P}) + \gamma + e_{[l,r)}$

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*Assumption: $\mathcal{P} \cup \{[l,r)\}$ is a partition

Questions?