# 4. Searching

Linear Search, Binary Search, (Interpolation Search,) Lower Bounds
[Ottman/Widmayer, Kap. 3.2, Cormen et al, Kap. 2: Problems 2.1-3,2.2-3,2.3-5]

# The Search Problem

Provided

- A set of data sets

  telephone book, dictionary, symbol table

- Each dataset has a key $k$.
- Keys are comparable: unique answer to the question $k_1 \leq k_2$ for keys $k_1$, $k_2$.

Task: find data set by key $k$.

# Search in Array

Provided

- Array $A$ with $n$ elements $(A[1], \dots, A[n])$.
- Key $b$

Wanted: index $k$, $1 \le k \le n$ with $A[k] = b$ or "not found".

| 22 | 20 | 32 | 10 | 35 | 24 | 42 | 38 | 28 | 41 |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

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$$\frac{1}{n}\sum_{i=1}^{n} i = \frac{n+1}{2}.$$

# Search in a Sorted Array

Provided

- Sorted array $A$ with $n$ elements $(A[1], \dots, A[n])$ with $A[1] \leq A[2] \leq \cdots \leq A[n]$.
- Key $b$

Wanted: index $k$, $1 \leq k \leq n$ with $A[k] = b$ or "not found".

| 10 | 20 | 22 | 24 | 28 | 32 | 35 | 38 | 41 | 42 |
|---|---|---|---|---|---|---|---|---|---|
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# Divide and Conquer!

Search $b = 23$.

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| | | | | | | | | | |
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$b < 28$

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$b < 28$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 10 | 20 | 22 | 24 | 28 | 32 | 35 | 38 | 41 | 42 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |

$b > 20$

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| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 10 | 20 | 22 | 24 | 28 | 32 | 35 | 38 | 41 | 42 | $b < 28$ |
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| | | | | | | | | | | |
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| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | |

| | | | | | | | | | | |
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| | | | | | | | | | | |
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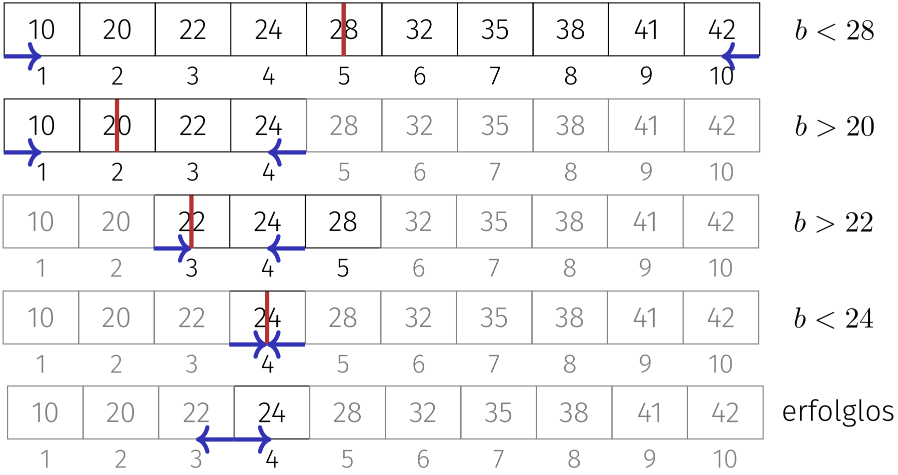

# Binary Search Algorithm BSearch$(A, l, r, b)$

**Input:** Sorted array $A$ of $n$ keys. Key $b$. Bounds $1 \leq l, r \leq n$ mit $l \leq r$ or $l = r + 1$.

**Output:** Index $m \in [l, \ldots, r+1]$, such that $A[i] \leq b$ for all $l \leq i < m$ and $A[i] \geq b$ for all $m < i \leq r$.

$m \leftarrow \lfloor (l + r)/2 \rfloor$
**if** $l > r$ **then** // Unsuccessful search
    **return** l
**else if** $b = A[m]$ **then**// found
    **return** $m$
**else if** $b < A[m]$ **then**// element to the left
    **return** BSearch$(A, l, m-1, b)$
**else** // $b > A[m]$: element to the right
    **return** BSearch$(A, m+1, r, b)$

# Analysis (worst case)

Recurrence ($n = 2^k$)

$$T(n) = \begin{cases} d & \text{falls } n = 1, \\ T(n/2) + c & \text{falls } n > 1. \end{cases}$$

Compute:

$$T(n) = T\left(\frac{n}{2}\right) + c$$

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Compute:

$$\begin{aligned} T(n) &= T\left(\frac{n}{2}\right) + c = T\left(\frac{n}{4}\right) + 2c = \ldots \\ &= T\left(\frac{n}{2^i}\right) + i \cdot c \end{aligned}$$

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**Guess** : $T(n) = d + c \cdot \log_2 n$

**Proof by induction:**

- Base clause: $T(1) = d$.
- Hypothesis: $T(n/2) = d + c \cdot \log_2 n/2$

# Analysis (worst case)

$$T(n) = \begin{cases} d & \text{if } n = 1, \\ T(n/2) + c & \text{if } n > 1. \end{cases}$$

**Guess** : $T(n) = d + c \cdot \log_2 n$

**Proof by induction:**

- Base clause: $T(1) = d$.
- Hypothesis: $T(n/2) = d + c \cdot \log_2 n/2$
- Step: $(n/2 \to n)$

$$T(n) = T(n/2) + c = d + c \cdot (\log_2 n - 1) + c = d + c \log_2 n.$$

# Result

*Theorem 8*

*The binary sorted search algorithm requires* $\Theta(\log n)$ *fundamental operations.*

# Iterative Binary Search Algorithm

**Input:** Sorted array $A$ of $n$ keys. Key $b$.
**Output:** Index of the found element. $0$, if unsuccessful.

```
l ← 1; r ← n
while l ≤ r do
    m ← ⌊(l + r)/2⌋
    if A[m] = b then
        return m
    else if A[m] < b then
        l ← m + 1
    else
        r ← m − 1
return NotFound;
```

# Correctness

Algorithm terminates only if $A$ is empty or $b$ is found.

**Invariant:** If $b$ is in $A$ then $b$ is in domain $A[l..r]$

**Proof by induction**

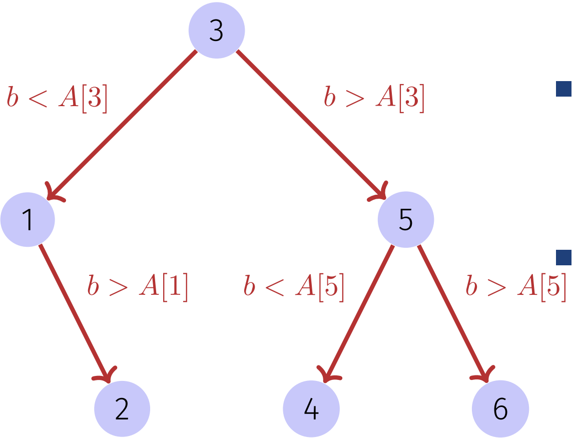
- Base clause $b \in A[1..n]$ (oder nicht)
- Hypothesis: invariant holds after $i$ steps.
- Step:
  $b < A[m] \Rightarrow b \in A[l..m-1]$
  $b > A[m] \Rightarrow b \in A[m+1..r]$

# Lower Bounds

Binary Search (worst case): $\Theta(\log n)$ comparisons.

Does for *any* search algorithm in a sorted array (worst case) hold that number comparisons = $\Omega(\log n)$?

# Decision tree




- For any input $b = A[i]$ the algorithm must succeed $\Rightarrow$ decision tree comprises at least $n$ nodes.
- Number comparisons in worst case = height of the tree = maximum number nodes from root to leaf.

# Decision Tree

Binary tree with height $h$ has at most $2^0 + 2^1 + \cdots + 2^{h-1} = 2^h - 1 < 2^h$ nodes.

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Decision tree with $n$ node has at least height $\log_2 n$.

Number decisions = $\Omega(\log n)$.

**Theorem 9**

*Any comparison-based search algorithm on sorted data with length $n$ requires in the worst case $\Omega(\log n)$ comparisons.*

# Lower bound for Search in Unsorted Array

**Theorem 10**

*Any comparison-based search algorithm with unsorted data of length $n$ requires in the worst case $\Omega(n)$ comparisons.*

# Attempt

**Correct?**

”Proof”: to find $b$ in $A$, $b$ must be compared with each of the $n$ elements $A[i]$ $(1 \leq i \leq n)$.

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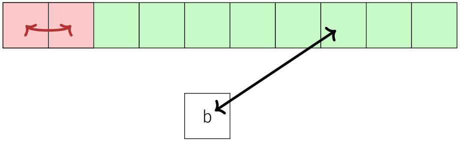
**Correct?**

”Proof”: to find $b$ in $A$, $b$ must be compared with each of the $n$ elements $A[i]$ $(1 \leq i \leq n)$.
Wrong argument! It is still possible to compare elements within $A$.

# Better Argument




- Different comparisons: Number comparisons with $b$: $e$ Number comparisons without $b$: $i$
- Comparisons induce $g$ groups. Initially $g = n$.

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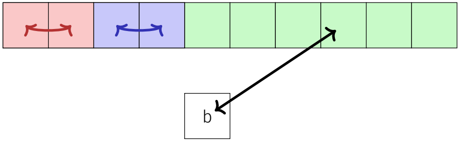


- Different comparisons: Number comparisons with $b$: $e$ Number comparisons without $b$: $i$
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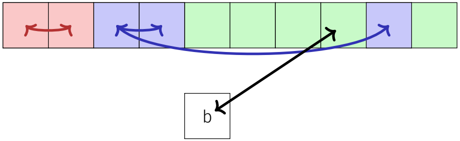


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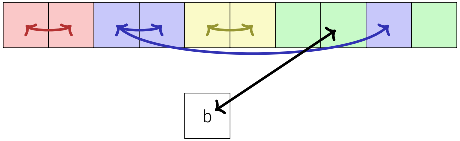


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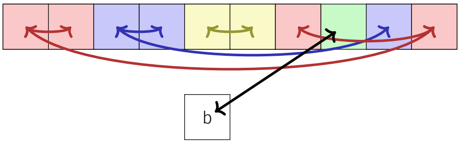


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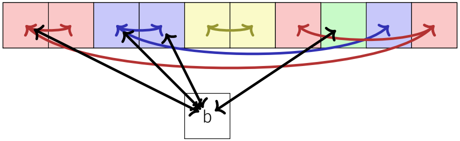


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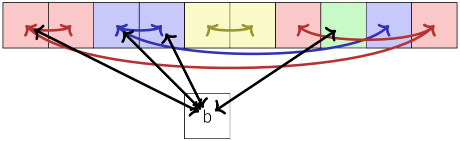


- Different comparisons: Number comparisons with $b$: $e$ Number comparisons without $b$: $i$
- Comparisons induce $g$ groups. Initially $g = n$.
- To connect two groups at least one comparison is needed: $n - g \leq i$.
- At least one element per group must be compared with $b$.
- Number comparisons $i + e \geq n - g + g = n$. ■

# 5. Selection

The Selection Problem, Randomised Selection, Linear Worst-Case Selection
[Ottman/Widmayer, Kap. 3.1, Cormen et al, Kap. 9]

# The Problem of Selection

Input

- unsorted array $A = (A_1, \ldots, A_n)$ with pairwise different values
- Number $1 \le k \le n$.

Output $A[i]$ with $|\{j : A[j] < A[i]\}| = k - 1$

**Special cases**

$k = 1$: Minimum: Algorithm with $n$ comparison operations trivial.
$k = n$: Maximum: Algorithm with $n$ comparison operations trivial.
$k = \lfloor n/2 \rfloor$: Median.

# Naive Algorithm

# Naive Algorithm

Repeatedly find and remove the minimum $\Theta(k \cdot n)$.
$\rightarrow$ Median in $\Theta(n^2)$

# Better Approaches

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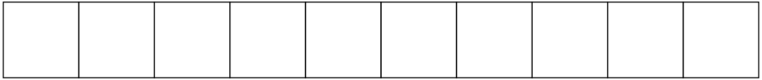
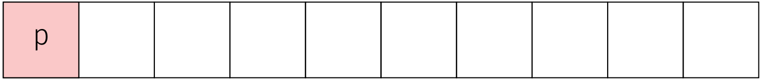
- Sorting (covered soon): $\Theta(n \log n)$
- Use a pivot: $\Theta(n)$ !

# Use a pivot



# Use a pivot

1. Choose a (an arbitrary) **pivot** $p$



# Use a pivot

1. Choose a (an arbitrary) **pivot** $p$
2. Partition $A$ in two parts, and determine the rank of $p$ by counting the indices $i$ with $A[i] \leq p$.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| p | $>$ | $\leq$ | $\leq$ | $>$ | $>$ | $\leq$ | $\leq$ | $>$ | $\leq$ |

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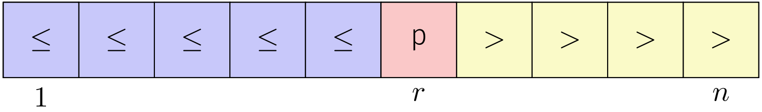
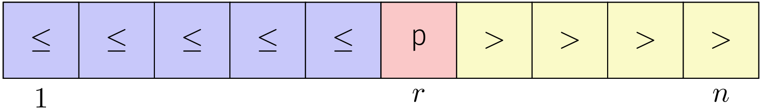


# Use a pivot

1. Choose a (an arbitrary) **pivot** $p$
2. Partition $A$ in two parts, and determine the rank of $p$ by counting the indices $i$ with $A[i] \leq p$.
3. Recursion on the relevant part. If $k = r$ then found.



# Algorithm `Partition`$(A, l, r, p)$

**Input:** Array $A$, that contains the pivot $p$ in $A[l, \ldots, r]$ at least once.
**Output:** Array $A$ partitioned in $[l, \ldots, r]$ around $p$. Returns position of $p$.

```
while l ≤ r do
    while A[l] < p do
        l ← l + 1
    while A[r] > p do
        r ← r − 1
    swap(A[l], A[r])
    if A[l] = A[r] then
        l ← l + 1
return l-1
```

# Correctness: Invariant

Invariant $I$: $A_i \leq p \; \forall i \in [0, l)$, $A_i \geq p \; \forall i \in (r, n]$, $\exists k \in [l, r] : A_k = p$.

**while** $l \leq r$ **do**

$I$

- **while** $A[l] < p$ **do**
  - $l \leftarrow l + 1$

$I$ und $A[l] \geq p$

- **while** $A[r] > p$ **do**
  - $r \leftarrow r - 1$

$I$ und $A[r] \leq p$

- swap($A[l]$, $A[r]$)

$I$ und $A[l] \leq p \leq A[r]$

- **if** $A[l] = A[r]$ **then**
  - $l \leftarrow l + 1$

$I$

**return** l-1

# Correctness: progress

**while** $l \leq r$ **do**
    **while** $A[l] < p$ **do** — progress if $A[l] < p$
        $l \leftarrow l + 1$
    **while** $A[r] > p$ **do** — progress if $A[r] > p$
        $r \leftarrow r - 1$
    swap($A[l]$, $A[r]$) — progress if $A[l] > p$ oder $A[r] < p$
    **if** $A[l] = A[r]$ **then** — progress if $A[l] = A[r] = p$
        $l \leftarrow l + 1$

**return** l-1

# Choice of the pivot.

The minimum is a bad pivot: worst case $\Theta(?)$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $p_1$ | | | | | | | | | |

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|---|---|---|---|---|---|---|---|---|---|
| $p_1$ | $p_2$ | | | | | | | | |

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| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $p_1$ | $p_2$ | $p_3$ | | | | | | | |

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The minimum is a bad pivot: worst case $\Theta(?)$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $p_1$ | $p_2$ | $p_3$ | $p_4$ | | | | | | |

# Choice of the pivot.

The minimum is a bad pivot: worst case $\Theta(n^2)$

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $p_1$ | $p_2$ | $p_3$ | $p_4$ | $p_5$ | | | | | |

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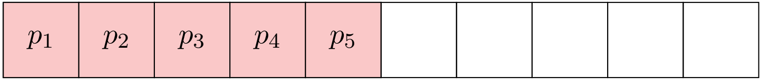
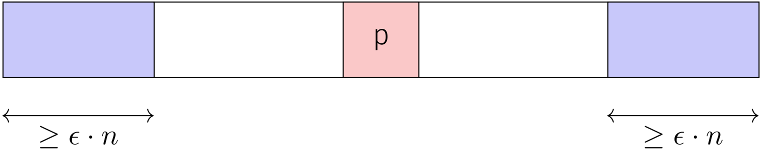
The minimum is a bad pivot: worst case $\Theta(n^2)$




A good pivot has a linear number of elements on both sides.



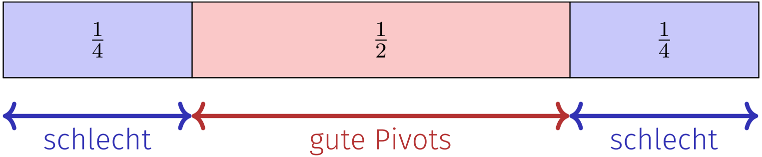
# Analysis

Partitioning with factor $q$ $(0 < q < 1)$: two groups with $q \cdot n$ and $(1-q) \cdot n$ elements (without loss of generality $g \geq 1 - q$).

$$
\begin{aligned}
T(n) &\leq T(q \cdot n) + c \cdot n \\
&\leq c \cdot n + q \cdot c \cdot n + T(q^2 \cdot n) \leq \ldots = c \cdot n \sum_{i=0}^{\log_q(n)-1} q^i + T(1) \\
&\leq c \cdot n \underbrace{\sum_{i=0}^{\infty} q^i}_{\text{geom. Reihe}} + d = c \cdot n \cdot \frac{1}{1-q} + d = \mathcal{O}(n)
\end{aligned}
$$

# How can we achieve this?

Randomness to our rescue (Tony Hoare, 1961). In each step choose a random pivot.




Probability for a good pivot in one trial: $\frac{1}{2} =: \rho$.
Probability for a good pivot after $k$ trials: $(1-\rho)^{k-1} \cdot \rho$.
Expected number of trials: $1/\rho = 2$ (Expected value of the geometric distribution:)

# Algorithm `Quickselect` $(A, l, r, k)$

**Input:** Array $A$ with length $n$. Indices $1 \leq l \leq k \leq r \leq n$, such that for all $x \in A[l..r] : |\{j|A[j] \leq x\}| \geq l$ and $|\{j|A[j] \leq x\}| \leq r$.

**Output:** Value $x \in A[l..r]$ with $|\{j|A[j] \leq x\}| \geq k$ and $|\{j|x \leq A[j]\}| \geq n-k+1$

```
if l=r then
    return A[l];
x ← RandomPivot(A, l, r)
m ← Partition(A, l, r, x)
if k < m then
    return QuickSelect(A, l, m − 1, k)
else if k > m then
    return QuickSelect(A, m + 1, r, k)
else
    return A[k]
```

# Algorithm `RandomPivot` $(A, l, r)$

**Input:** Array $A$ with length $n$. Indices $1 \leq l \leq r \leq n$
**Output:** Random "good" pivot $x \in A[l, \ldots, r]$

**repeat**
    choose a random pivot $x \in A[l..r]$
    $p \leftarrow l$
    **for** $j = l$ **to** $r$ **do**
        **if** $A[j] \leq x$ **then** $p \leftarrow p + 1$
**until** $\left\lfloor \frac{3l+r}{4} \right\rfloor \leq p \leq \left\lceil \frac{l+3r}{4} \right\rceil$
**return** $x$

*This algorithm is only of theoretical interest and delivers a good pivot in 2 expected iterations. Practically, in algorithm QuickSelect a uniformly chosen random pivot can be chosen or a deterministic one such as the median of three elements.*

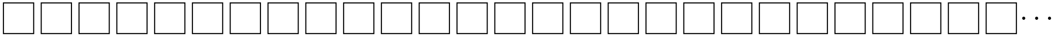
# Median of medians

Goal: find an algorithm that even in worst case requires only linearly many steps.

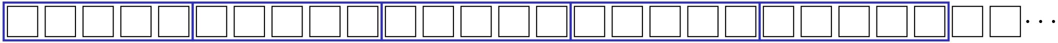
Algorithm Select ($k$-smallest)

- Consider groups of five elements.
- Compute the median of each group (straighforward)
- Apply Select recursively on the group medians.
- Partition the array around the found median of medians. Result: $i$
- If $i = k$ then result. Otherwise: select recursively on the proper side.

# Median of medians

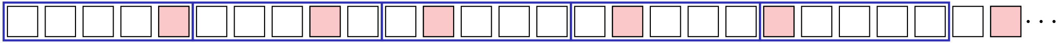


# Median of medians



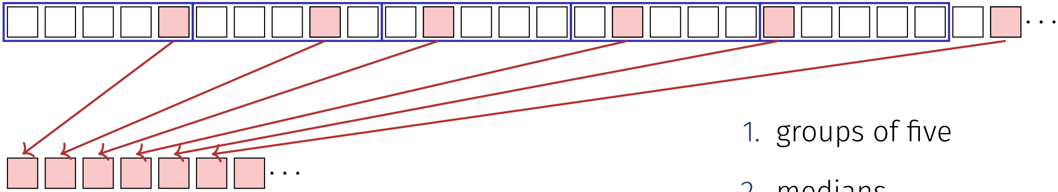
1. groups of five

# Median of medians



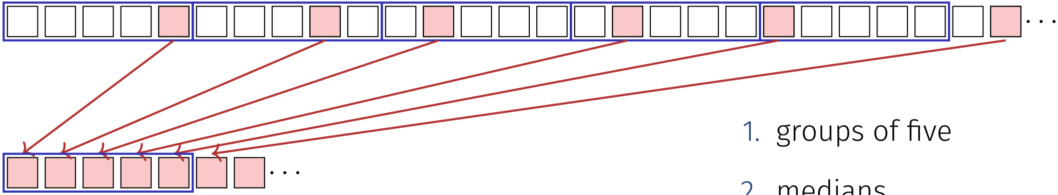
1. groups of five
2. medians

# Median of medians



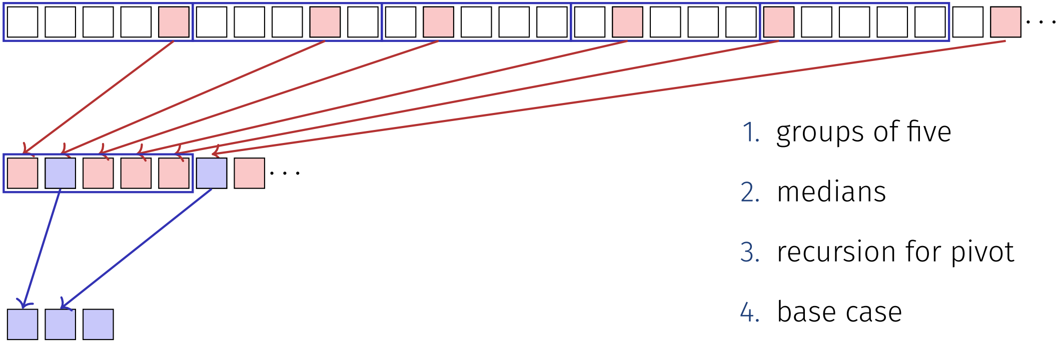
1. groups of five
2. medians
3. recursion for pivot

# Median of medians



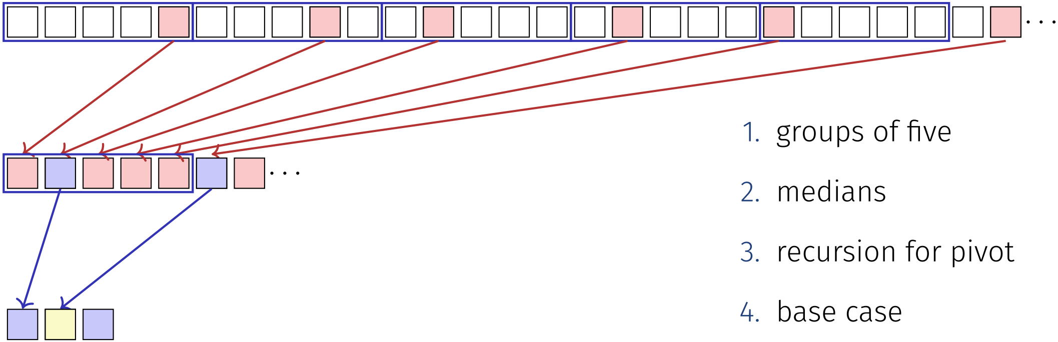
1. groups of five
2. medians
3. recursion for pivot

# Median of medians



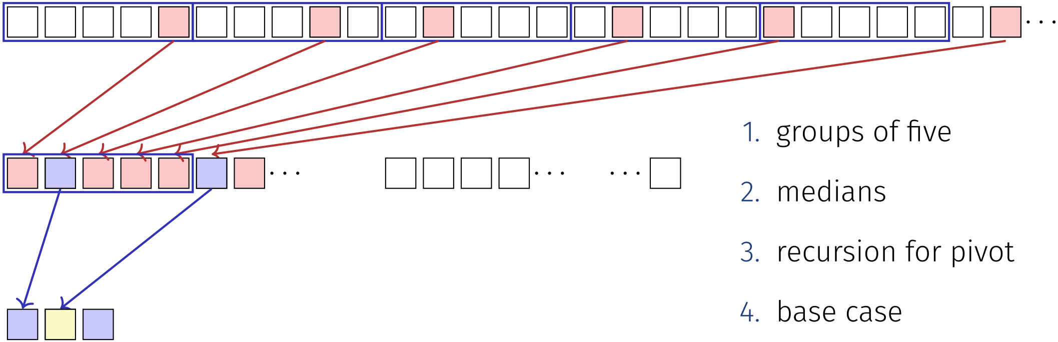
1. groups of five
2. medians
3. recursion for pivot
4. base case

# Median of medians



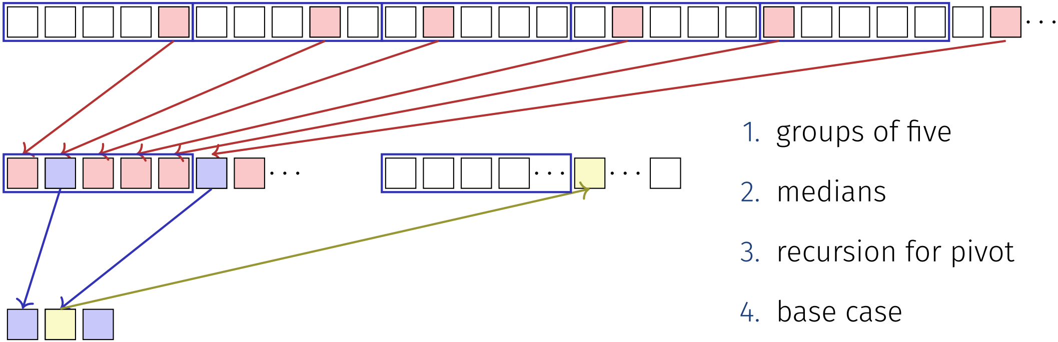

1. groups of five
2. medians
3. recursion for pivot
4. base case
5. pivot (level 1)

# Median of medians



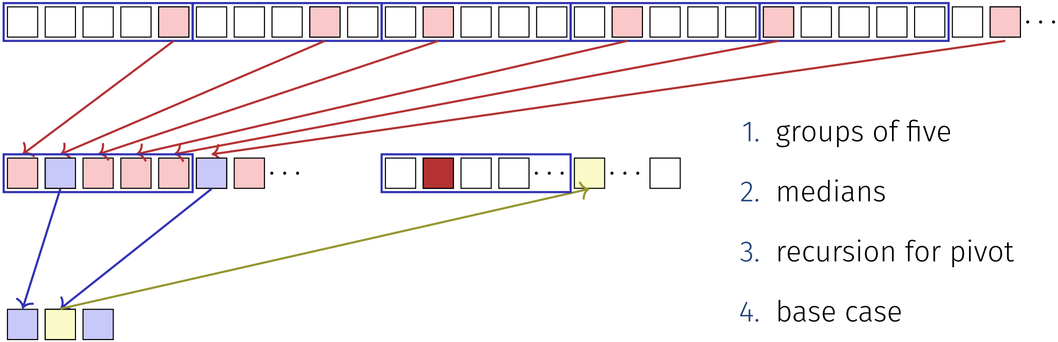
1. groups of five
2. medians
3. recursion for pivot
4. base case
5. pivot (level 1)

# Median of medians



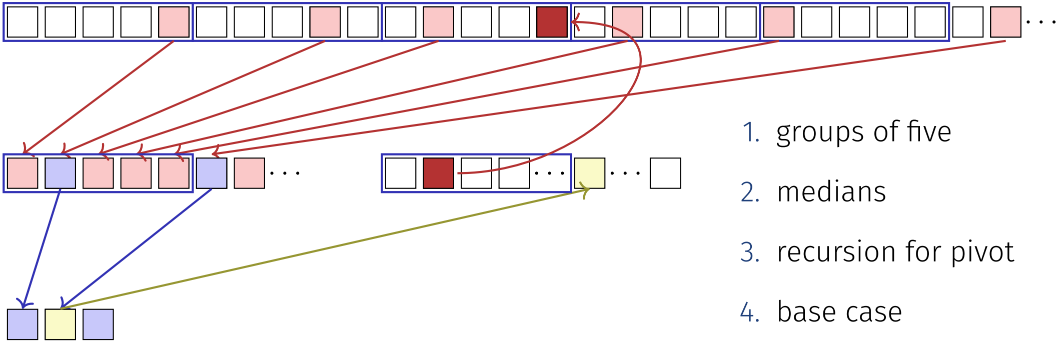
1. groups of five
2. medians
3. recursion for pivot
4. base case
5. pivot (level 1)
6. partition (level 1)

# Median of medians



1. groups of five
2. medians
3. recursion for pivot
4. base case
5. pivot (level 1)
6. partition (level 1)
7. median = pivot level 0

# Median of medians




1. groups of five
2. medians
3. recursion for pivot
4. base case
5. pivot (level 1)
6. partition (level 1)
7. median = pivot level 0
8. 2. recursion starts

# Algorithmus $\texttt{MMSelect}(A, l, r, k)$

**Input:** Array $A$ with length $n$ with pair-wise different entries. $1 \leq l \leq k \leq r \leq n$,
$A[i] < A[k] \; \forall \; 1 \leq i < l$, $A[i] > A[k] \; \forall \; r < i \leq n$

**Output:** Value $x \in A$ with $|\{j | A[j] \leq x\}| = k$

$m \leftarrow \texttt{MMChoose}(A, l, r)$

$i \leftarrow \texttt{Partition}(A, l, r, m)$

**if** $k < i$ **then**

  **return** $\texttt{MMSelect}(A, l, i-1, k)$

**else if** $k > i$ **then**

  **return** $\texttt{MMSelect}(A, i+1, r, k)$

**else**

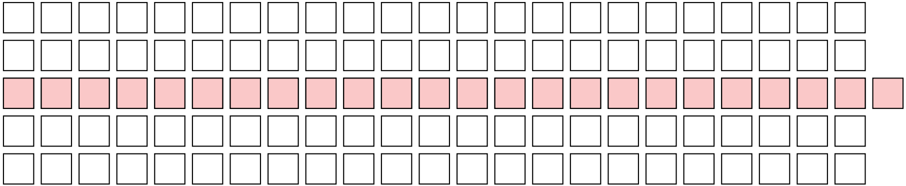
  **return** $A[i]$

# Algorithmus `MMChoose`$(A, l, r)$

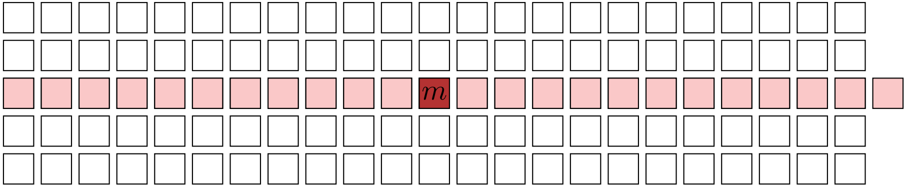
**Input:** Array $A$ with length $n$ with pair-wise different entries. $1 \leq l \leq r \leq n$.
**Output:** Median $m$ of medians

**if** $r - l \leq 5$ **then**
    return MedianOf5($A[l, \dots, r]$)
**else**
    $A' \leftarrow$ MedianOf5Array($A[l, \dots, r]$)
    **return** `MMSelect`$(A', 1, |A'|, \left\lfloor \frac{|A'|}{2} \right\rfloor)$

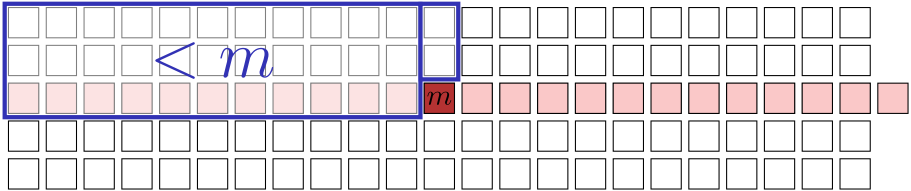
# How good is this?



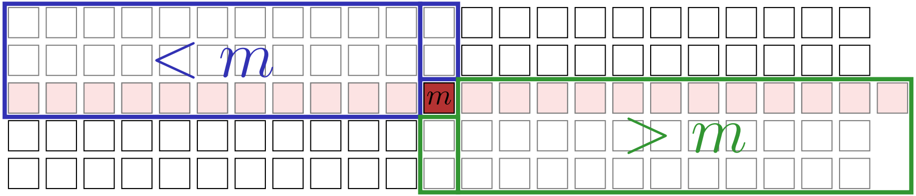
# How good is this?



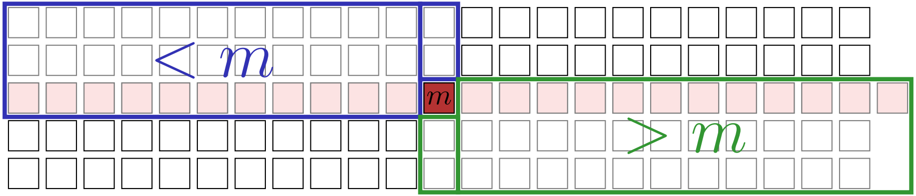
# How good is this?



# How good is this?



# How good is this?

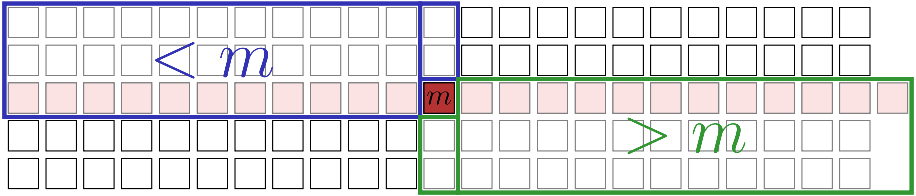



- Number groups of five: $\lceil \frac{n}{5} \rceil$, without median group: $\lceil \frac{n}{5} \rceil - 1$
- Minimal number groups left / right of Mediangroup $\left\lfloor \frac{1}{2}\left(\left\lceil \frac{n}{5} \right\rceil - 1\right)\right\rfloor$
- Minimal number of points less than / greater than $m$

$$3\left\lfloor \frac{1}{2}\left(\left\lceil \frac{n}{5} \right\rceil - 1\right)\right\rfloor \geq 3\left\lfloor \frac{1}{2}\left(\frac{n}{5} - 1\right)\right\rfloor \geq 3\left(\frac{n}{10} - \frac{1}{2} - 1\right) > \frac{3n}{10} - 6$$

(Fill rest group with points from the median group)

# How good is this?




- Number groups of five: $\lceil \frac{n}{5} \rceil$, without median group: $\lceil \frac{n}{5} \rceil - 1$
- Minimal number groups left / right of Mediangroup $\left\lfloor \frac{1}{2} \left( \lceil \frac{n}{5} \rceil - 1 \right) \right\rfloor$
- Minimal number of points less than / greater than $m$

$$3 \left\lfloor \frac{1}{2} \left( \left\lceil \frac{n}{5} \right\rceil - 1 \right) \right\rfloor \geq 3 \left\lfloor \frac{1}{2} \left( \frac{n}{5} - 1 \right) \right\rfloor \geq 3 \left( \frac{n}{10} - \frac{1}{2} - 1 \right) > \frac{3n}{10} - 6$$

(Fill rest group with points from the median group)

$\Rightarrow$ Recursive call with maximally $\lceil \frac{7n}{10} + 6 \rceil$ elements.

# Analysis

Recursion inequality:

$$T(n) \leq T\left(\left\lceil \frac{n}{5} \right\rceil\right) + T\left(\left\lceil \frac{7n}{10} + 6 \right\rceil\right) + d \cdot n.$$

with some constant $d$.
Claim:

$$T(n) = \mathcal{O}(n).$$

# Proof

Base clause:[6] choose $c$ large enough such that

$$T(n) \leq c \cdot n \text{ für alle } n \leq n_0.$$

Induction hypothesis: $H(n)$

$$T(i) \leq c \cdot i \text{ für alle } i < n.$$

Induction step: $H(k)_{k<n} \to H(n)$

$$\begin{aligned} T(n) &\leq T\left(\left\lceil \frac{n}{5} \right\rceil\right) + T\left(\left\lceil \frac{7n}{10} + 6 \right\rceil\right) + d \cdot n \\ &\leq c \cdot \left\lceil \frac{n}{5} \right\rceil + c \cdot \left\lceil \frac{7n}{10} + 6 \right\rceil + d \cdot n \qquad (\text{for } n > 20). \end{aligned}$$

[6]It will turn out in the induction step that the base case has to hold of some fixed $n_0 > 0$. Because an arbitrarily large value can be chosen for $c$ and because there is a limited number of terms, this is a simple extension of the base case for $n = 1$

# Proof

Induction step:

$$T(n) \overset{n>20}{\leq} c \cdot \left\lceil \frac{n}{5} \right\rceil + c \cdot \left\lceil \frac{7n}{10} + 6 \right\rceil + d \cdot n$$
$$\leq c \cdot \frac{n}{5} + c + c \cdot \frac{7n}{10} + 6c + c + d \cdot n = \frac{9}{10} \cdot c \cdot n + 8c + d \cdot n.$$

To show

$$\exists n_0, \exists c \quad | \quad \frac{9}{10} \cdot c \cdot n + 8c + d \cdot n \leq cn \quad \forall n \geq n_0$$

thus

$$8c + d \cdot n \leq \frac{1}{10} cn \quad \Leftrightarrow \quad n \geq \frac{80c}{c - 10d}$$

Set, for example $c = 90d, n_0 = 91$ $\qquad \Rightarrow T(n) \leq cn \; \forall \; n \geq n_0$ $\blacksquare$

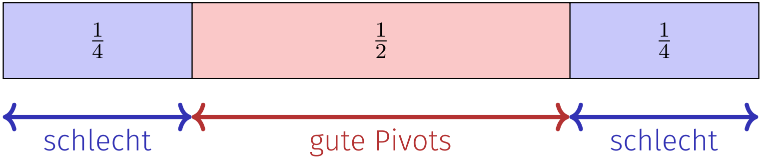
# Result

**Theorem 11**

*The $k$-th element of a sequence of $n$ elements can, in the worst case, be found in $\Theta(n)$ steps.*

# Overview

| | | |
|---|---|---|
| 1. | Repeatedly find minimum | $\mathcal{O}(n^2)$ |
| 2. | Sorting and choosing $A[i]$ | $\mathcal{O}(n \log n)$ |
| 3. | Quickselect with random pivot | $\mathcal{O}(n)$ expected |
| 4. | Median of Medians (Blum) | $\mathcal{O}(n)$ worst case |



# 5.1 Appendix

Derivation of some mathemmatical formulas

# [Expected value of the Geometric Distribution]

Random variable $X \in \mathbb{N}^+$ with $\mathbb{P}(X = k) = (1-p)^{k-1} \cdot p$.

Expected value

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} \cdot p = \sum_{k=1}^{\infty} k \cdot q^{k-1} \cdot (1-q) \\
&= \sum_{k=1}^{\infty} k \cdot q^{k-1} - k \cdot q^k = \sum_{k=0}^{\infty} (k+1) \cdot q^k - k \cdot q^k \\
&= \sum_{k=0}^{\infty} q^k = \frac{1}{1-q} = \frac{1}{p}.
\end{aligned}$$