4. Searching

Linear Search, Binary Search, (Interpolation Search,) Lower Bounds
[Ottman/Widmayer, Kap. 3.2, Cormen et al, Kap. 2: Problems 2.1-3,2.2-3,2.3-5]

The Search Problem

Provided

- A set of data sets

  telephone book, dictionary, symbol table

- Each dataset has a key k.
- Keys are comparable: unique answer to the question $k_1 \leq k_2$ for keys k_1, k_2.

Task: find data set by key k.

Search in Array

Provided

- Array A with n elements $(A[1], \dots, A[n])$.
- Key b

Wanted: index k, $1 \leq k \leq n$ with $A[k] = b$ or "not found".

22	20	32	10	35	24	42	38	28	41
1	2	3	4	5	6	7	8	9	10

Linear Search

Traverse the array from $A[1]$ to $A[n]$.

- **Best case:** 1 comparison.
- **Worst case:** n comparisons.
- Assumption: each permutation of the n keys with same probability. **Expected** number of comparisons for the successful search:

$$\frac{1}{n}\sum_{i=1}^{n} i = \frac{n+1}{2}.$$

Search in a Sorted Array

Provided

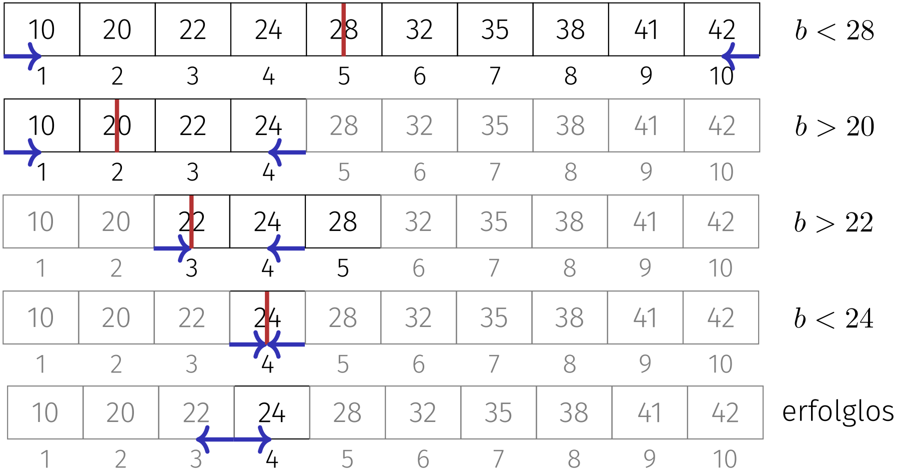
- Sorted array A with n elements $(A[1], \dots, A[n])$ with $A[1] \leq A[2] \leq \cdots \leq A[n]$.
- Key b

Wanted: index k, $1 \leq k \leq n$ with $A[k] = b$ or "not found".

10	20	22	24	28	32	35	38	41	42
1	2	3	4	5	6	7	8	9	10

Divide and Conquer!

Search $b = 23$.



Binary Search Algorithm BSearch(A, l, r, b)

Input: Sorted array A of n keys. Key b. Bounds $1 \leq l, r \leq n$ mit $l \leq r$ or $l = r + 1$.

Output: Index $m \in [l, \dots, r+1]$, such that $A[i] \leq b$ for all $l \leq i < m$ and $A[i] \geq b$ for all $m < i \leq r$.

$m \leftarrow \lfloor (l+r)/2 \rfloor$
if $l > r$ **then** // Unsuccessful search
  return l
else if $b = A[m]$ **then**// found
  return m
else if $b < A[m]$ **then**// element to the left
  return BSearch$(A, l, m-1, b)$
else // $b > A[m]$: element to the right
  return BSearch$(A, m+1, r, b)$

Analysis (worst case)

Recurrence ($n = 2^k$)

$$T(n) = \begin{cases} d & \text{falls } n = 1, \\ T(n/2) + c & \text{falls } n > 1. \end{cases}$$

Compute:

$$\begin{aligned} T(n) &= T\left(\frac{n}{2}\right) + c = T\left(\frac{n}{4}\right) + 2c = \dots \\ &= T\left(\frac{n}{2^i}\right) + i \cdot c \\ &= T\left(\frac{n}{n}\right) + \log_2 n \cdot c = d + c \cdot \log_2 n \in \Theta(\log n) \end{aligned}$$

Analysis (worst case)

$$T(n) = \begin{cases} d & \text{if } n = 1, \\ T(n/2) + c & \text{if } n > 1. \end{cases}$$

Guess : $T(n) = d + c \cdot \log_2 n$

Proof by induction:

- Base clause: $T(1) = d$.
- Hypothesis: $T(n/2) = d + c \cdot \log_2 n/2$
- Step: $(n/2 \to n)$

$$T(n) = T(n/2) + c = d + c \cdot (\log_2 n - 1) + c = d + c \log_2 n.$$

Result

Theorem 8

The binary sorted search algorithm requires $\Theta(\log n)$ fundamental operations.

Iterative Binary Search Algorithm

Input: Sorted array A of n keys. Key b.
Output: Index of the found element. 0, if unsuccessful.

```
l ← 1; r ← n
while l ≤ r do
    m ← ⌊(l + r)/2⌋
    if A[m] = b then
        return m
    else if A[m] < b then
        l ← m + 1
    else
        r ← m − 1
return NotFound;
```

Correctness

Algorithm terminates only if A is empty or b is found.

Invariant: If b is in A then b is in domain $A[l..r]$

Proof by induction

- Base clause $b \in A[1..n]$ (oder nicht)
- Hypothesis: invariant holds after i steps.
- Step:
  $b < A[m] \Rightarrow b \in A[l..m-1]$
  $b > A[m] \Rightarrow b \in A[m+1..r]$

[Can this be improved?]

Assumption: *values* of the array are uniformly distributed.

Example

Search for ”Becker” at the very beginning of a telephone book while search for ”Wawrinka" rather close to the end.
Binary search always starts in the middle.

Binary search always takes $m = \left\lfloor l + \frac{r-l}{2} \right\rfloor$.

[Interpolation search]

Expected relative position of b in the search interval $[l, r]$

$$\rho = \frac{b - A[l]}{A[r] - A[l]} \in [0, 1].$$

New 'middle': $l + \rho \cdot (r - l)$

Expected number of comparisons $\mathcal{O}(\log \log n)$ (without proof).

Would you always prefer interpolation search?

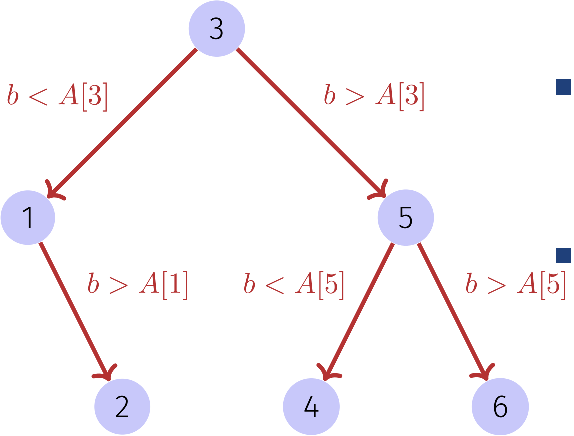
No: worst case number of comparisons $\Omega(n)$.

Lower Bounds

Binary Search (worst case): $\Theta(\log n)$ comparisons.

Does for *any* search algorithm in a sorted array (worst case) hold that number comparisons = $\Omega(\log n)$?

Decision tree



- For any input $b = A[i]$ the algorithm must succeed $\Rightarrow$ decision tree comprises at least n nodes.
- Number comparisons in worst case = height of the tree = maximum number nodes from root to leaf.

Decision Tree

Binary tree with height h has at most $2^0 + 2^1 + \cdots + 2^{h-1} = 2^h - 1 < 2^h$ nodes.

$$2^h > n \Rightarrow h > \log_2 n$$

Decision tree with n node has at least height $\log_2 n$.
Number decisions = $\Omega(\log n)$.

Theorem 9

Any comparison-based search algorithm on sorted data with length n requires in the worst case $\Omega(\log n)$ comparisons.

Lower bound for Search in Unsorted Array

Theorem 10

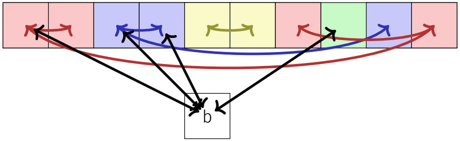
Any comparison-based search algorithm with unsorted data of length n requires in the worst case $\Omega(n)$ comparisons.

Attempt

Correct?

”Proof”: to find b in A, b must be compared with each of the n elements $A[i]$ $(1 \leq i \leq n)$.
Wrong argument! It is still possible to compare elements within A.

Better Argument



- Different comparisons: Number comparisons with b: e Number comparisons without b: i
- Comparisons induce g groups. Initially $g = n$.
- To connect two groups at least one comparison is needed: $n - g \leq i$.
- At least one element per group must be compared with b.
- Number comparisons $i + e \geq n - g + g = n$. ∎

5. Selection

The Selection Problem, Randomised Selection, Linear Worst-Case Selection
[Ottman/Widmayer, Kap. 3.1, Cormen et al, Kap. 9]

The Problem of Selection

Input

- unsorted array $A = (A_1, \dots, A_n)$ with pairwise different values
- Number $1 \leq k \leq n$.

Output $A[i]$ with $|\{j : A[j] < A[i]\}| = k - 1$

Special cases

$k = 1$: Minimum: Algorithm with n comparison operations trivial.
$k = n$: Maximum: Algorithm with n comparison operations trivial.
$k = \lfloor n/2 \rfloor$: Median.

Naive Algorithm

Repeatedly find and remove the minimum $\Theta(k \cdot n)$.
$\rightarrow$ Median in $\Theta(n^2)$

Min and Max

❓ To separately find minimum an maximum in $(A[1], \ldots, A[n])$, $2n$ comparisons are required. (How) can an algorithm with less than $2n$ comparisons for both values at a time can be found?

❗ Possible with $\frac{3}{2}n$ comparisons: compare 2 elements each and then the smaller one with min and the greater one with max.[5]

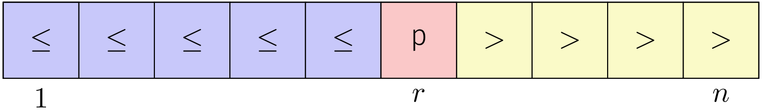
[5] An indication that the naive algorithm can be improved.

Better Approaches

- Sorting (covered soon): $\Theta(n \log n)$
- Use a pivot: $\Theta(n)$!

Use a pivot

1. Choose a (an arbitrary) **pivot** p
2. Partition A in two parts, and determine the rank of p by counting the indices i with $A[i] \leq p$.
3. Recursion on the relevant part. If $k = r$ then found.



Algorithm `Partition`(A, l, r, p)

Input: Array A, that contains the pivot p in $A[l, \ldots, r]$ at least once.
Output: Array A partitioned in $[l, \ldots, r]$ around p. Returns position of p.

```
while l ≤ r do
    while A[l] < p do
        l ← l + 1
    while A[r] > p do
        r ← r − 1
    swap(A[l], A[r])
    if A[l] = A[r] then
        l ← l + 1
return l-1
```

Correctness: Invariant

Invariant I: $A_i \leq p \; \forall i \in [0, l)$, $A_i \geq p \; \forall i \in (r, n]$, $\exists k \in [l, r] : A_k = p$.

while $l \leq r$ **do**

I

- **while** $A[l] < p$ **do**
  - $l \leftarrow l + 1$

I und $A[l] \geq p$

- **while** $A[r] > p$ **do**
  - $r \leftarrow r - 1$

I und $A[r] \leq p$

- swap($A[l]$, $A[r]$)

I und $A[l] \leq p \leq A[r]$

- **if** $A[l] = A[r]$ **then**
  - $l \leftarrow l + 1$

I

return l-1

Correctness: progress

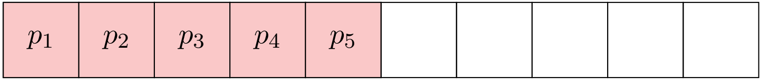
while $l \leq r$ **do**

- **while** $A[l] < p$ **do** — progress if $A[l] < p$
  - $l \leftarrow l + 1$
- **while** $A[r] > p$ **do** — progress if $A[r] > p$
  - $r \leftarrow r - 1$
- swap($A[l]$, $A[r]$) — progress if $A[l] > p$ oder $A[r] < p$
- **if** $A[l] = A[r]$ **then** — progress if $A[l] = A[r] = p$
  - $l \leftarrow l + 1$

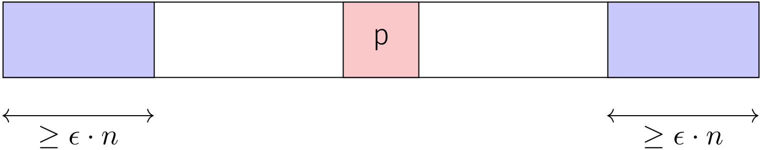
return l-1

Choice of the pivot.

The minimum is a bad pivot: worst case $\Theta(n^2)$



A good pivot has a linear number of elements on both sides.



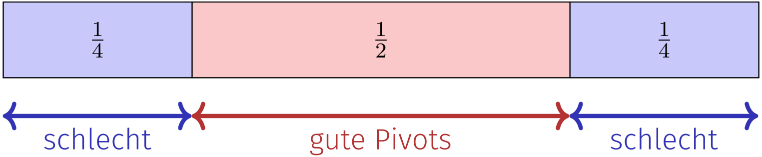
Analysis

Partitioning with factor q $(0 < q < 1)$: two groups with $q \cdot n$ and $(1-q) \cdot n$ elements (without loss of generality $g \geq 1 - q$).

$$
\begin{aligned}
T(n) &\leq T(q \cdot n) + c \cdot n \\
&\leq c \cdot n + q \cdot c \cdot n + T(q^2 \cdot n) \leq \ldots = c \cdot n \sum_{i=0}^{\log_q(n)-1} q^i + T(1) \\
&\leq c \cdot n \underbrace{\sum_{i=0}^{\infty} q^i}_{\text{geom. Reihe}} + d = c \cdot n \cdot \frac{1}{1-q} + d = \mathcal{O}(n)
\end{aligned}
$$

How can we achieve this?

Randomness to our rescue (Tony Hoare, 1961). In each step choose a random pivot.



Probability for a good pivot in one trial: $\frac{1}{2} =: \rho$.
Probability for a good pivot after k trials: $(1-\rho)^{k-1} \cdot \rho$.
Expected number of trials: $1/\rho = 2$ (Expected value of the geometric distribution:)

Algorithm `Quickselect` (A, l, r, k)

Input: Array A with length n. Indices $1 \leq l \leq k \leq r \leq n$, such that for all $x \in A[l..r] : |\{j|A[j] \leq x\}| \geq l$ and $|\{j|A[j] \leq x\}| \leq r$.

Output: Value $x \in A[l..r]$ with $|\{j|A[j] \leq x\}| \geq k$ and $|\{j|x \leq A[j]\}| \geq n - k + 1$

```
if l=r then
    return A[l];
x ← RandomPivot(A, l, r)
m ← Partition(A, l, r, x)
if k < m then
    return QuickSelect(A, l, m − 1, k)
else if k > m then
    return QuickSelect(A, m + 1, r, k)
else
    return A[k]
```

Algorithm `RandomPivot` (A, l, r)

Input: Array A with length n. Indices $1 \leq l \leq r \leq n$
Output: Random "good" pivot $x \in A[l, \dots, r]$

repeat
    choose a random pivot $x \in A[l..r]$
    $p \leftarrow l$
    for $j = l$ **to** r **do**
        if $A[j] \leq x$ **then** $p \leftarrow p + 1$
until $\left\lfloor \frac{3l+r}{4} \right\rfloor \leq p \leq \left\lceil \frac{l+3r}{4} \right\rceil$
return x

This algorithm is only of theoretical interest and delivers a good pivot in 2 expected iterations. Practically, in algorithm QuickSelect a uniformly chosen random pivot can be chosen or a deterministic one such as the median of three elements.

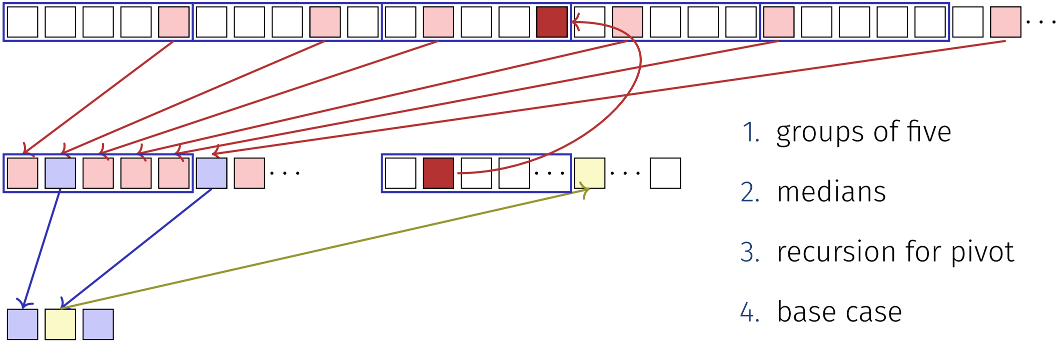
Median of medians

Goal: find an algorithm that even in worst case requires only linearly many steps.

Algorithm Select (k-smallest)

- Consider groups of five elements.
- Compute the median of each group (straighforward)
- Apply Select recursively on the group medians.
- Partition the array around the found median of medians. Result: i
- If $i = k$ then result. Otherwise: select recursively on the proper side.

Median of medians



1. groups of five
2. medians
3. recursion for pivot
4. base case
5. pivot (level 1)
6. partition (level 1)
7. median = pivot level 0
8. 2. recursion starts

Algorithmus `MMSelect`(A, l, r, k)

Input: Array A with length n with pair-wise different entries. $1 \leq l \leq k \leq r \leq n$,
$A[i] < A[k] \; \forall \; 1 \leq i < l$, $A[i] > A[k] \; \forall \; r < i \leq n$

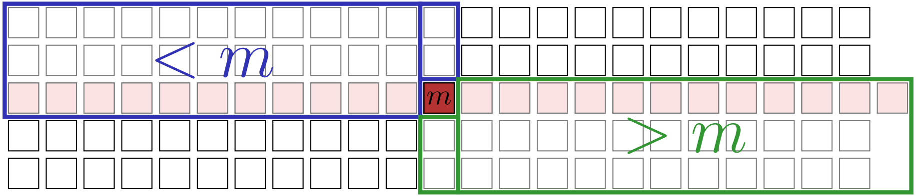
Output: Value $x \in A$ with $|\{j | A[j] \leq x\}| = k$

```
m ← MMChoose(A, l, r)
i ← Partition(A, l, r, m)
if k < i then
    return MMSelect(A, l, i − 1, k)
else if k > i then
    return MMSelect(A, i + 1, r, k)
else
    return A[i]
```

Algorithmus `MMChoose`(A, l, r)

Input: Array A with length n with pair-wise different entries. $1 \leq l \leq r \leq n$.
Output: Median m of medians
if $r - l \leq 5$ **then**
  return MedianOf5($A[l, \dots, r]$)
else
  $A' \leftarrow$ MedianOf5Array$(A[l, \dots, r])$
  return `MMSelect`$(A', 1, |A'|, \left\lfloor \frac{|A'|}{2} \right\rfloor)$

How good is this?



- Number groups of five: $\lceil \frac{n}{5} \rceil$, without median group: $\lceil \frac{n}{5} \rceil - 1$
- Minimal number groups left / right of Mediangroup $\left\lfloor \frac{1}{2}\left(\lceil \frac{n}{5} \rceil - 1\right)\right\rfloor$
- Minimal number of points less than / greater than m

$$3\left\lfloor \frac{1}{2}\left(\left\lceil \frac{n}{5} \right\rceil - 1\right)\right\rfloor \geq 3\left\lfloor \frac{1}{2}\left(\frac{n}{5} - 1\right)\right\rfloor \geq 3\left(\frac{n}{10} - \frac{1}{2} - 1\right) > \frac{3n}{10} - 6$$

(Fill rest group with points from the median group)

$\Rightarrow$ Recursive call with maximally $\lceil \frac{7n}{10} + 6 \rceil$ elements.

Analysis

Recursion inequality:

$$T(n) \leq T\left(\left\lceil \frac{n}{5} \right\rceil\right) + T\left(\left\lceil \frac{7n}{10} + 6 \right\rceil\right) + d \cdot n.$$

with some constant d.
Claim:

$$T(n) = \mathcal{O}(n).$$

Proof

Base clause:[6] choose c large enough such that

$$T(n) \le c \cdot n \text{ für alle } n \le n_0.$$

Induction hypothesis: $H(n)$

$$T(i) \le c \cdot i \text{ für alle } i < n.$$

Induction step: $H(k)_{k<n} \to H(n)$

$$\begin{aligned} T(n) &\le T\left(\left\lceil \frac{n}{5} \right\rceil\right) + T\left(\left\lceil \frac{7n}{10} + 6 \right\rceil\right) + d \cdot n \\ &\le c \cdot \left\lceil \frac{n}{5} \right\rceil + c \cdot \left\lceil \frac{7n}{10} + 6 \right\rceil + d \cdot n \qquad \text{(for } n > 20\text{).} \end{aligned}$$

[6] It will turn out in the induction step that the base case has to hold of some fixed $n_0 > 0$. Because an arbitrarily large value can be chosen for c and because there is a limited number of terms, this is a simple extension of the base case for $n = 1$

Proof

Induction step:

$$T(n) \overset{n>20}{\leq} c \cdot \left\lceil \frac{n}{5} \right\rceil + c \cdot \left\lceil \frac{7n}{10} + 6 \right\rceil + d \cdot n$$

$$\leq c \cdot \frac{n}{5} + c + c \cdot \frac{7n}{10} + 6c + c + d \cdot n = \frac{9}{10} \cdot c \cdot n + 8c + d \cdot n.$$

To show

$$\exists n_0, \exists c \quad | \quad \frac{9}{10} \cdot c \cdot n + 8c + d \cdot n \leq cn \quad \forall n \geq n_0$$

thus

$$8c + d \cdot n \leq \frac{1}{10} cn \quad \Leftrightarrow \quad n \geq \frac{80c}{c - 10d}$$

Set, for example $c = 90d, n_0 = 91$ $\qquad \Rightarrow T(n) \leq cn \; \forall \; n \geq n_0$ ■

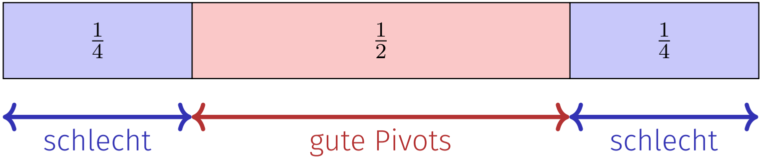
Result

Theorem 11

The k-th element of a sequence of n elements can, in the worst case, be found in $\Theta(n)$ steps.

Overview

1.	Repeatedly find minimum	$\mathcal{O}(n^2)$
2.	Sorting and choosing $A[i]$	$\mathcal{O}(n \log n)$
3.	Quickselect with random pivot	$\mathcal{O}(n)$ expected
4.	Median of Medians (Blum)	$\mathcal{O}(n)$ worst case



5.1 Appendix

Derivation of some mathemmatical formulas

[Expected value of the Geometric Distribution]

Random variable $X \in \mathbb{N}^+$ with $\mathbb{P}(X = k) = (1-p)^{k-1} \cdot p$.
Expected value

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} \cdot p = \sum_{k=1}^{\infty} k \cdot q^{k-1} \cdot (1-q) \\
&= \sum_{k=1}^{\infty} k \cdot q^{k-1} - k \cdot q^k = \sum_{k=0}^{\infty} (k+1) \cdot q^k - k \cdot q^k \\
&= \sum_{k=0}^{\infty} q^k = \frac{1}{1-q} = \frac{1}{p}.
\end{aligned}$$