# 26.8 A*-Algorithm

# Disclaimer

These slides contain the most important formalities around the A*-algorithm and its correctness. We motivate the algorithm in the lectures and give more examples there.

Another nice motivation of the algorithm can found here: `https://www.youtube.com/watch?v=bRvs8rOQU-Q`

# A*-Algorithm

Prerequisites

- Positively weighted graph $G = (V, E, c)$
- $G$ finite or $\delta$-Graph: $\exists\, \delta > 0 : c(e) \geq \delta$ for all $e \in E$
- $s \in V$, $t \in V$
- Distance estimate $\widehat{h}_t(v) \leq h_t(v) := \delta(v, t)\ \forall\, v \in V$.
- Wanted: shortest path $p : s \rightsquigarrow t$

# A*-Algorithm($G, s, t, \hat{h}$)

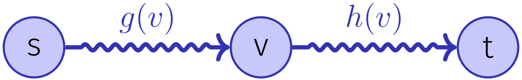
**Input:** Positively weighted Graph $G = (V, E, c)$, starting point $s \in V$, end point $t \in V$, estimate $\widehat{h}(v) \leq \delta(v, t)$
**Output:** Existence and value of a shortest path from $s$ to $t$

**foreach** $u \in V$ **do**
    $d[u] \leftarrow \infty$; $\widehat{f}[u] \leftarrow \infty$; $\pi[u] \leftarrow$ null
$d[s] \leftarrow 0$; $\widehat{f}[s] \leftarrow \widehat{h}(s)$; $R \leftarrow \{s\}$; $M \leftarrow \{\}$
**while** $R \neq \emptyset$ **do**
    $u \leftarrow$ ExtractMin$_{\widehat{f}}(R)$; $M \leftarrow M \cup \{u\}$
    **if** $u = t$ **then return** success
    **foreach** $v \in N^+(u)$ with $d[v] > d[u] + c(u, v)$ **do**
        $d[v] \leftarrow d[u] + c(u, v)$; $\widehat{f}[v] \leftarrow d[v] + \widehat{h}(v)$; $\pi[v] \leftarrow u$
        $R \leftarrow R \cup \{v\}$; $M \leftarrow M - \{v\}$
**return** failure

# Notation

Let $f(v)$ be the distance of a shortest path from $s$ to $t$ via $v$, thus

$$f(v) := \underbrace{\delta(s,v)}_{g(v)} + \underbrace{\delta(v,t)}_{h(v)}$$




let $p$ be a shortest path from $s$ to $t$.

It holds that $f(s) = \delta(s,t)$ and $f(v) = f(s)$ for all $v \in p$.

Let $\hat{g}(v) := d[v]$ be an estimate of $g(v)$ in the algorithm above. It holds that $\hat{g}(v) \geq g(v)$.

$\hat{h}(v)$ is an estimate of $h(v)$ with $\hat{h}(v) \leq h(v)$.

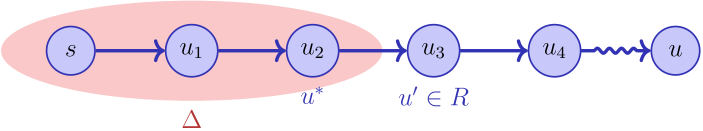
# Why the Algorithm Works

**Lemma 26**

*Let $u \in V$ and, at a time during the execution of the algorithm, $u \notin M$. Let $p$ be a shortest path from $s$ to $u$. Then there is a $u' \in p$ with $\widehat{g}(u') = g(u')$ and $u' \in R$.*

The lemma states that there is always a node in the open set $R$ with the minimal distance from $s$ already computed and that belongs to a shortest path (if existing).

# Illustration and Proof




Proof: If $s \in R$, then $\hat{g}(s) = g(s) = 0$. Therefore, let $s \notin R$.

Let $p = \langle s = u_0, u_1, \ldots, u_k = u \rangle$ and $\Delta = \{u_i \in p, u_i \in M, \hat{g}(u_i) = g(u_i)\}$.
$\Delta \neq \emptyset$, because $s \in \Delta$.

Let $m = \max\{i : u_i \in \Delta\}$, $u^* = u_m$. Then $u^* \neq u$, since $u \notin M$. Let $u' = u_{m+1}$.

1. $\hat{g}(u') \leq \hat{g}(u^*) + c(u^*, u')$ (construction of $\hat{g}$)
2. $\hat{g}(u^*) = g(u^*)$ (because $u^* \in \Delta$)
3. $g(u') = g(u^*) + c(u^*, u')$ (because $p$ optimal)
4. $\hat{g}(u') \geq g(u')$ (construction of $\hat{g}$)

Therefore: $\hat{g}(u') = g(u')$ and thus also $u' \in R$. ■

# Corollary

**Corollary 27**

*Wenn $\widehat{h}(u) \leq h(u)$ für alle $u \in V$ und A*- Algorithmus hat noch nicht terminiert. Dann existiert für jeden kürzesten Pfad $p$ von $s$ nach $t$ ein Knoten $u' \in p$ mit $\hat{f}(u') \leq \delta(s,t)$.*

If there is a shortest path $p$ from $s$ to $t$, then there is always a node in the open set $T$ that underestimates the overal distance and that is on the shortest path.

# Proof of the Corollary

Proof:
From the lemma: $\exists u' \in p$ with $\widehat{g}(u') = g(u')$.
Therefore:

$$\begin{aligned}\widehat{f}(u') &= \widehat{g}(u') + \widehat{h}(u') \\ &= g(u') + \widehat{h}(u') \\ &\leq g(u') + h(u') = f(u')\end{aligned}$$

Because $p$ is shortest path: $f(u') = \delta(s,t)$. $\blacksquare$

# Zulässigkeit

***Theorem 28***

*Under the conditions stated on page 787 the A*-algorithm is admissible: if there is a shortest path from $s$ to $t$ then A* terminates with $\hat{g}(t) = \delta(s,t)$*

Proof: If the algorithm terminates, then it termines with $t$ with $f(t) = \widehat{g}(t) + 0 = g(t)$. That is because $\widehat{g}$ overestimates $g$ at most and by the corollary above that algorithm always finds an element $v \in R$ with $f(v) \leq \delta(s,t)$.

The algorithm terminates in finitely many steps. For finite graphs the maximal number of relaxing steps is bounded.[45] ■

[45] For a $\delta$-graph the maximum number of relaxing steps before $R$ contains only nodes with $\hat{f}(s) > \delta(s,t)$ is limited as well. The exact argument can be found in the seminal article Hart, P. E.; Nilsson, N. J.; Raphael, B. (1968). "A Formal Basis for the Heuristic Determination of Minimum Cost Paths".

# Revisiting nodes

- The A*-algorithm can re-insert nodes that had been extracted from $R$ before.
- This can lead to suboptimal behavior (w.r.t. running time of the algorithm).
- If $\widehat{h}$, in addition to being admissible ($\widehat{h}(v) \leq h(v)$ for all $v \in V$), fulfils monotonicity, i.e. if for all $(u, u') \in E$:

$$\widehat{h}(u') \leq \widehat{h}(u) + c(u', u)$$

  then the A*-Algorithm is equivalent to the Dijsktra-algorithm with edge weights $\tilde{c}(u, v) = c(u, v) + \widehat{h}(u) - \widehat{h}(v)$, and no node is re-inserted into $R$.
- It is not always possible to find monotone heuristics.