23. Greedy Algorithms

Fractional Knapsack Problem, Huffman Coding [Cormen et al, Kap. 16.1, 16.3]

The Fractional Knapsack Problem

set of $n \in \mathbb{N}$ items $\{1, \ldots, n\}$ Each item i has value $v_i \in \mathbb{N}$ and weight $w_i \in \mathbb{N}$. The maximum weight is given as $W \in \mathbb{N}$. Input is denoted as $E = (v_i, w_i)_{i=1,\ldots,n}$.

Wanted: Fractions $0 \le q_i \le 1$ $(1 \le i \le n)$ that maximise the sum $\sum_{i=1}^{n} q_i \cdot v_i$ under $\sum_{i=1}^{n} q_i \cdot w_i \le W$.

Greedy heuristics

Sort the items decreasingly by value per weight v_i/w_i.

Assumption $v_i/w_i \geq v_{i+1}/w_{i+1}$

Let $j = \max\{0 \leq k \leq n : \sum_{i=1}^{k} w_i \leq W\}$. Set

- $q_i = 1$ for all $1 \leq i \leq j$.
- $q_{j+1} = \frac{W - \sum_{i=1}^{j} w_i}{w_{j+1}}$.
- $q_i = 0$ for all $i > j + 1$.

That is fast: $\Theta(n \log n)$ for sorting and $\Theta(n)$ for the computation of the q_i.

Correctness

Assumption: optimal solution (r_i) $(1 \leq i \leq n)$.
The knapsack is full: $\sum_i r_i \cdot w_i = \sum_i q_i \cdot w_i = W$.
Consider k: smallest i with $r_i \neq q_i$ Definition of greedy: $q_k > r_k$. Let $x = q_k - r_k > 0$.
Construct a new solution (r'_i): $r'_i = r_i \forall i < k$. $r'_k = q_k$. Remove weight $\sum_{i=k+1}^{n} \delta_i = x \cdot w_k$ from items $k+1$ to n. This works because $\sum_{i=k}^{n} r_i \cdot w_i = \sum_{i=k}^{n} q_i \cdot w_i$.

Correctness

$$
\begin{aligned}
\sum_{i=k}^{n} r'_i v_i &= r_k v_k + x w_k \frac{v_k}{w_k} + \sum_{i=k+1}^{n} (r_i w_i - \delta_i) \frac{v_i}{w_i} \\
&\geq r_k v_k + x w_k \frac{v_k}{w_k} + \sum_{i=k+1}^{n} r_i w_i \frac{v_i}{w_i} - \delta_i \frac{v_k}{w_k} \\
&= r_k v_k + x w_k \frac{v_k}{w_k} - x w_k \frac{v_k}{w_k} + \sum_{i=k+1}^{n} r_i w_i \frac{v_i}{w_i} = \sum_{i=k}^{n} r_i v_i .
\end{aligned}
$$

Thus (r'_i) is also optimal. Iterative application of this idea generates the solution (q_i).

Huffman-Codes

Goal: memory-efficient saving of a sequence of characters using a binary code with code words..

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Example

File consisting of 100.000 characters from the alphabet $\{a, \ldots, f\}$.

	a	b	c	d	e	f
Frequency (Thousands)	45	13	12	16	9	5
Code word with fix length	000	001	010	011	100	101
Code word variable length	0	101	100	111	1101	1100

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File size (code with fix length): 300.000 bits.
File size (code with variable length): 224.000 bits.

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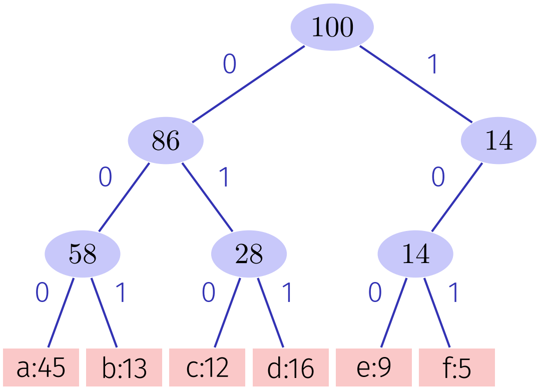
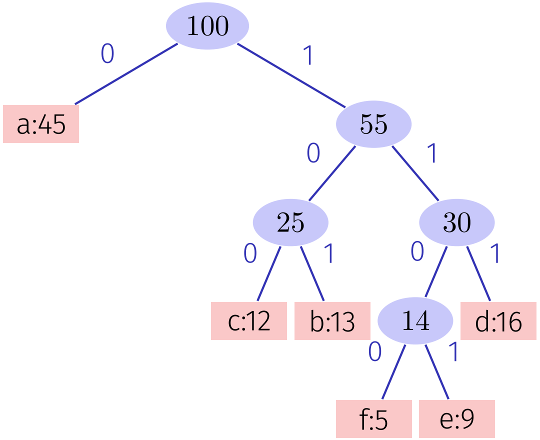
  $affe \rightarrow 0 \cdot 1100 \cdot 1100 \cdot 1101 \rightarrow 0110011001101$
- Decoding simple because prefixcode
  $0110011001101 \rightarrow 0 \cdot 1100 \cdot 1100 \cdot 1101 \rightarrow affe$

Code trees



Code words with fixed length



Code words with variable length

Properties of the Code Trees

- An optimal coding of a file is alway represented by a complete binary tree: every inner node has two children.

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- Let C be the set of all code words, $f(c)$ the frequency of a codeword c and $d_T(c)$ the depth of a code word in tree T. Define the cost of a tree as

$$B(T) = \sum_{c \in C} f(c) \cdot d_T(c).$$

(cost = number bits of the encoded file)

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In the following a code tree is called optimal when it minimizes the costs.

Algorithm Idea

Tree construction bottom up

- Start with the set C of code words
- Replace iteriatively the two nodes with smallest frequency by a new parent node.

a:45 b:13 c:12 d:16 e:9 f:5

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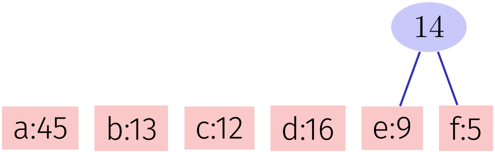

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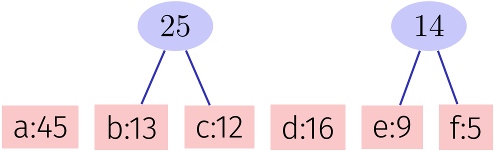

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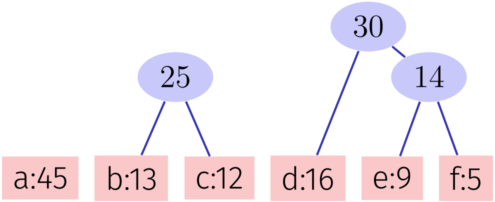

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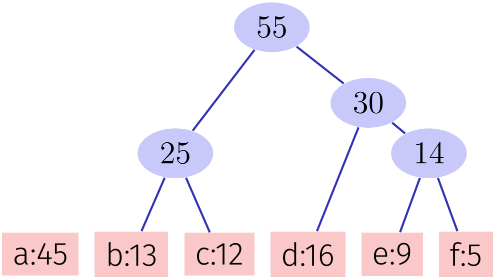

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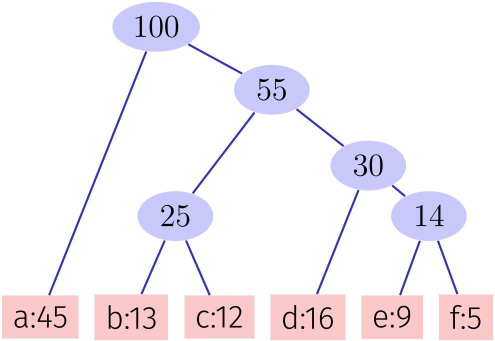

Algorithm Huffman(C)

Input: code words $c \in C$
Output: Root of an optimal code tree

$n \leftarrow |C|$
$Q \leftarrow C$
for $i = 1$ **to** $n - 1$ **do**
  allocate a new node z
  z.left $\leftarrow$ ExtractMin(Q) // extract word with minimal frequency.
  z.right $\leftarrow$ ExtractMin(Q)
  z.freq $\leftarrow$ z.left.freq $+$ z.right.freq
  Insert(Q, z)
return ExtractMin(Q)

Analyse

Use a heap: build Heap in $\mathcal{O}(n)$. Extract-Min in $O(\log n)$ for n Elements. Yields a runtime of $O(n \log n)$.

The greedy approach is correct

Theorem 21

Let x, y be two symbols with smallest frequencies in C and let $T'(C')$ be an optimal code tree to the alphabet $C' = C - \{x, y\} + \{z\}$ with a new symbol z with $f(z) = f(x) + f(y)$. Then the tree $T(C)$ that is constructed from $T'(C')$ by replacing the node z by an inner node with children x and y is an optimal code tree for the alphabet C.

Proof

It holds that
$f(x)\cdot d_T(x)+f(y)\cdot d_T(y) = (f(x)+f(y))\cdot(d_{T'}(z)+1) = f(z)\cdot d_{T'}(x)+f(x)+f(y)$.
Thus $B(T') = B(T) - f(x) - f(y)$.

Assumption: T is not optimal. Then there is an optimal tree T'' with $B(T'') < B(T)$. We assume that x and y are brothers in T''. Let T''' be the tree where the inner node with children x and y is replaced by z. Then it holds that $B(T''') = B(T'') - f(x) - f(y) < B(T) - f(x) - f(y) = B(T')$. Contradiction to the optimality of T'.

The assumption that x and y are brothers in T'' can be justified because a swap of elements with smallest frequency to the lowest level of the tree can at most decrease the value of B.

24. C++ advanced (IV): Exceptions

Some operations that can fail

- Opening files for reading and writing

```
std::ifstream input("myfile.txt");
```

- Parsing

```
int value = std::stoi("12-8");
```

- Memory allocation

```
std::vector<double> data(ManyMillions);
```

- Invalid data

```
int a = b/x; // what if x is zero?
```

Possibilities of Error Handling

- None (inacceptable)
- Global error variable (flags)
- Functions returning Error Codes
- Objects that keep error status
- Exceptions

Global error variables

- Common in older C-Code
- Concurrency is a problem.
- Error handling at good will. Requires extreme discipline, documentation and litters the code with seemingly unrelated checks.

Functions Returning Error Codes

- Every call to a function yields a result.
- Typical for large APIs (e.g. OS level). Often combined with global error code.[40]
- Caller can check the return value of a function in order to check the correct execution.

[40] Global error code thread-safety provided via thread-local storage.

Functions Returning Error Codes

Example

```
#include <errno.h>
...

pf = fopen ("notexisting.txt", "r+");
if (pf == NULL) {
  fprintf(stderr, "Error opening file: %s\n", strerror( errno ));
}
else { // ...
  fclose (pf);
}
```

Error state Stored in Object

- Error state of an object stored internally in the object.

Example

```
int i;
std::cin >> i;
if (std::cin.good()){// success, continue
  ...
}
```

Exceptions

- Exceptions break the normal control flow
- Exceptions can be thrown (throw) and catched (catch)
- Exceptions can become effective accross function boundaries.

Example: throw exception

```
class MyException{};

void f(int i){
  if (i==0) throw MyException();
  f(i-1);
}

int main()
{
  f(4);
  return 0;
}
```

Example: throw exception

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class MyException{};

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int main()
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  f(4);
  return 0;
}
```

terminate called after throwing an instance of 'MyException'
Aborted

Example: catch exception

```
class MyException{};

void f(int i){
  if (i==0) throw MyException();
  f(i-1);
}

int main(){
  try{
    f(4);
  }
  catch (MyException e){
      std::cout << "exception caught\n";
  }
}
```

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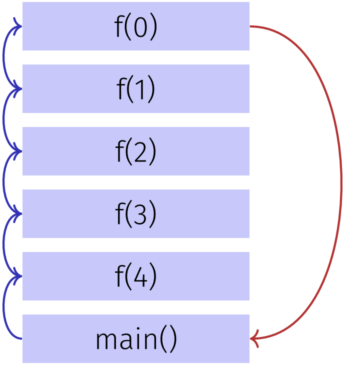

exception caught

Resources get closed

```
class MyException{};
struct SomeResource{
    ~SomeResource(){std::cout << "closed resource\n";}
};
void f(int i){
  if (i==0) throw MyException();
  SomeResource x;
  f(i-1);
}
int main(){
  try{f(5);}
  catch (MyException e){
      std::cout << "exception caught\n";
  }
}
```

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int main(){
  try{f(5);}
  catch (MyException e){
      std::cout << "exception caught\n";
  }
}
```

```
closed resource
closed resource
closed resource
closed resource
closed resource
exception caught
```

When Exceptions?

Exceptions are used for **error handling** exclusively.

- Use **`throw`** only in order to identify an error that violates the post-condition of a function or that makes the continued execution of the code impossible in an other way.
- Use **`catch`** only when it is clear how to handle the error (potentially re-throwing the exception)
- Do **not** use **`throw`** in order to show a programming error or a violation of invariants, use **`assert`** instead.
- Do **not** use exceptions in order to change the control flow. Throw is **not** a better return.

Why Exceptions?

This:

```
int ret = f();
if (ret == 0) {
  // ...
} else {
  // ...code that handles the error...
}
```

may look better than this on a first sight:

```
try {
  f();
  // ...
} catch (std::exception& e) {
  // ...code that handles the error...
}
```

Why exceptions?

Truth is that toy examples do not necessarily hit the point.

Using return-codes for error handling either pollutes the code with checks or the error handling is not done right in the first place.

That's why

Example 1: Expression evaluation (expression parser from Introduction to programming)

Input: `1 + (3 * 6 / (/ 7 ))`

Error is deap in the recursion hierarchy. How to produce a meaningful error message (and continue execution)? Would have to pass error code over recursion steps.

Second Example

Value type with guarantee: values in range provided.

```
template <typename T, T min, T max>
class Range{
public:
  Range(){}
  Range (const T& v) : value (v) {
    if (value < min) throw Underflow ();
    if (value > max) throw Overflow ();
  }
  operator const T& () const {return value;}
private:
  T value;
};
```

Error handling in the constructor.

Types of Exceptions, Hierarchical

```
class RangeException {};
class Overflow : public RangeException {};
class Underflow : public RangeException {};
class DivisionByZero: public RangeException {};
class FormatError: public RangeException {};
```

Operators

```
template <typename T, T min, T max>
Range<T, min, max> operator/ (const Range<T, min, max>& a,
                              const Range<T, min, max>& b){
  if (b == 0) throw DivisionByZero();
  return T (a) * T(b);
}

template <typename T, T min, T max>
std::istream& operator >> (std::istream& is, Range<T, min, max>& a){
  T value;
  if (!(is >> value)) throw FormatError();
  a = value;
  return is;
}
```

Error handling in the operator.

Error handling (central)

```
Range<int,-10,10> a,b,c;
try{
  std::cin >> a;
  std::cin >> b;
  std::cin >> c;
  a = a / b + 4 * (b - c);
  std::cout << a;
}
catch(FormatError& e){ std::cout << "Format error\n"; }
catch(Underflow& e){ std::cout << "Underflow\n"; }
catch(Overflow& e){ std::cout << "Overflow\n"; }
catch(DivisionByZero& e){ std::cout << "Divison By Zero\n"; }
```