22. Dynamic Programming III

FPTAS [Ottman/Widmayer, Kap. 7.2, 7.3, Cormen et al, Kap. 15,35.5], Optimal Search Tree [Ottman/Widmayer, Kap. 5.7]

Approximation

Let $\varepsilon \in (0, 1)$ given. Let I_{opt} an optimal selection.
No try to find a valid selection I with

$$\sum_{i \in I} v_i \geq (1 - \varepsilon) \sum_{i \in I_{\text{opt}}} v_i.$$

Sum of weights may not violate the weight limit.

Different formulation of the algorithm

Before: weight limit $w \to$ maximal value v
Reversed: value $v \to$ minimal weight w

$\Rightarrow$ **alternative table** $g[i, v]$ provides the minimum weight with

- a selection of the first i items ($0 \leq i \leq n$) that
- provide a value of exactly v ($0 \leq v \leq \sum_{i=1}^{n} v_i$).

Computation

Initially

- $g[0, 0] \leftarrow 0$
- $g[0, v] \leftarrow \infty$ (Value v cannot be achieved with 0 items.).

Computation

$$g[i, v] \leftarrow \begin{cases} g[i-1, v] & \text{falls } v < v_i \\ \min\{g[i-1, v], g[i-1, v-v_i] + w_i\} & \text{sonst.} \end{cases}$$

incrementally in i and for fixed i increasing in v.
Solution can be found at largest index v with $g[n, v] \leq w$.

Example

$E = \{(2,3),(4,5),(1,1)\}$

$v \longrightarrow$

$i \downarrow$	0	1	2	3	4	5	6	7	8	9
$\emptyset$	0	∞	∞	∞	∞	∞	∞	∞	∞	∞
$(2,3)$	0	∞	∞	2	∞	∞	∞	∞	∞	∞
$(4,5)$	0	∞	∞	2	∞	4	∞	∞	6	∞
$(1,1)$	0	1	∞	2	3	4	5	∞	6	7

Read out the solution: if $g[i,v] = g[i-1,v]$ then item i unused and continue with $g[i-1,v]$ otherwise used and continue with $g[i-1,b-v_i]$.

The approximation trick

Pseduopolynomial run time gets polynmial if the number of occuring values can be bounded by a polynom of the input length.

Let $K > 0$ be chosen *appropriately*. Replace values v_i by “rounded values” $\tilde{v}_i = \lfloor v_i/K \rfloor$ delivering a new input $E' = (w_i, \tilde{v}_i)_{i=1\dots n}$.

Apply the algorithm on the input E' with the same weight limit W.

Idea

Example $K = 5$

Values

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots, 98, 99, 100$$
$$\rightarrow$$
$$0, 0, 0, 0, 1, 1, 1, 1, 1, 2, \ldots, 19, 19, 20$$

Obviously less different values

Properties of the new algorithm

- Selection of items in E' is also admissible in E. Weight remains unchanged!
- Run time of the algorithm is bounded by $\mathcal{O}(n^2 \cdot v_{\max}/K)$ $(v_{\max} := \max\{v_i | 1 \leq i \leq n\})$

How good is the approximation?

It holds that

$$v_i - K \le K \cdot \left\lfloor \frac{v_i}{K} \right\rfloor = K \cdot \tilde{v}_i \le v_i$$

Let I'_{opt} be an optimal solution of E'. Then

$$\left(\sum_{i \in I_{\text{opt}}} v_i\right) - n \cdot K \overset{|I_{\text{opt}}| \le n}{\le} \sum_{i \in I_{\text{opt}}} (v_i - K) \le \sum_{i \in I_{\text{opt}}} (K \cdot \tilde{v}_i) = K \sum_{i \in I_{\text{opt}}} \tilde{v}_i$$

$$\underset{I'_{\text{opt}}\,\text{optimal}}{\le} K \sum_{i \in I'_{\text{opt}}} \tilde{v}_i = \sum_{i \in I'_{\text{opt}}} K \cdot \tilde{v}_i \le \sum_{i \in I'_{\text{opt}}} v_i.$$

Choice of K

Requirement:

$$\sum_{i \in I'} v_i \geq (1-\varepsilon) \sum_{i \in I_{\text{opt}}} v_i.$$

Inequality from above:

$$\sum_{i \in I'_{\text{opt}}} v_i \geq \left(\sum_{i \in I_{\text{opt}}} v_i \right) - n \cdot K$$

thus: $K = \varepsilon \frac{\sum_{i \in I_{\text{opt}}} v_i}{n}$.

Choice of K

Choose $K = \varepsilon \frac{\sum_{i \in I_{\text{opt}}} v_i}{n}$. The optimal sum is unknown. Therefore we choose $K' = \varepsilon \frac{v_{\max}}{n}$.[38]

It holds that $v_{\max} \le \sum_{i \in I_{\text{opt}}} v_i$ and thus $K' \le K$ and the approximation is even slightly better.

The run time of the algorithm is bounded by

$$\mathcal{O}(n^2 \cdot v_{\max}/K') = \mathcal{O}(n^2 \cdot v_{\max}/(\varepsilon \cdot v_{\max}/n)) = \mathcal{O}(n^3/\varepsilon).$$

[38] We can assume that items i with $w_i > W$ have been removed in the first place.

FPTAS

Such a family of algorithms is called an **approximation scheme**: the choice of ε controls both running time and approximation quality.

The runtime $\mathcal{O}(n^3/\varepsilon)$ is a polynom in n and in $\frac{1}{\varepsilon}$. The scheme is therefore also called a **FPTAS - Fully Polynomial Time Approximation Scheme**

22.2 Optimale Suchbäume

Optimal binary Search Trees

Given: search probabilities p_i for each key k_i ($i = 1, \dots, n$) and q_i of each interval d_i ($i = 0, \dots, n$) between search keys of a binary search tree.
$\sum_{i=1}^{n} p_i + \sum_{i=0}^{n} q_i = 1$.
Wanted: optimal search tree T with key depths $\operatorname{depth}(\cdot)$, that minimizes the expected search costs

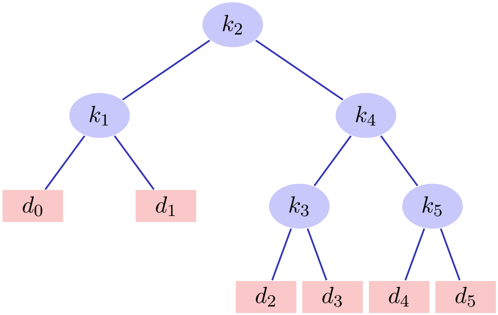
$$\begin{aligned} C(T) &= \sum_{i=1}^{n} p_i \cdot (\operatorname{depth}(k_i) + 1) + \sum_{i=0}^{n} q_i \cdot (\operatorname{depth}(d_i) + 1) \\ &= 1 + \sum_{i=1}^{n} p_i \cdot \operatorname{depth}(k_i) + \sum_{i=0}^{n} q_i \cdot \operatorname{depth}(d_i) \end{aligned}$$

Example

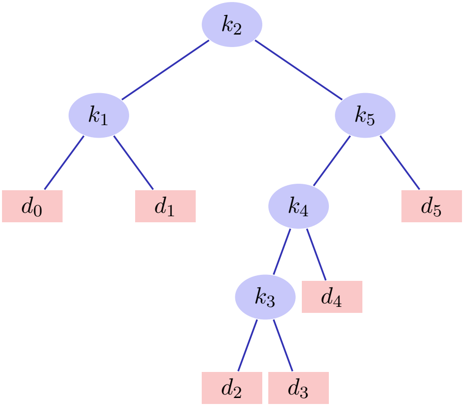
Expected Frequencies

i	0	1	2	3	4	5
p_i		0.15	0.10	0.05	0.10	0.20
q_i	0.05	0.10	0.05	0.05	0.05	0.10

Example



Search tree with expected costs 2.8



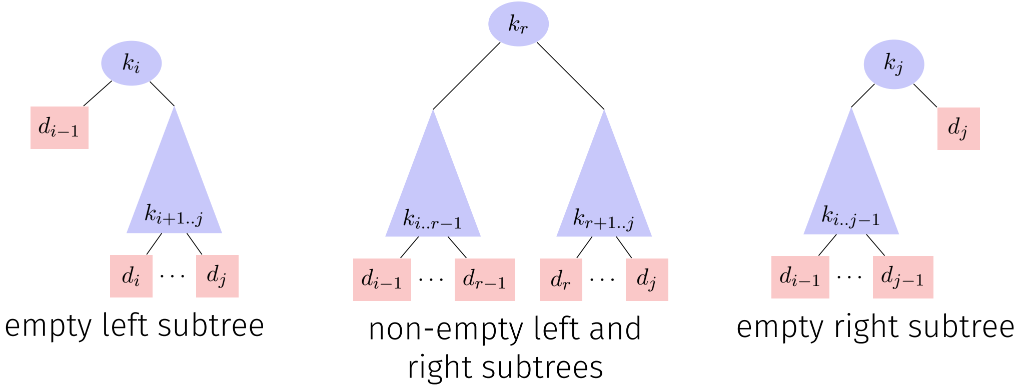
Search tree with expected costs 2.75

Structure of a optimal binary search tree

- Subtree with keys $k_i, \ldots, k_j$ and intervals $d_{i-1}, \ldots, d_j$ must be optimal for the respective sub-problem.[39]
- Consider all subtrees with roots k_r and optimal subtrees for keys $k_i, \ldots, k_{r-1}$ and $k_{r+1}, \ldots, k_j$

[39]The usual argument: if it was not optimal, it could be replaced by a better solution improving the overal solution.

Sub-trees for Searching



Expected Search Costs

Let $\mathrm{depth}_T(k)$ be the depth of a node k in the sub-tree T. Let k be the root of subtrees T_r and T_{L_r} and T_{R_r} be the left and right sub-tree of T_r. Then

$$\mathrm{depth}_T(k_i) = \mathrm{depth}_{T_{L_r}}(k_i) + 1,\ (i < r)$$
$$\mathrm{depth}_T(k_i) = \mathrm{depth}_{T_{R_r}}(k_i) + 1,\ (i > r)$$

Expected Search Costs

Let $e[i,j]$ be the costs of an optimal search tree with nodes $k_i, \ldots, k_j$.
Base case $e[i,i-1]$, expected costs d_{i-1}
Let $w(i,j) = \sum_{l=i}^{j} p_l + \sum_{l=i-1}^{j} q_l$.
If k_r is the root of an optimal search tree with keys $k_i, \ldots, k_j$, then

$$e[i,j] = p_r + (e[i,r-1] + w(i,r-1)) + (e[r+1,j] + w(r+1,j))$$

with $w(i,j) = w(i,r-1) + p_r + w(r+1,j)$:

$$e[i,j] = e[i,r-1] + e[r+1,j] + w(i,j).$$

Dynamic Programming

$$e[i,j] = \begin{cases} q_{i-1} & \text{if } j = i-1, \\ \min_{i \le r \le j}\{e[i,r-1] + e[r+1,j] + w[i,j]\} & \text{if } i \le j \end{cases}$$

Computation

Tables $e[1 \ldots n+1, 0 \ldots n], w[1 \ldots n+1, 0 \ldots m], r[1 \ldots n, 1 \ldots n]$ Initially

- $e[i, i-1] \leftarrow q_{i-1}$, $w[i, i-1] \leftarrow q_{i-1}$ for all $1 \leq i \leq n+1$.

We compute

$$
\begin{aligned}
w[i,j] &= w[i,j-1] + p_j + q_j \\
e[i,j] &= \min_{i \leq r \leq j} \{e[i,r-1] + e[r+1,j] + w[i,j]\} \\
r[i,j] &= \arg\min_{i \leq r \leq j} \{e[i,r-1] + e[r+1,j] + w[i,j]\}
\end{aligned}
$$

for intervals $[i,j]$ with increasing lengths $l = 1, \ldots, n$, each for $i = 1, \ldots, n-l+1$. Result in $e[1,n]$, reconstruction via r. Runtime $\Theta(n^3)$.

Example

i	0	1	2	3	4	5
p_i		0.15	0.10	0.05	0.10	0.20
q_i	0.05	0.10	0.05	0.05	0.05	0.10

w

j \ i	1	2	3	4	5	6
0	0.05					
1	0.30	0.10				
2	0.45	0.25	0.05			
3	0.55	0.35	0.15	0.05		
4	0.70	0.50	0.30	0.20	0.05	
5	1.00	0.80	0.60	0.50	0.35	0.10

e

j \ i	1	2	3	4	5	6
0	0.05					
1	0.45	0.10				
2	0.90	0.40	0.05			
3	1.25	0.70	0.25	0.05		
4	1.75	1.20	0.60	0.30	0.05	
5	2.75	2.00	1.30	0.90	0.50	0.10

r

j \ i	1	2	3	4	5
1	1				
2	1	2			
3	2	2	3		
4	2	2	4	4	
5	2	4	5	5	5