

20. Dynamic Programming I

Memoization, Optimal Substructure, Overlapping Sub-Problems, Dependencies, General Procedure. Examples: Fibonacci, Rod Cutting, Longest Ascending Subsequence, Longest Common Subsequence, Edit Distance, Matrix Chain Multiplication (Strassen)

[Ottman/Widmayer, Kap. 1.2.3, 7.1, 7.4, Cormen et al, Kap. 15]

Fibonacci Numbers



(again)

$$F_n := \begin{cases} n & \text{if } n < 2 \\ F_{n-1} + F_{n-2} & \text{if } n \geq 2. \end{cases}$$

Analysis: why is the recursive algorithm so slow?

Algorithm FibonacciRecursive(n)

Input: $n \geq 0$

Output: n -th Fibonacci number

if $n < 2$ **then**

 | $f \leftarrow n$

else

 | $f \leftarrow \text{FibonacciRecursive}(n - 1) + \text{FibonacciRecursive}(n - 2)$

return f

Analysis

$T(n)$: Number executed operations.

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$$T(n) = T(n - 2) + T(n - 1) + c \geq 2T(n - 2) + c \geq 2^{n/2}c' = (\sqrt{2})^n c'$$

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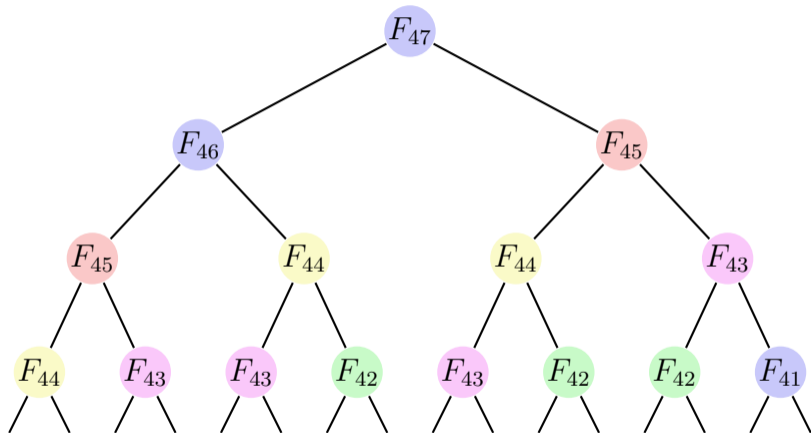
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Algorithm is **exponential** in n .

Reason (visual)



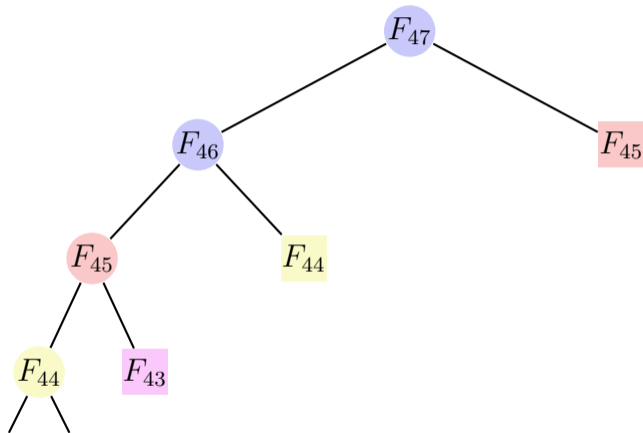
Nodes with same values are evaluated (too) often.

Memoization

Memoization (sic) saving intermediate results.

- Before a subproblem is solved, the existence of the corresponding intermediate result is checked.
- If an intermediate result exists then it is used.
- Otherwise the algorithm is executed and the result is saved accordingly.

Memoization with Fibonacci



Rechteckige Knoten wurden bereits ausgewertet.

Algorithm FibonacciMemoization(n)

Input: $n \geq 0$

Output: n -th Fibonacci number

if $n \leq 2$ **then**

| $f \leftarrow 1$

else if $\exists \text{memo}[n]$ **then**

| $f \leftarrow \text{memo}[n]$

else

| $f \leftarrow \text{FibonacciMemoization}(n - 1) + \text{FibonacciMemoization}(n - 2)$

| $\text{memo}[n] \leftarrow f$

return f

Computational complexity:

$$T(n) = T(n - 1) + c = \dots = \mathcal{O}(n).$$

because after the call to $f(n - 1)$, $f(n - 2)$ has already been computed.
A different argument: $f(n)$ is computed exactly once recursively for each n .
Runtime costs: n calls with $\Theta(1)$ costs per call $n \cdot c \in \Theta(n)$. The recursion vanishes from the running time computation.
Algorithm requires $\Theta(n)$ memory.³³

³³But the naive recursive algorithm also requires $\Theta(n)$ memory implicitly.

Looking closer ...

... the algorithm computes the values of F_1, F_2, F_3, \dots in the **top-down** approach of the recursion.

Can write the algorithm **bottom-up**. This is characteristic for **dynamic programming**.

Algorithm FibonacciBottomUp(n)

Input: $n \geq 0$

Output: n -th Fibonacci number

$F[1] \leftarrow 1$

$F[2] \leftarrow 1$

for $i \leftarrow 3, \dots, n$ **do**

$F[i] \leftarrow F[i - 1] + F[i - 2]$

return $F[n]$

Dynamic Programming: Idea

- Divide a complex problem into a reasonable number of sub-problems
- The solution of the sub-problems will be used to solve the more complex problem
- Identical problems will be computed only once

Dynamic Programming Consequence

Identical problems will be computed only once

⇒ Results are saved

Arbeitsspeicher



192.-

HyperX Fury (2x, 8GB,
DDR4-2400, DIMM 288)

★★★★★ 16

We trade speed against
memory consumption

Dynamic Programming: Description

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Runtime (typical) = number entries of the table times required operations per entry.

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F_i with increasing i .

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 F_n ist die n -te Fibonacci-Zahl.

Dynamic Programming = Divide-And-Conquer ?

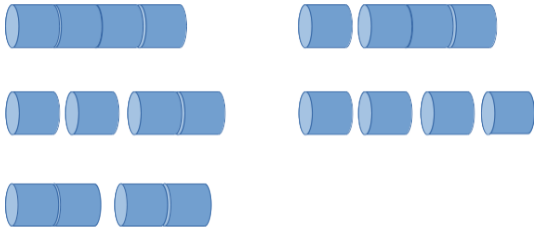
- In both cases the original problem can be solved (more easily) by utilizing the solutions of sub-problems. The problem provides **optimal substructure**.
- Divide-And-Conquer algorithms (such as Mergesort): sub-problems are independent; their solutions are required only once in the algorithm.
- DP: sub-problems are dependent. The problem is said to have **overlapping sub-problems** that are required multiple-times in the algorithm.
- In order to avoid redundant computations, results are tabulated. For **sub-problems there must not be any circular dependencies**.

Rod Cutting

- Rods (metal sticks) are cut and sold.
- Rods of length $n \in \mathbb{N}$ are available. A cut does not provide any costs.
- For each length $l \in \mathbb{N}, l \leq n$ known is the value $v_l \in \mathbb{R}^+$
- Goal: cut the rods such (into $k \in \mathbb{N}$ pieces) that

$$\sum_{i=1}^k v_{l_i} \text{ is maximized subject to } \sum_{i=1}^k l_i = n.$$

Rod Cutting: Example



Possibilities to cut a rod of length 4 (without permutations)

Length	0	1	2	3	4
Price	0	2	3	8	9

\Rightarrow Best cut: 3 + 1 with value 10.

Wie findet man den DP Algorithms

0. Exact formulation of the wanted solution
1. Define sub-problems (and compute the cardinality)
2. Guess / Enumerate (and determine the running time for guessing)
3. Recursion: relate sub-problems
4. Memoize / Tabularize. Determine the dependencies of the sub-problems
5. Solve the problem
Running time = #sub-problems \times time/sub-problem

Structure of the problem

0. **Wanted:** r_n = maximal value of rod (cut or as a whole) with length n .
1. **sub-problems:** maximal value r_k for each $0 \leq k < n$
2. **Guess** the length of the first piece
3. **Recursion**

$$r_k = \max\{v_i + r_{k-i} : 0 < i \leq k\}, \quad k > 0$$
$$r_0 = 0$$

4. **Dependency:** r_k depends (only) on values v_i , $1 \leq i \leq k$ and the optimal cuts r_i , $i < k$
5. **Solution** in r_n

Algorithm RodCut(v, n)

Input: $n \geq 0$, Prices v

Output: best value

$q \leftarrow 0$

if $n > 0$ **then**

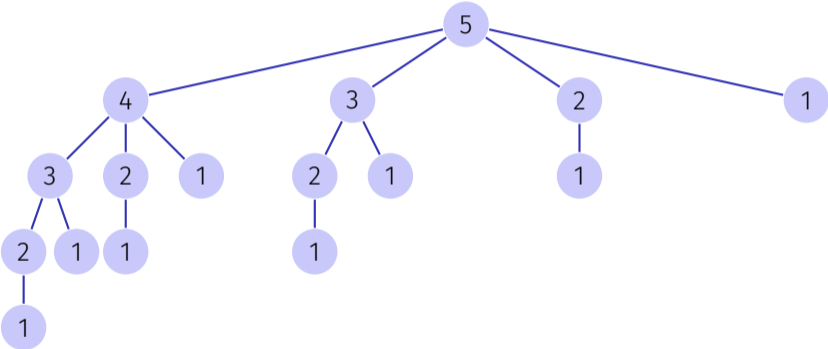
for $i \leftarrow 1, \dots, n$ **do**
 $q \leftarrow \max\{q, v_i + \text{RodCut}(v, n - i)\};$

return q

Running time $T(n) = \sum_{i=0}^{n-1} T(i) + c \Rightarrow^{34} T(n) \in \Theta(2^n)$

$$^{34}T(n) = T(n-1) + \sum_{i=0}^{n-2} T(i) + c = T(n-1) + (T(n-1) - c) + c = 2T(n-1) \quad (n > 0)$$

Recursion Tree



Algorithm RodCutMemoized(m, v, n)

Input: $n \geq 0$, Prices v , Memoization Table m

Output: best value

$q \leftarrow 0$

if $n > 0$ **then**

if $\exists m[n]$ **then**

$q \leftarrow m[n]$

else

for $i \leftarrow 1, \dots, n$ **do**

$q \leftarrow \max\{q, v_i + \text{RodCutMemoized}(m, v, n - i)\};$

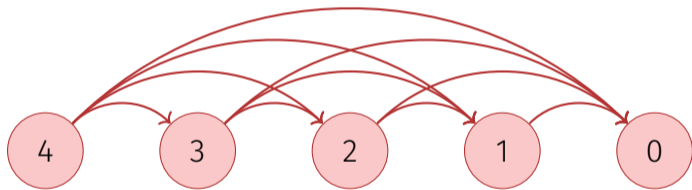
$m[n] \leftarrow q$

return q

Running time $\sum_{i=1}^n i = \Theta(n^2)$

Subproblem-Graph

Describes the mutual dependencies of the subproblems



and must not contain cycles

Construction of the Optimal Cut

- During the (recursive) computation of the optimal solution for each $k \leq n$ the recursive algorithm determines the optimal length of the first rod
- Store the length of the first rod in a separate table of length n

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Dimension of the table? Semantics of the entries?

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$r_i, i = 1, \dots, n.$

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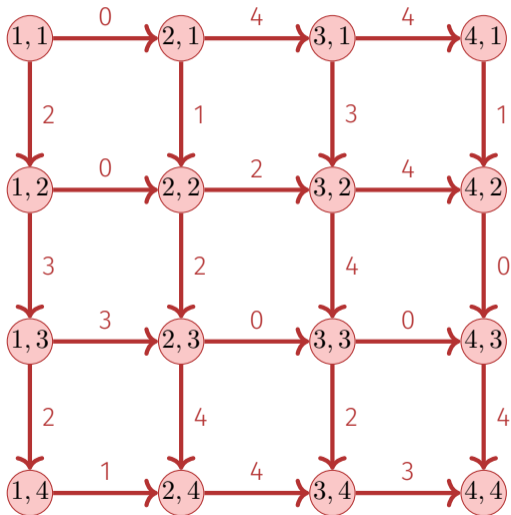
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4.

r_n is the best value for the rod of length n .

Rabbit!

A rabbit sits on cite (1, 1) of an $n \times n$ grid. It can only move to east or south. On each pathway there is a number of carrots. How many carrots does the rabbit collect maximally?

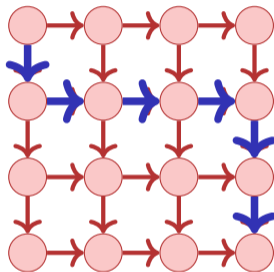


Rabbit!

Number of possible paths?

- Choice of $n - 1$ ways to south out of $2n - 2$ ways overall.

⇒ No chance for a naive algorithm



The path 100011
(1:to south, 0: to east)

Rabbit!

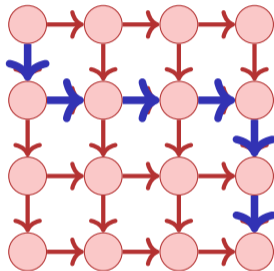
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$$\binom{2n - 2}{n - 1} \in \Omega(2^n)$$

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Recursion

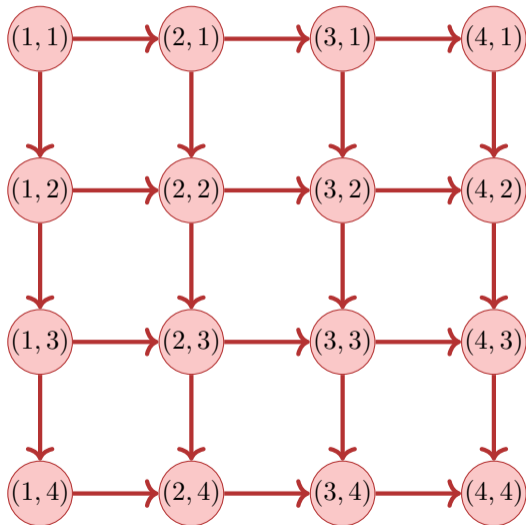
Wanted: $T_{0,0}$ = **maximal number carrots from** $(0, 0)$ **to** (n, n) .

Let $w_{(i,j)-(i',j')}$ number of carrots on egde from (i, j) to (i', j') .

Recursion (maximal number of carrots from (i, j) to (n, n))

$$T_{ij} = \begin{cases} \max\{w_{(i,j)-(i,j+1)} + T_{i,j+1}, w_{(i,j)-(i+1,j)} + T_{i+1,j}\}, & i < n, j < n \\ w_{(i,j)-(i,j+1)} + T_{i,j+1}, & i = n, j < n \\ w_{(i,j)-(i+1,j)} + T_{i+1,j}, & i < n, j = n \\ 0 & i = j = n \end{cases}$$

Graph of Subproblem Dependencies



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3. $T_{i,j}$ with $i = n \searrow 1$ and for each i : $j = n \searrow 1$, (or vice-versa: $j = n \searrow 1$ and for each j : $i = n \searrow 1$).

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Reconstruction of a solution?

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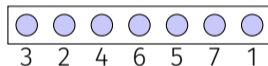
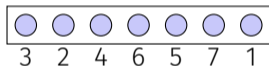
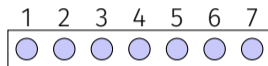
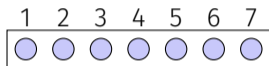
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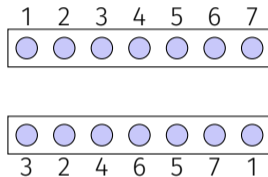
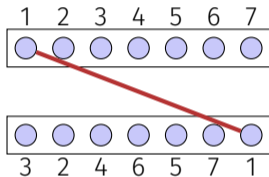
4. $T_{1,1}$ provides the maximal number of carrots.

Longest Ascending Sequence (LAS)



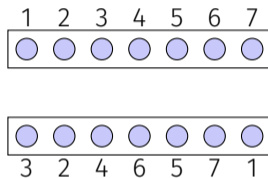
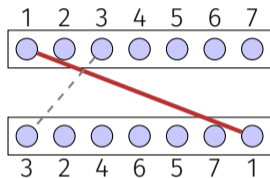
Connect as many as possible fitting ports without lines crossing.

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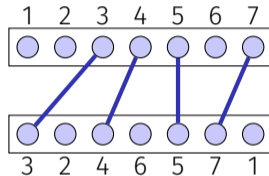
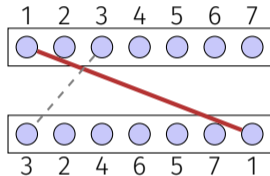
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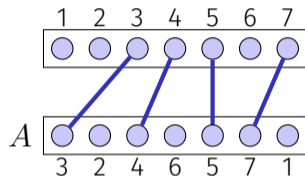
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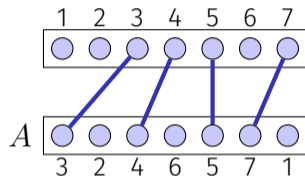
Formally

- Consider Sequence $A_n = (a_1, \dots, a_n)$.
- Search for a longest increasing subsequence of A_n .
- Examples of increasing subsequences: $(3, 4, 5)$, $(2, 4, 5, 7)$, $(3, 4, 5, 7)$, $(3, 7)$.



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Generalization: allow any numbers, even with duplicates (still only strictly increasing subsequences permitted). Example: $(2, 3, 3, 3, 5, 1)$ with increasing subsequence $(2, 3, 5)$.

First idea

Let L_i = **longest ascending subsequence of** A_i ($1 \leq i \leq n$)

Assumption: LAS L_k of A_k known for Now want to compute L_{k+1} for A_{k+1} .

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Counterexample $A_5 = (1, 2, 5, 3, 4)$. Let $A_3 = (1, 2, 5)$ with $L_3 = A_3$ and $L_4 = A_3$. Determine L_5 from L_4 ?

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It does not work this way, we cannot infer L_{k+1} from L_k .

Second idea.

Let $L_i =$ **longest ascending subsequence of** A_i ($1 \leq i \leq n$)

Assumption: a LAS L_j is known for each $j \leq k$. Now compute LAS L_{k+1} for $k + 1$.

Second idea.

Let $L_i =$ **longest ascending subsequence of** A_i ($1 \leq i \leq n$)

Assumption: a LAS L_j is known for each $j \leq k$. Now compute LAS L_{k+1} for $k + 1$.

Look at all fitting $L_{k+1} = L_j \oplus a_{k+1}$ ($j \leq k$) and choose a longest sequence.

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Counterexample: $A_5 = (1, 2, 5, 3, 4)$. Let $A_4 = (1, 2, 5, 3)$ with $L_1 = (1)$, $L_2 = (1, 2)$, $L_3 = (1, 2, 5)$, $L_4 = (1, 2, 5)$. Determine L_5 from L_1, \dots, L_4 ?

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That does not work either: cannot infer L_{k+1} from only **an arbitrary solution** L_j . We need to consider all LAS. Too many.

Third approach

Let $M_{n,i}$ = **longest ascending subsequence of** A_i ($1 \leq i \leq n$)

Assumption: the LAS M_j for A_k , **that end with smallest element** are known for each of the lengths $1 \leq j \leq k$.

Third approach

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Assumption: the LAS M_j for A_k , **that end with smallest element** are known for each of the lengths $1 \leq j \leq k$.

Consider all fitting $M_{k,j} \oplus a_{k+1}$ ($j \leq k$) and update the table of the LAS, that end with smallest possible element.

Third approach Example

Example: $A = (1, 1000, 1001, 4, 5, 2, 6, 7)$

A	LAT $M_{k,\cdot}$
1	(1)

Third approach Example

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1	(1)
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+ 1001	(1), (1, 1000), (1, 1000, 1001)

Third approach Example

Example: $A = (1, 1000, 1001, 4, 5, 2, 6, 7)$

A	LAT $M_{k,\cdot}$
1	(1)
+ 1000	(1), (1, 1000)
+ 1001	(1), (1, 1000), (1, 1000, 1001)
+ 4	(1), (1, 4), (1, 1000, 1001)

Third approach Example

Example: $A = (1, 1000, 1001, 4, 5, 2, 6, 7)$

A	LAT $M_{k,\cdot}$
1	(1)
+ 1000	(1), (1, 1000)
+ 1001	(1), (1, 1000), (1, 1000, 1001)
+ 4	(1), (1, 4), (1, 1000, 1001)
+ 5	(1), (1, 4), (1, 4, 5)

Third approach Example

Example: $A = (1, 1000, 1001, 4, 5, 2, 6, 7)$

A	LAT $M_{k,\cdot}$
1	(1)
+ 1000	(1), (1, 1000)
+ 1001	(1), (1, 1000), (1, 1000, 1001)
+ 4	(1), (1, 4), (1, 1000, 1001)
+ 5	(1), (1, 4), (1, 4, 5)
+ 2	(1), (1, 2), (1, 4, 5)

Third approach Example

Example: $A = (1, 1000, 1001, 4, 5, 2, 6, 7)$

A	LAT $M_{k,\cdot}$
1	(1)
+ 1000	(1), (1, 1000)
+ 1001	(1), (1, 1000), (1, 1000, 1001)
+ 4	(1), (1, 4), (1, 1000, 1001)
+ 5	(1), (1, 4), (1, 4, 5)
+ 2	(1), (1, 2), (1, 4, 5)
+ 6	(1), (1, 2), (1, 4, 5), (1, 4, 5, 6)

Third approach Example

Example: $A = (1, 1000, 1001, 4, 5, 2, 6, 7)$

A	LAT $M_{k,\cdot}$
1	(1)
+ 1000	(1), (1, 1000)
+ 1001	(1), (1, 1000), (1, 1000, 1001)
+ 4	(1), (1, 4), (1, 1000, 1001)
+ 5	(1), (1, 4), (1, 4, 5)
+ 2	(1), (1, 2), (1, 4, 5)
+ 6	(1), (1, 2), (1, 4, 5), (1, 4, 5, 6)
+ 7	(1), (1, 2), (1, 4, 5), (1, 4, 5, 6), (1, 4, 5, 6, 7)

DP Table

- Idea: save the last element of the increasing sequence $M_{k,j}$ at slot j .

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Index	0	1	2	3	4	...
$(L_j)_j$	$-\infty$	∞	∞	∞	∞	

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Index	0	1	2	3	4	...
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DP Table

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- Example: 3 2 5 1 6 4

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Wert	3	2	5	1	6	4

Index	0	1	2	3	4	...
$(L_j)_j$	$-\infty$	2	5	∞	∞	

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- Problem: Table does not contain the subsequence, only the last value.

Index	1	2	3	4	5	6
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Index	0	1	2	3	4	...
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- Idea: save the last element of the increasing sequence $M_{k,j}$ at slot j .
- Example: 3 2 5 1 6 4
- Problem: Table does not contain the subsequence, only the last value.
- Solution: second table with the predecessors.

Index	1	2	3	4	5	6
Wert	3	2	5	1	6	4

Index	0	1	2	3	4	...
$(L_j)_j$	$-\infty$	1	4	6	∞	

DP Table

- Idea: save the last element of the increasing sequence $M_{k,j}$ at slot j .
- Example: 3 2 5 1 6 4
- Problem: **Table** does not contain the subsequence, only the last value.
- Solution: **second table** with the predecessors.

Index	1	2	3	4	5	6
Wert	3	2	5	1	6	4
Predecessor	$-\infty$	$-\infty$	2	$-\infty$	5	1

Index	0	1	2	3	4	...
$(L_j)_j$	$-\infty$	1	4	6	∞	

Dynamic Programming Algorithm LAS

Table dimension? Semantics?

1.

Dynamic Programming Algorithm LAS

Table dimension? Semantics?

1. Two tables $T[0, \dots, n]$ and $V[1, \dots, n]$.
 $T[j]$: last Element of the increasing subsequence $M_{n,j}$
 $V[j]$: Value of the predecessor of a_j .
Start with $T[0] \leftarrow -\infty, T[i] \leftarrow \infty \forall i > 1$

Dynamic Programming Algorithm LAS

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Computation of an entry

- 2.

Dynamic Programming Algorithm LAS

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 $V[j]$: Value of the predecessor of a_j .
Start with $T[0] \leftarrow -\infty, T[i] \leftarrow \infty \forall i > 1$

Computation of an entry

2. Entries in T sorted in ascending order. For each new entry a_k binary search for l , such that $T[l] < a_k < T[l + 1]$. Set $T[l + 1] \leftarrow a_k$. Set $V[k] = T[l]$.

Dynamic Programming algorithm LAS

3.

Computation order

Dynamic Programming algorithm LAS

Computation order

3.

Traverse the list and compute $T[k]$ and $V[k]$ with ascending k

Dynamic Programming algorithm LAS

Computation order

3.

Traverse the list and compute $T[k]$ and $V[k]$ with ascending k

Reconstruction of a solution?

4.

Dynamic Programming algorithm LAS

Computation order

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Traverse the list and compute $T[k]$ and $V[k]$ with ascending k

Reconstruction of a solution?

4.

Search the largest l with $T[l] < \infty$. l is the last index of the LAS. Starting at l search for the index $i < l$ such that $V[l] = a_i$, i is the predecessor of l . Repeat with $l \leftarrow i$ until $T[l] = -\infty$

Analysis

■ Computation of the table:

- Initialization: $\Theta(n)$ Operations
- Computation of the k th entry: binary search on positions $\{1, \dots, k\}$ plus constant number of assignments.

$$\sum_{k=1}^n (\log k + \mathcal{O}(1)) = \mathcal{O}(n) + \sum_{k=1}^n \log(k) = \Theta(n \log n).$$

- **Reconstruction:** traverse A from right to left: $\mathcal{O}(n)$.

Overall runtime:

$$\Theta(n \log n).$$

Minimal Editing Distance

Editing distance of two sequences $A_n = (a_1, \dots, a_n)$, $B_m = (b_1, \dots, b_m)$.

Editing operations:

- Insertion of a character
- Deletion of a character
- Replacement of a character

Question: how many editing operations at least required in order to transform string A into string B .

TIGER ZIGER ZIEGER ZIEGE

Minimal Editing Distance

Wanted: cheapest character-wise transformation $A_n \rightarrow B_m$ with costs

operation	Levenshtein	LCS ³⁵	general
Insert c	1	1	ins(c)
Delete c	1	1	del(c)
Replace $c \rightarrow c'$	$\mathbb{1}(c \neq c')$	$\infty \cdot \mathbb{1}(c \neq c')$	repl(c, c')

Beispiel

T	I	G	E	R	T	I	_	G	E	R	T \rightarrow Z	+E	-R
Z	I	E	G	E	Z	I	E	G	E	_	Z \rightarrow T	-E	+R

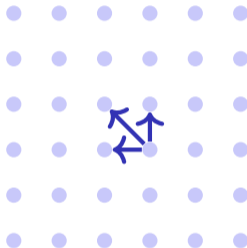
³⁵Longest common subsequence – A special case of an editing problem

0. $E(n, m)$ = minimum number edit operations (ED cost) $a_{1\dots n} \rightarrow b_{1\dots m}$
1. Subproblems $E(i, j)$ = ED von $a_{1\dots i}$. $b_{1\dots j}$. #SP = $n \cdot m$
2. Guess Costs $\Theta(1)$
- $a_{1\dots i} \rightarrow a_{1\dots i-1}$ (delete)
 - $a_{1\dots i} \rightarrow a_{1\dots i}b_j$ (insert)
 - $a_{1\dots i} \rightarrow a_{1\dots i-1}b_j$ (replace)

3. Rekursion

$$E(i, j) = \min \begin{cases} \text{del}(a_i) + E(i-1, j), \\ \text{ins}(b_j) + E(i, j-1), \\ \text{repl}(a_i, b_j) + E(i-1, j-1) \end{cases}$$

4. Dependencies



⇒ Computation from left top to bottom right. Row- or column-wise.

5. Solution in $E(n, m)$

Example (Levenshtein Distance)

$$E[i, j] \leftarrow \min \{ E[i-1, j] + 1, E[i, j-1] + 1, E[i-1, j-1] + \mathbb{1}(a_i \neq b_j) \}$$

	\emptyset	Z	I	E	G	E
\emptyset	0	1	2	3	4	5
T	1	1	2	3	4	5
I	2	2	1	2	3	4
G	3	3	2	2	2	3
E	4	4	3	2	3	2
R	5	5	4	3	3	3

Editing steps: from bottom right to top left, following the recursion.
Bottom-Up description of the algorithm: exercise

Bottom-Up DP algorithm ED

Dimension of the table? Semantics?

1.

Bottom-Up DP algorithm ED

Dimension of the table? Semantics?

1. Table $E[0, \dots, m][0, \dots, n]$. $E[i, j]$: minimal edit distance of the strings (a_1, \dots, a_i) and (b_1, \dots, b_j)

Bottom-Up DP algorithm ED

Dimension of the table? Semantics?

1. Table $E[0, \dots, m][0, \dots, n]$. $E[i, j]$: minimal edit distance of the strings (a_1, \dots, a_i) and (b_1, \dots, b_j)

Computation of an entry

- 2.

Bottom-Up DP algorithm ED

Dimension of the table? Semantics?

1. Table $E[0, \dots, m][0, \dots, n]$. $E[i, j]$: minimal edit distance of the strings (a_1, \dots, a_i) and (b_1, \dots, b_j)

Computation of an entry

2. $E[0, i] \leftarrow i \forall 0 \leq i \leq m$, $E[j, 0] \leftarrow j \forall 0 \leq j \leq n$. Computation of $E[i, j]$ otherwise via $E[i, j] = \min\{\text{del}(a_i) + E(i-1, j), \text{ins}(b_j) + E(i, j-1), \text{repl}(a_i, b_j) + E(i-1, j-1)\}$

Bottom-Up DP algorithm ED

Computation order

3.

Bottom-Up DP algorithm ED

Computation order

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Rows increasing and within columns increasing (or the other way round).

Bottom-Up DP algorithm ED

Computation order

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Rows increasing and within columns increasing (or the other way round).

Reconstruction of a solution?

4.

Bottom-Up DP algorithm ED

Computation order

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Rows increasing and within columns increasing (or the other way round).

Reconstruction of a solution?

4.

Start with $j = m, i = n$. If $E[i, j] = \text{repl}(a_i, b_j) + E(i - 1, j - 1)$ then output $a_i \rightarrow b_j$ and continue with $(j, i) \leftarrow (j - 1, i - 1)$; otherwise, if $E[i, j] = \text{del}(a_i) + E(i - 1, j)$ output $\text{del}(a_i)$ and continue with $j \leftarrow j - 1$ otherwise, if $E[i, j] = \text{ins}(b_j) + E(i, j - 1)$, continue with $i \leftarrow i - 1$. Terminate for $i = 0$ and $j = 0$.

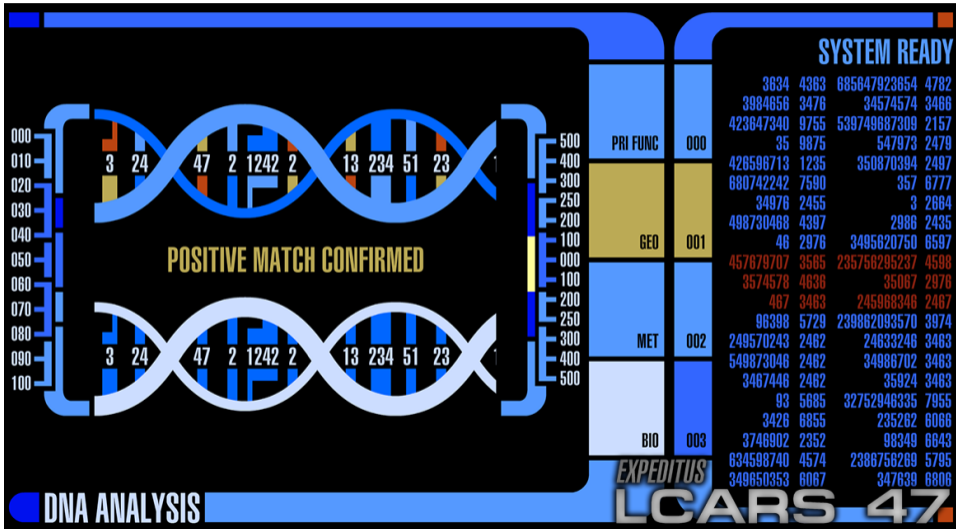
Analysis ED

- Number table entries: $(m + 1) \cdot (n + 1)$.
- Constant number of assignments and comparisons each. Number steps: $\mathcal{O}(mn)$
- Determination of solution: decrease i or j . Maximally $\mathcal{O}(n + m)$ steps.

Runtime overall:

$$\mathcal{O}(mn).$$

DNA - Comparison (Star Trek)



DNA - Comparison

- DNA consists of sequences of four different nucleotides **A**denine **G**uanine **T**hymine **C**ytosine
- DNA sequences (genes) thus can be described with strings of A, G, T and C.
- Possible comparison of two genes: determine the **longest common subsequence**

The longest common subsequence problem is a special case of the minimal edit distance problem.

Matrix-Chain-Multiplication

Task: Computation of the product $A_1 \cdot A_2 \cdot \dots \cdot A_n$ of matrices A_1, \dots, A_n .

Matrix multiplication is associative, i.e. the order of evaluation can be chosen arbitrarily

Goal: efficient computation of the product.

Assumption: multiplication of an $(r \times s)$ -matrix with an $(s \times u)$ -matrix provides costs $r \cdot s \cdot u$.

Does it matter?

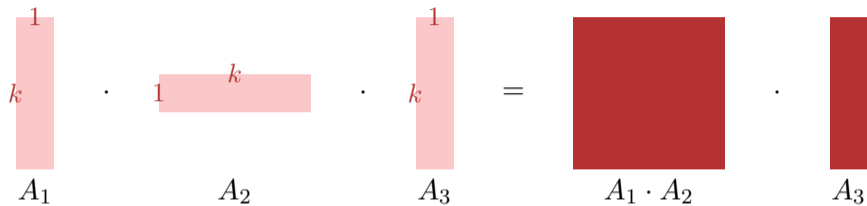
A diagram illustrating the multiplication of three matrices, A_1 , A_2 , and A_3 , represented by red rectangles. A_1 is a vertical rectangle with height k and width 1 . A_2 is a horizontal rectangle with height 1 and width k . A_3 is a vertical rectangle with height k and width 1 . The matrices are arranged in a row, separated by multiplication dots, followed by an equals sign.

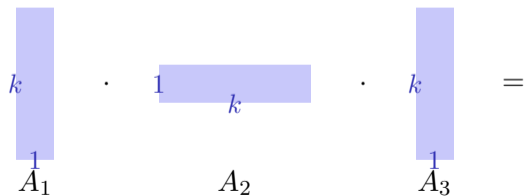
$$\begin{matrix} 1 \\ k \end{matrix} A_1 \cdot \begin{matrix} 1 & k \end{matrix} A_2 \cdot \begin{matrix} 1 \\ k \end{matrix} A_3 =$$

A diagram illustrating the multiplication of three matrices, A_1 , A_2 , and A_3 , represented by blue rectangles. A_1 is a vertical rectangle with height k and width 1 . A_2 is a horizontal rectangle with height 1 and width k . A_3 is a vertical rectangle with height k and width 1 . The matrices are arranged in a row, separated by multiplication dots, followed by an equals sign.

$$\begin{matrix} k \\ 1 \end{matrix} A_1 \cdot \begin{matrix} 1 & k \end{matrix} A_2 \cdot \begin{matrix} k \\ 1 \end{matrix} A_3 =$$

Does it matter?


$$\begin{matrix} 1 \\ k \\ A_1 \end{matrix} \cdot \begin{matrix} 1 & k \\ A_2 \end{matrix} \cdot \begin{matrix} 1 \\ k \\ A_3 \end{matrix} = \begin{matrix} A_1 \cdot A_2 \end{matrix} \cdot \begin{matrix} A_3 \end{matrix}$$


$$\begin{matrix} k \\ 1 \\ A_1 \end{matrix} \cdot \begin{matrix} 1 & k \\ A_2 \end{matrix} \cdot \begin{matrix} k \\ 1 \\ A_3 \end{matrix} =$$

Does it matter?

A_1 (dimensions $k \times 1$) \cdot A_2 (dimensions $1 \times k$) \cdot A_3 (dimensions $k \times 1$) $=$ $(A_1 \cdot A_2)$ (dimensions $k \times k$) \cdot A_3 (dimensions $k \times 1$) $=$ $(A_1 \cdot A_2 \cdot A_3)$ (dimensions $k \times 1$)

A_1 (dimensions $k \times 1$) \cdot A_2 (dimensions $1 \times k$) \cdot A_3 (dimensions $k \times 1$) $=$

Does it matter?

$A_1 \cdot A_2 \cdot A_3 = (A_1 \cdot A_2) \cdot A_3 = A_1 \cdot A_2 \cdot A_3$

$A_1 \cdot A_2 \cdot A_3 = A_1 \cdot A_2 \cdot A_3$

Does it matter?

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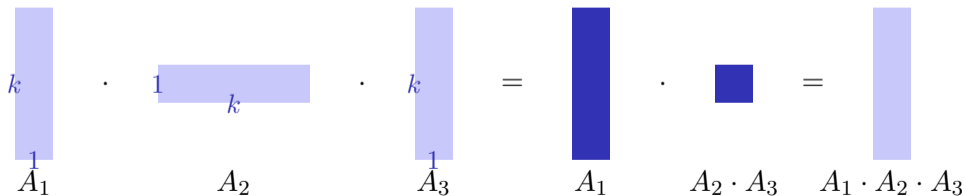
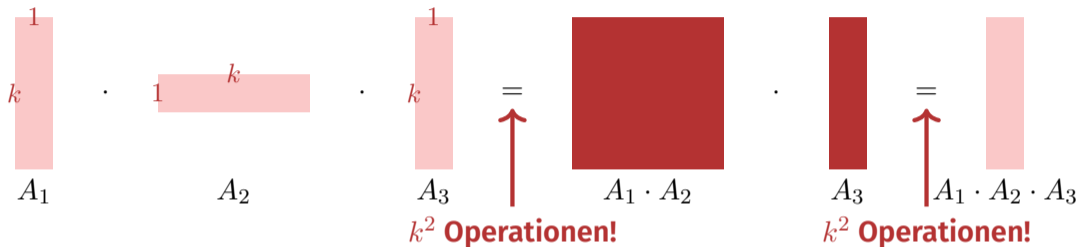
$A_1 \cdot A_2 \cdot A_3 = A_1 \cdot (A_2 \cdot A_3)$

Does it matter?

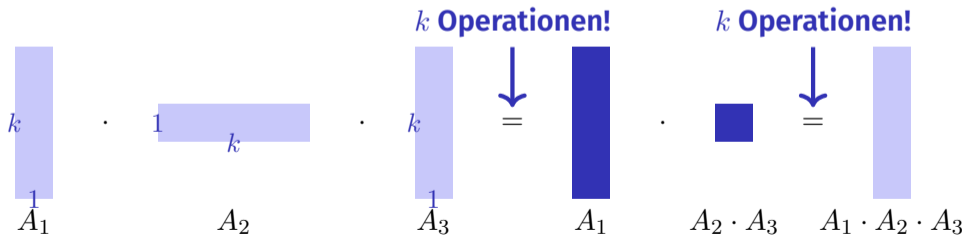
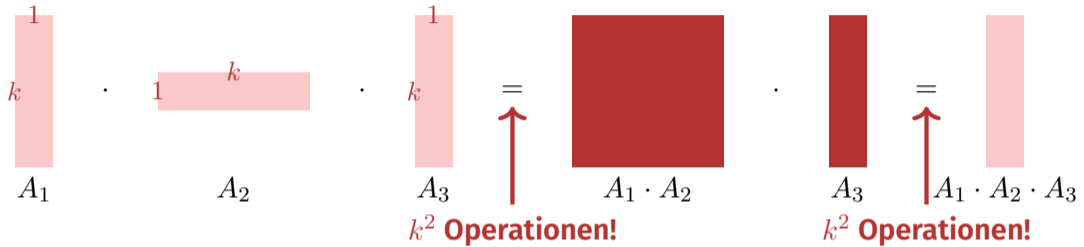
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$A_1 \cdot A_2 \cdot A_3 = A_1 \cdot (A_2 \cdot A_3) = A_1 \cdot A_2 \cdot A_3$

Does it matter?



Does it matter?



Recursion

- Assume that the best possible computation of $(A_1 \cdot A_2 \cdots A_i)$ and $(A_{i+1} \cdot A_{i+2} \cdots A_n)$ is known for each i .
- Compute best i , done.

$n \times n$ -table M . entry $M[p, q]$ provides costs of the best possible bracketing $(A_p \cdot A_{p+1} \cdots A_q)$.

$$M[p, q] \leftarrow \min_{p \leq i < q} (M[p, i] + M[i + 1, q] + \text{costs of the last multiplication})$$

Computation of the DP-table

- Base cases $M[p, p] \leftarrow 0$ for all $1 \leq p \leq n$.
- Computation of $M[p, q]$ depends on $M[i, j]$ with $p \leq i \leq j \leq q$, $(i, j) \neq (p, q)$.
In particular $M[p, q]$ depends at most from entries $M[i, j]$ with $i - j < q - p$.
Consequence: fill the table from the diagonal.

Analysis

DP-table has n^2 entries. Computation of an entry requires considering up to $n - 1$ other entries.

Overall runtime $\mathcal{O}(n^3)$.

Readout the order from M : exercise!

Digression: matrix multiplication

Consider the multiplication of two $n \times n$ matrices.

Let

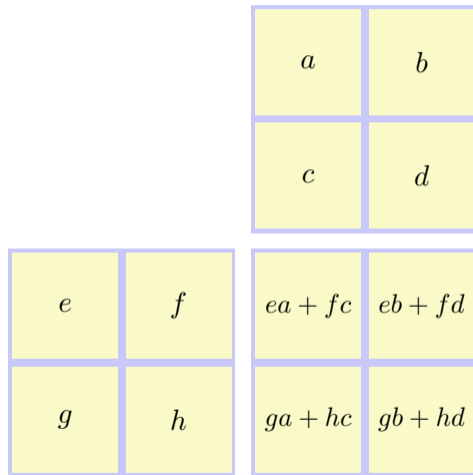
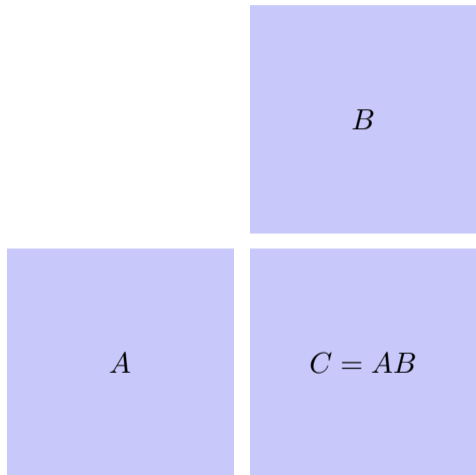
$$A = (a_{ij})_{1 \leq i, j \leq n}, B = (b_{ij})_{1 \leq i, j \leq n}, C = (c_{ij})_{1 \leq i, j \leq n}, \\ C = A \cdot B$$

then

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}.$$

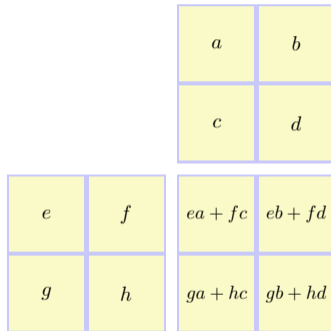
Naive algorithm requires $\Theta(n^3)$ elementary multiplications.

Divide and Conquer



Divide and Conquer

- Assumption $n = 2^k$.
- Number of elementary multiplications:
 $M(n) = 8M(n/2)$, $M(1) = 1$.
- yields $M(n) = 8^{\log_2 n} = n^{\log_2 8} = n^3$. No advantage 😞



Strassen's Matrix Multiplication

- **Nontrivial observation by Strassen (1969):** It suffices to compute the seven products
 $A = (e + h) \cdot (a + d)$, $B = (g + h) \cdot a$, $C = e \cdot (b - d)$,
 $D = h \cdot (c - a)$, $E = (e + f) \cdot d$, $F = (g - e) \cdot (a + b)$,
 $G = (f - h) \cdot (c + d)$. Denn:
 $ea + fc = A + D - E + G$, $eb + fd = C + E$,
 $ga + hc = B + D$, $gb + hd = A - B + C + F$.
- This yields $M'(n) = 7M(n/2)$, $M'(1) = 1$.
Thus $M'(n) = 7^{\log_2 n} = n^{\log_2 7} \approx n^{2.807}$.
- Fastest currently known algorithm: $\mathcal{O}(n^{2.37})$

