Felix Friedrich

Data Structures and Algorithms

Course at D-MATH (CSE) of ETH Zurich

Spring 2020

1. Introduction

Overview, Algorithms and Data Structures, Correctness, First Example

Goals of the course

- Understand the design and analysis of fundamental algorithms and data structures.
- An advanced insight into a modern programming model (with C++).
- Knowledge about chances, problems and limits of the parallel and concurrent computing.

Contents

data structures / algorithms

The notion invariant, cost model, Landau notation
algorithms design, induction
searching, selection and sorting
amortized analysis
dynamic programming

Minimum Spanning Trees, Fibonacci Heaps
shortest paths, Max-Flow
Fundamental algorithms on graphs,
dictionaries: hashing and search trees
van-Emde Boas Trees, Splay-Trees

prorgamming with C++

RAII, Move Konstruktion, Smart Pointers,
Templates and generic programming
Exceptions
functors and lambdas

promises and futures
threads, mutex and monitors

parallel programming

parallelism vs. concurrency, speedup (Amdahl/Gustavson), races, memory reordering, atomir registers, RMW (CAS,TAS), deadlock/starvation

1.2 Algorithms

[Cormen et al, Kap. 1; Ottman/Widmayer, Kap. 1.1]

Algorithm

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Well-defined procedure to compute **output** data from **input** data

Example Problem: Sorting

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Possible input

$(1, 7, 3)$, $(15, 13, 12, -0.5)$, $(999, 998, 997, 996, \ldots, 2, 1)$, (1), $()$ …

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Every example represents a **problem instance**

The performance (speed) of an algorithm usually depends on the problem instance. Often there are “good” and “bad” instances.

Therefore we consider algorithms sometimes **”in the average“** and most often in the **”worst case“**.

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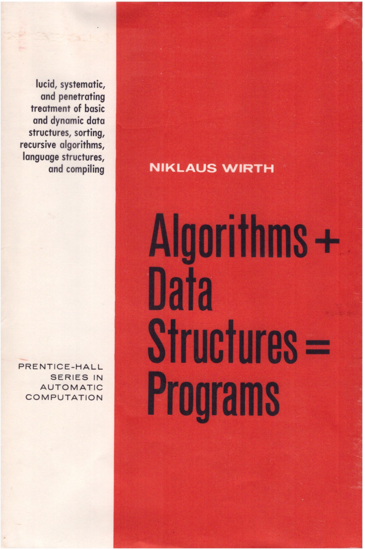
- Fast Lookup : Hash-Tables
- The travelling Salesman: Dynamic Programming, Minimum Spanning Tree, Simulated Annealing

Characteristics

- Extremely large number of potential solutions
- Practical applicability

Data Structures

- A data structure is a particular way of **organizing data** in a computer so that they can be **used efficiently** (in the algorithms operating on them).
- Programs = algorithms + data structures.



Efficiency

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Reality: resources are bounded and not free:

- Computing time $\rightarrow$ Efficiency
- Storage space $\rightarrow$ Efficiency

Actually, this course is nearly only about efficiency.

Hard problems.

- NP-complete problems: no known efficient solution (the existence of such a solution is very improbable – but it has not yet been proven that there is none!)
- Example: travelling salesman problem

This course is *mostly* about problems that can be solved efficiently (in polynomial time).

2. Efficiency of algorithms

Efficiency of Algorithms, Random Access Machine Model, Function Growth, Asymptotics [Cormen et al, Kap. 2.2,3,4.2-4.4 | Ottman/Widmayer, Kap. 1.1]

Efficiency of Algorithms

Goals

- Quantify the runtime behavior of an algorithm independent of the machine.
- Compare efficiency of algorithms.
- Understand dependece on the input size.

Programs and Algorithms

Technology

$$\text{program} \xrightarrow{\text{implemented in}} \text{programming language} \xrightarrow{\text{specified for}} \text{computer}$$

Abstraction

$$\text{algorithm} \xrightarrow{\text{specified in}} \text{pseudo-code} \xrightarrow{\text{based on}} \text{computation model}$$

Technology Model

Random Access Machine (RAM) Model

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- Unit cost model: fundamental operations provide a cost of 1.
- Data types: fundamental types like size-limited integer or floating point number.

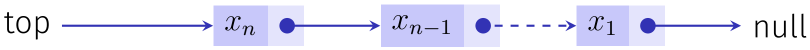
Size of the Input Data

- Typical: number of input objects (of fundamental type).
- Sometimes: number bits for a *reasonable* / *cost-effective* representation of the data.
- fundamental types fit into word of size : $w \geq \log(\text{sizeof(mem)})$ bits.

For Dynamic Data Strcutures

Pointer Machine Model

- Objects bounded in size can be dynamically allocated in constant time
- Fields (with word-size) of the objects can be accessed in constant time 1.



Asymptotic behavior

An exact running time of an algorithm can normally not be predicted even for small input data.

- We consider the asymptotic behavior of the algorithm.
- And ignore all constant factors.

An operation with cost 20 is no worse than one with cost 1
Linear growth with gradient 5 is as good as linear growth with gradient 1.

2.2 Function growth

$\mathcal{O}$, Θ, Ω [Cormen et al, Kap. 3; Ottman/Widmayer, Kap. 1.1]

Superficially

Use the asymptotic notation to specify the execution time of algorithms.
We write $\Theta(n^2)$ and mean that the algorithm behaves for large n like n^2:
when the problem size is doubled, the execution time multiplies by four.

More precise: asymptotic upper bound

provided: a function $g : \mathbb{N} \to \mathbb{R}$.

Definition:[1]

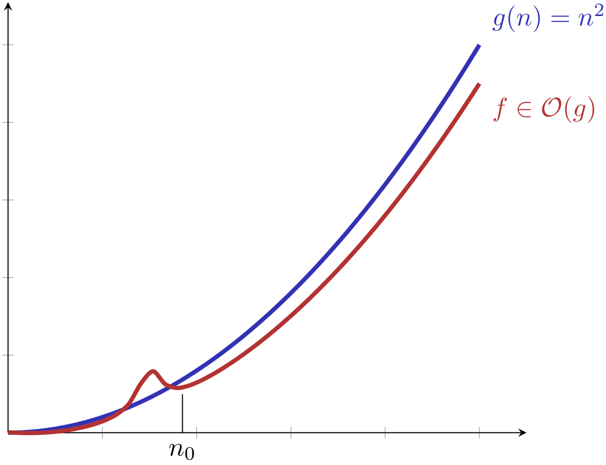
$$\begin{aligned}\mathcal{O}(g) = \{f : \mathbb{N} \to \mathbb{R} \,|\\ &\exists\, c > 0, \exists n_0 \in \mathbb{N} :\\ &\forall\, n \geq n_0 : 0 \leq f(n) \leq c \cdot g(n)\}\end{aligned}$$

Notation:

$$\mathcal{O}(g(n)) := \mathcal{O}(g(\cdot)) = \mathcal{O}(g).$$

[1] Ausgesprochen: Set of all functions $f : \mathbb{N} \to \mathbb{R}$ that satisfy: there is some (real valued) $c > 0$ and some $n_0 \in \mathbb{N}$ such that $0 \leq f(n) \leq n \cdot g(n)$ for all $n \geq n_0$.

Graphic



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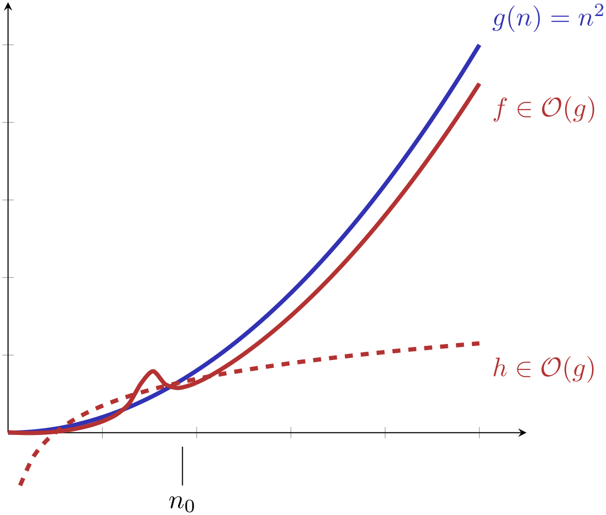


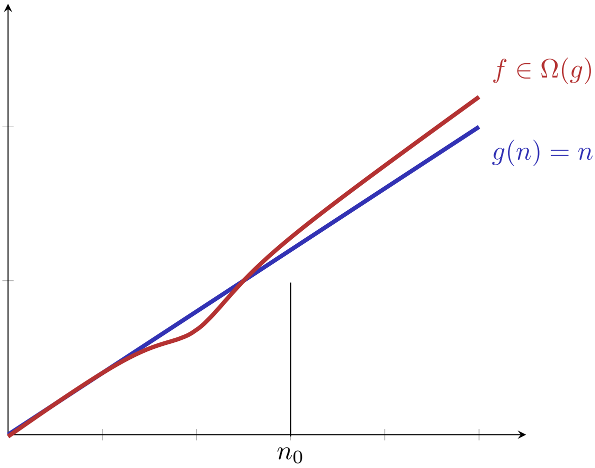
Converse: asymptotic lower bound

Given: a function $g : \mathbb{N} \to \mathbb{R}$.

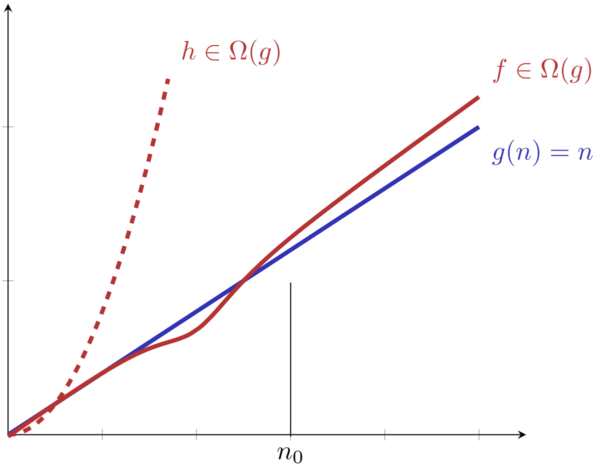
Definition:

$$\begin{aligned}\Omega(g) = \{f : \mathbb{N} \to \mathbb{R} |\\ &\exists\, c > 0, \exists n_0 \in \mathbb{N} :\\ &\forall\, n \geq n_0 : 0 \leq c \cdot g(n) \leq f(n)\}\end{aligned}$$

Example



Example



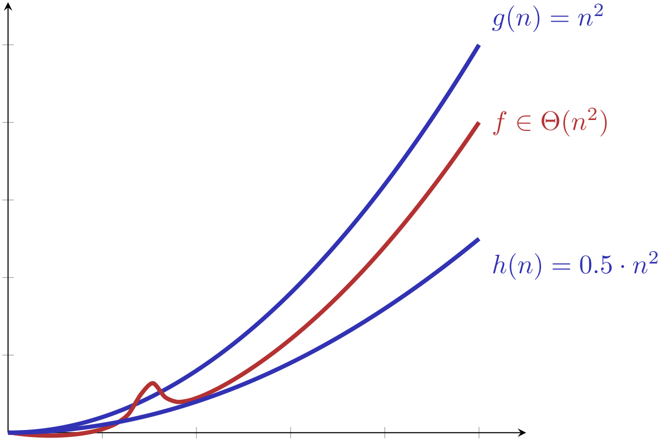
Asymptotic tight bound

Given: function $g : \mathbb{N} \to \mathbb{R}$.
Definition:

$$\Theta(g) := \Omega(g) \cap \mathcal{O}(g).$$

Simple, closed form: exercise.

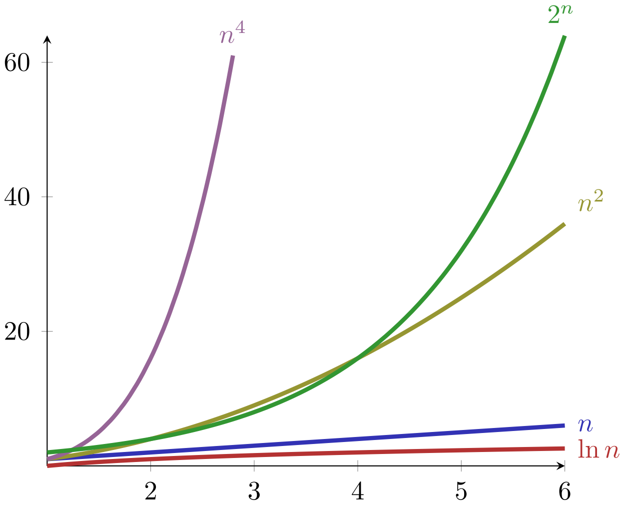
Example



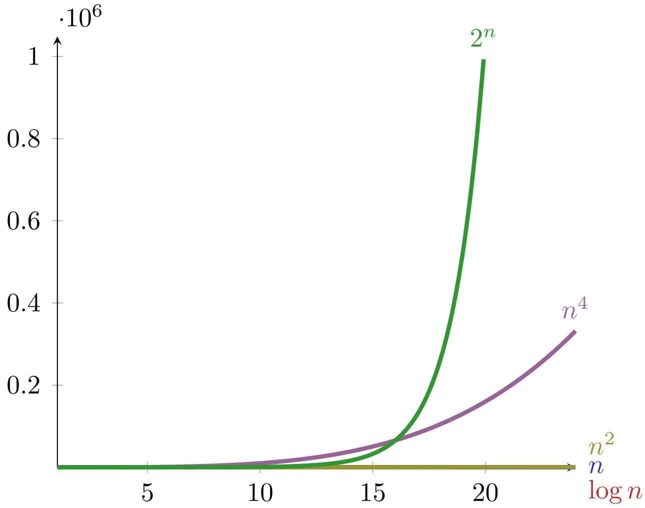
Notions of Growth

$\mathcal{O}(1)$	bounded	array access
$\mathcal{O}(\log\log n)$	double logarithmic	interpolated binary sorted sort
$\mathcal{O}(\log n)$	logarithmic	binary sorted search
$\mathcal{O}(\sqrt{n})$	like the square root	naive prime number test
$\mathcal{O}(n)$	linear	unsorted naive search
$\mathcal{O}(n\log n)$	superlinear / loglinear	good sorting algorithms
$\mathcal{O}(n^2)$	quadratic	simple sort algorithms
$\mathcal{O}(n^c)$	polynomial	matrix multiply
$\mathcal{O}(2^n)$	exponential	Travelling Salesman Dynamic Programming
$\mathcal{O}(n!)$	factorial	Travelling Salesman naively

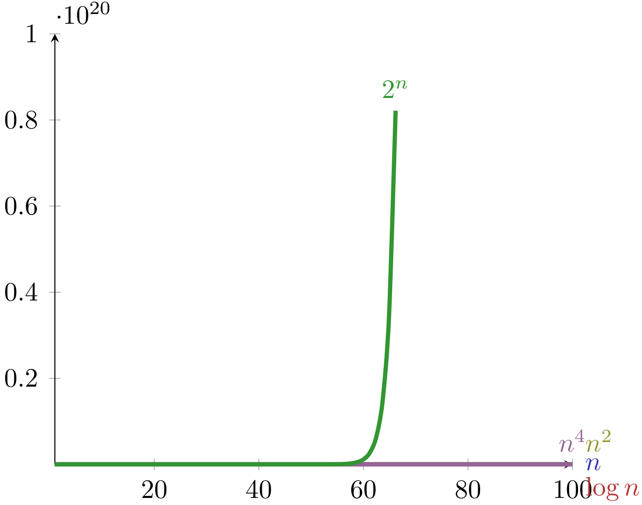
Small n



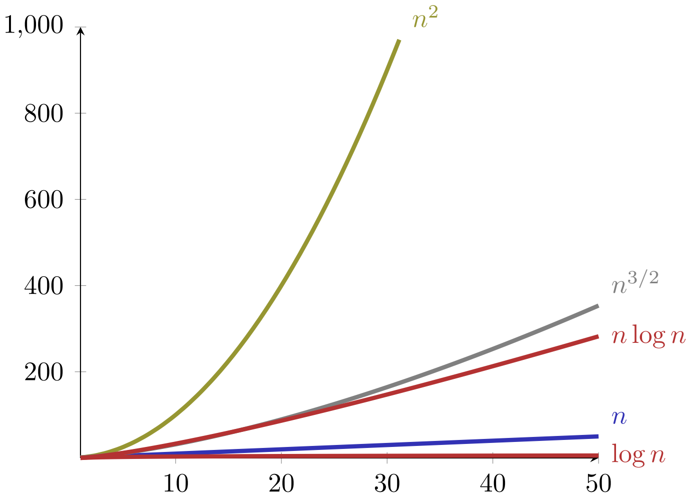
Larger n



“Large” n



Logarithms



Time Consumption

Assumption 1 Operation = $1\mu s$.

problem size	1	100	10000	10^6	10^9
$\log_2 n$	$1\mu s$				
n	$1\mu s$				
$n \log_2 n$	$1\mu s$				
n^2	$1\mu s$				
2^n	$1\mu s$				

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n	$1\mu s$	$100\mu s$	$1/100s$	$1s$	17 minutes
$n \log_2 n$	$1\mu s$	$700\mu s$	$13/100\mu s$	$20s$	8.5 hours
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n^2	$1\mu s$	$1/100s$	1.7 minutes	11.5 days	317 centuries
2^n	$1\mu s$	10^{14} centuries	$\approx \infty$	$\approx \infty$	$\approx \infty$

About the Notation

Common casual notation

$$f = \mathcal{O}(g)$$

should be read as $f \in \mathcal{O}(g)$.
Clearly it holds that

$$f_1 = \mathcal{O}(g), f_2 = \mathcal{O}(g) \not\Rightarrow f_1 = f_2!$$

$n = \mathcal{O}(n^2), n^2 = \mathcal{O}(n^2)$ but naturally $n \neq n^2$.

We avoid this notation where it could lead to ambiguities.

Reminder: Efficiency: Arrays vs. Linked Lists

- Memory: our **avec** requires roughly n ints (vector size n), our **llvec** roughly $3n$ ints (a pointer typically requires 8 byte)
- Runtime (with **avec** = **std::vector**, **llvec** = **std::list**):

```
prepending (insert at front) [100,000x]:
    ► avec:    675 ms
    ► llvec:    10 ms
appending (insert at back) [100,000x]:
    ► avec:      2 ms
    ► llvec:     9 ms
removing first [100,000x]:
    ► avec:    675 ms
    ► llvec:     4 ms
removing last [100,000x]:
    ► avec:      0 ms
    ► llvec:     4 ms
removing randomly [10,000x]:
    ► avec:      3 ms
    ► llvec:   113 ms
inserting randomly [10,000x]:
    ► avec:     16 ms
    ► llvec:   117 ms
fully iterate sequentially (5000 elements) [5,000x]:
    ► avec:    354 ms
    ► llvec:   525 ms
```

Asymptotic Runtimes

With our new language ($\Omega, \mathcal{O}, \Theta$), we can now **state the behavior of the data structures and their algorithms more precisely**

Typical asymptotic running times (Anticipation!)

Data structure	Random Access	Insert	Next	Insert After Element	Search
`std::vector`	$\Theta(1)$	$\Theta(1)\ A$	$\Theta(1)$	$\Theta(n)$	$\Theta(n)$
`std::list`	$\Theta(n)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(n)$
`std::set`	–	$\Theta(\log n)$	$\Theta(\log n)$	–	$\Theta(\log n)$
`std::unordered_set`	–	$\Theta(1)\ P$	–	–	$\Theta(1)\ P$

A = amortized, P=expected, otherwise worst case

Complexity

Complexity of a problem P

minimal (asymptotic) costs over all algorithms A that solve P.

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Complexity of the single-digit multiplication of two numbers with n digits is $\Omega(n)$ and $\mathcal{O}(n^{\log_3 2})$ (Karatsuba Ofman).

Complexity

Problem	Complexity	$\mathcal{O}(n)$	$\mathcal{O}(n)$	$\mathcal{O}(n^2)$	$\Omega(n \log n)$
		$\Uparrow$	$\Uparrow$	$\Uparrow$	$\Downarrow$
Algorithm	Costs[2]	$3n-4$	$\mathcal{O}(n)$	$\Theta(n^2)$	$\Omega(n \log n)$
		$\Downarrow$	$\Updownarrow$	$\Updownarrow$	$\Downarrow$
Program	Execution time	$\Theta(n)$	$\mathcal{O}(n)$	$\Theta(n^2)$	$\Omega(n \log n)$

[2]Number fundamental operations