Datenstrukturen und Algorithmen

Exercise 5

FS 2020

Program of today

1. Feedback of last exercise
2. Repetition theory
3. Programming Task

Amortized analysis: push_back

Strategy: double if array is full.

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Let $i \in \mathbb{N}$ be the number of elements appended and let $n_i \in \mathbb{N}$ be the array size allocated after appending i.

It holds that

$$n_i = \begin{cases} 1 & \text{if } i = 1 \text{ [Start]} \\ 2 \cdot n_{i-1} & \text{if } i - 1 \in \{2^k : k \in \mathbb{N}\} \text{ [Array full]} \\ n_{i-1} & \text{otherwise} \end{cases}$$

$$n_i = 2^{\lceil \log_2 i \rceil}$$

i	n_i
1	1
2	2
3	4
4	4
5	8
6	8
..	..

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Real costs

$$t_i = \begin{cases} 1 & \text{if } i = 1 \text{ [Start]} \\ 3n_{i-1} + 1 & \text{if } i - 1 \in \{2^k : k \in \mathbb{N}\} \text{ [Array full]}^1 \\ 1 & \text{otherwise} \end{cases}$$

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Find potential function such that the amortized costs are constant:

$$a_i = t_i + \Phi_i - \Phi_{i-1}$$

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$$\begin{aligned}\Phi_i &= 6 \cdot \text{number of elements in the upper half of the array} \\ &= 6 \cdot (i - \frac{n_i}{2}) = 6i - 3n_i\end{aligned}$$

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$$\Phi_i - \Phi_{i-1} = \begin{cases} 6 + 3n_{i-1} - 3 \overbrace{n_i}^{2 \cdot n_{i-1}} & \text{if } i - 1 \in \{2^k : k \in \mathbb{N}\} \text{ [Array full]} \\ 6 & \text{otherwise} \end{cases}$$

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Strategy: double if array is full.

Find potential function such that the amortized costs are constant:

$$
\begin{aligned}
a_i &= t_i + \Phi_i - \Phi_{i-1} \\
&= \begin{cases} 3n_{i-1} + 1 + 6 - 3n_{i-1} & \text{if } i-1 \in \{2^k : k \in \mathbb{N}\} \text{ [Array full]} \\ 1 + 6 & \text{otherwise} \end{cases} \\
&\leq 7 \quad \text{for all } i
\end{aligned}
$$

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Strategy: halve if array is three quarters empty.

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Let k_i be the number of elements in the array in step i

$\Phi_i = 3 \cdot$ number of empty elements in the lower half of array $(1, \ldots, \dfrac{n}{2})$

$$= 3 \cdot (\frac{n_i}{2} - k_i)$$

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$$\Phi_i - \Phi_{i-1} = \begin{cases} 3 & \text{if array is more than quarter full} \\ 3 \cdot \left(1 + \frac{n_{i-1}}{4} - \frac{n_{i-1}}{2})\right) & \text{otherwise} \end{cases}$$

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$$\Rightarrow 4 \geq a_i \text{ (in both cases)}$$

Amortized analysis: pop and push

$$\Phi_i = 6 \cdot \text{number elements in the upper half} \\ + 3 \cdot \text{number empty slots in the lower half}$$

2. Repetition theory

Hashing well-done

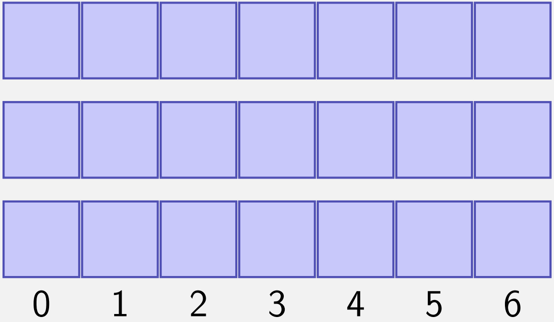
Useful Hashing. . .

- distributes the keys as uniformly as possible in the hash table.
- avoids probing over long areas of used entries (e.g. primary clustering).
- avoids using the same probing sequence for keys with the same hash value (e.g. secondary clustering).

Hashing Examples

Insert the keys $25, 4, 17, 45$ into the hash table, using the function $h(k) = k \bmod 7$ and probing to the right, $h(k) + s(j,k)$:

- linear probing,
  $s(j,k) = j$.
- quadratic probing,
  $s(j,k) = (-1)^{j+1}\lceil j/2 \rceil^2$.
- Double Hashing,
  $s(j,k) = j \cdot (1 + (k \bmod 5))$.



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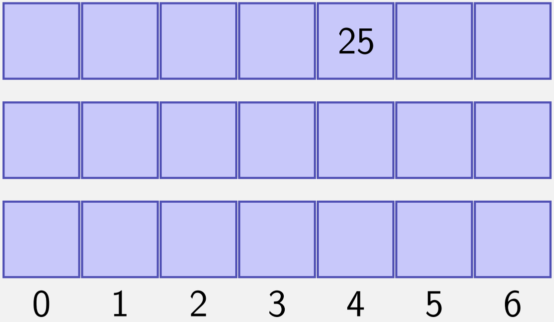

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				25	4	
0	1	2	3	4	5	6

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linear probing				17	25	4	45
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Double Hashing			4	17	25	45	

Simple Uniform Hashing

Statement about the uniform distribution and independence of the *keys*.

Property of closed addressing: simple uniform hashing $\Rightarrow$ expected length of the chains as good as possible $\leq \alpha = \frac{n}{m}$.

Uniform Hashing

Statement about the uniform distribution and independence of *key probing sequences*.

Property of open addressing: Uniform Hashing $\Rightarrow$ expected runtime costs $\leq \frac{1}{1-\alpha}$.

Universal Hashing

Property about the available, randomly chosen *hash-functions*

$$|\{h \in \mathcal{H} \text{ with } h(k_1) = h(k_2)\}| \leq \frac{|\mathcal{H}|}{m}$$

Property independent of chose sequence of keys: for hashing with chaining the expected chain length is $\leq \alpha = \frac{n}{m}$

Prerequisite for Perfect Hashing

3. Programming Task

Finding a Sub-Array

- Given: two integer arrays $A = (a_0, \ldots, a_{n-1})$ and $B = (b_0, \ldots, b_{k-1})$
- Task: Find position of B in A.

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  - $O(nk)$ comparison operations
- Solution using hashing: Calculate hash $h(B)$ and compare it to $h((a_i, a_{i+1}, \ldots, a_{i+k-1}))$.
- Avoid re-computing $h((a_i, a_{i+1}, \ldots, ai + k - 1)$ for each i $\implies O(n)$ expected

Sliding Window Hash

- Possible hash function: sum of all elements:
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Sliding Window Hash

- Possible hash function: sum of all elements:
  - Can be updated easily: subtract a_i and add a_{i+k}.
  - However: bad hash function
- Better:

$$H_{c,m}((a_i, \cdots, a_{i+k-1})) = \left(\sum_{j=0}^{k-1} a_{i+j} \cdot c^{k-j-1}\right) \bmod m$$

  - $c = 1021$ prime number
  - $m = 2^{15}$ `int`, no overflows at calculations

Computing with Modulo

$$(a+b) \bmod m = ((a \bmod m) + (b \bmod m)) \bmod m$$
$$(a-b) \bmod m = ((a \bmod m) - (b \bmod m) + m) \bmod m$$
$$(a \cdot b) \bmod m = ((a \bmod m) \cdot (b \bmod m)) \bmod m$$

Exercise: Compute

$$12746357 \bmod 11$$

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\begin{aligned}
&12746357 \bmod 11 \\
&= (7 + 5 \cdot 10 + 3 \cdot 10^2 + 6 \cdot 10^3 + 4 \cdot 10^4 + 7 \cdot 10^5 + 2 \cdot 10^6 + 1 \cdot 10^7) \bmod 11
\end{aligned}
$$

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For the second equality we used the fact that $10^2 \bmod 11 = 1$.

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&= (7 + 6 + 3 + 5 + 4 + 4 + 2 + 10) \bmod 11
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&= (7 + 50 + 3 + 60 + 4 + 70 + 2 + 10) \bmod 11 \\
&= (7 + 6 + 3 + 5 + 4 + 4 + 2 + 10) \bmod 11 \\
&= 8 \bmod 11.
\end{aligned}
$$

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Sliding Window Hash

```
template<typename It1, typename It2>
It1 findOccurrence(const It1 from, const It1 to,
                   const It2 begin, const It2 end)
{
  const unsigned k = end - begin;
  const unsigned M = 32768;
  const unsigned C = 1021;

  // your code here
  // ...
```

Sliding Window Hash

```
  // elements can be compared using std::equal:
  if(std::equal(window_left, window_right, begin, end))
       return current;

  // if no occurrence is found return end of array
  return to;
}
```

Sliding Window Hash

Make sure that

- the algorithm computes c^k only once,
- all computations are modulo m for all values in order not to get an overflow (recall the rules of modular arithmetic), and
- the values are always positive (e.g., by adding multiples of m).

Questions?