7. Sorting I

Simple Sorting

7.1 Simple Sorting

Selection Sort, Insertion Sort, Bubblesort [Ottman/Widmayer, Kap. 2.1, Cormen et al, Kap. 2.1, 2.2, Exercise 2.2-2, Problem 2-2

Problem

Input: An array $A = (A[1], ..., A[n])$ with length n.

Output: a permutation A' of A, that is sorted: $A'[i] \leq A'[j]$ for all $1 \leq i \leq j \leq n$.

Algorithm: IsSorted(A)

Input: Array $A = (A[1], ..., A[n])$ with length n.
Output: Boolean decision "sorted" or "not sorted"

for $i \leftarrow 1$ **to** $n - 1$ **do**
    if $A[i] > A[i+1]$ **then**
        return "not sorted";

return "sorted";

Observation

IsSorted(A):"not sorted", if $A[i] > A[i+1]$ for any i.

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$\Rightarrow$ idea:

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$\Rightarrow$ idea:

for $j \leftarrow 1$ **to** $n-1$ **do**
  if $A[j] > A[j+1]$ **then**
    swap$(A[j], A[j+1])$;

Give it a try

5 ⟷ 6 2 8 4 1 $(j = 1)$

Give it a try

5 ⟷ 6   2   8   4   1 $(j = 1)$

5   6 ⟷ 2   8   4   1 $(j = 2)$

Give it a try

5 ⟷ 6 2 8 4 1 $(j = 1)$

5 6 ⟷ 2 8 4 1 $(j = 2)$

5 2 6 ⟷ 8 4 1 $(j = 3)$

Give it a try

5 ↔ 6 2 8 4 1 $(j = 1)$

5 6 ↔ 2 8 4 1 $(j = 2)$

5 2 6 ↔ 8 4 1 $(j = 3)$

5 2 6 8 ↔ 4 1 $(j = 4)$

Give it a try

5 ⟷ 6 2 8 4 1 $(j = 1)$

5 6 ⟷ 2 8 4 1 $(j = 2)$

5 2 6 ⟷ 8 4 1 $(j = 3)$

5 2 6 8 ⟷ 4 1 $(j = 4)$

5 2 6 4 8 ⟷ 1 $(j = 5)$

Give it a try

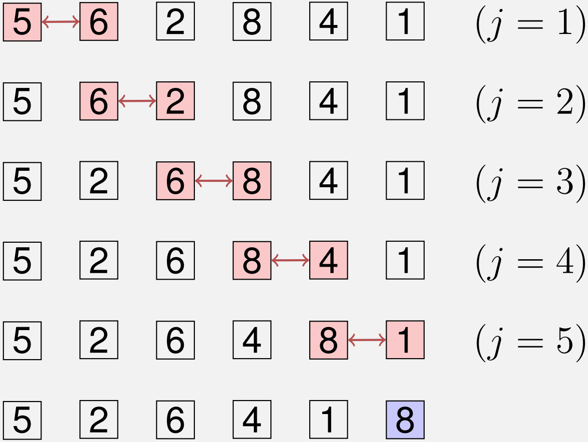
5 ⟷	**6**	2	8	4	1	$(j = 1)$
5	**6** ⟷	**2**	8	4	1	$(j = 2)$
5	2	**6** ⟷	**8**	4	1	$(j = 3)$
5	2	6	**8** ⟷	**4**	1	$(j = 4)$
5	2	6	4	**8** ⟷	**1**	$(j = 5)$
5	2	6	4	1	8	

Give it a try

5 ⟷	6	2	8	4	1	$(j = 1)$
5	6 ⟷	2	8	4	1	$(j = 2)$
5	2	6 ⟷	8	4	1	$(j = 3)$
5	2	6	8 ⟷	4	1	$(j = 4)$
5	2	6	4	8 ⟷	1	$(j = 5)$
5	2	6	4	1	8	

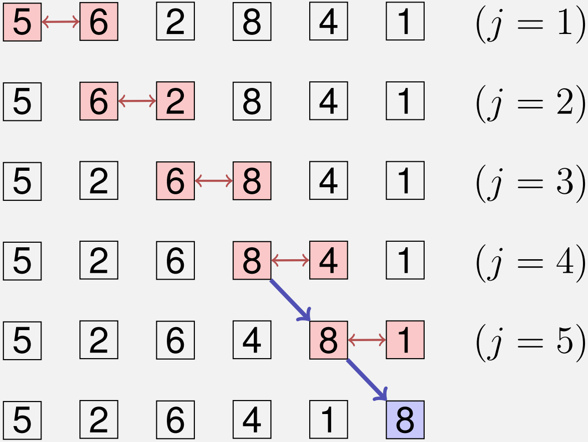
- Not sorted! ☹.

Give it a try



- Not sorted! ☹.

Give it a try



- Not sorted! ☹.
- But the greatest element moves to the right

  $\Rightarrow$ new idea! ☺

Try it out

5	**6**	2	8	4	1	$(j = 1, i = 1)$
5	**6**	**2**	8	4	1	$(j = 2)$
5	2	**6**	**8**	4	1	$(j = 3)$
5	2	6	**8**	**4**	1	$(j = 4)$
5	2	6	4	**8**	**1**	$(j = 5)$

- Apply the procedure iteratively.

Try it out

5	6	2	8	4	1	$(j = 1, i = 1)$
5	6	2	8	4	1	$(j = 2)$
5	2	6	8	4	1	$(j = 3)$
5	2	6	8	4	1	$(j = 4)$
5	2	6	4	8	1	$(j = 5)$
5	2	6	4	1	8	$(j = 1, i = 2)$

- Apply the procedure iteratively.
- For $A[1, \ldots, n]$,

Try it out

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5	6	2	8	4	1	$(j = 2)$
5	2	6	8	4	1	$(j = 3)$
5	2	6	8	4	1	$(j = 4)$
5	2	6	4	8	1	$(j = 5)$
5	2	6	4	1	8	$(j = 1, i = 2)$
2	5	6	4	1	8	$(j = 2)$

- Apply the procedure iteratively.
- For $A[1, \ldots, n]$, then $A[1, \ldots, n-1]$,

Try it out

5	6	2	8	4	1	$(j = 1, i = 1)$
5	6	2	8	4	1	$(j = 2)$
5	2	6	8	4	1	$(j = 3)$
5	2	6	8	4	1	$(j = 4)$
5	2	6	4	8	1	$(j = 5)$
5	2	6	4	1	8	$(j = 1, i = 2)$
2	5	6	4	1	8	$(j = 2)$
2	5	6	4	1	8	$(j = 3)$

- Apply the procedure iteratively.
- For $A[1, \ldots, n]$, then $A[1, \ldots, n-1]$,

Try it out

5	6	2	8	4	1	$(j = 1, i = 1)$
5	6	2	8	4	1	$(j = 2)$
5	2	6	8	4	1	$(j = 3)$
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5	2	6	4	8	1	$(j = 5)$
5	2	6	4	1	8	$(j = 1, i = 2)$
2	5	6	4	1	8	$(j = 2)$
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2	5	4	6	1	8	$(j = 4)$

- Apply the procedure iteratively.
- For $A[1, \ldots, n]$, then $A[1, \ldots, n-1]$,

Try it out

5	6	2	8	4	1	$(j = 1, i = 1)$
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5	2	6	8	4	1	$(j = 3)$
5	2	6	8	4	1	$(j = 4)$
5	2	6	4	8	1	$(j = 5)$
5	2	6	4	1	8	$(j = 1, i = 2)$
2	5	6	4	1	8	$(j = 2)$
2	5	6	4	1	8	$(j = 3)$
2	5	4	6	1	8	$(j = 4)$
2	5	4	1	6	8	$(j = 1, i = 3)$

- Apply the procedure iteratively.
- For $A[1, \ldots, n]$, then $A[1, \ldots, n-1]$, then $A[1, \ldots, n-2]$,

Try it out

5	6	2	8	4	1	$(j = 1, i = 1)$
5	6	2	8	4	1	$(j = 2)$
5	2	6	8	4	1	$(j = 3)$
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5	2	6	4	1	8	$(j = 1, i = 2)$
2	5	6	4	1	8	$(j = 2)$
2	5	6	4	1	8	$(j = 3)$
2	5	4	6	1	8	$(j = 4)$
2	5	4	1	6	8	$(j = 1, i = 3)$
2	5	4	1	6	8	$(j = 2)$
2	4	5	1	6	8	$(j = 3)$
2	4	1	5	6	8	$(j = 1, i = 4)$

- Apply the procedure iteratively.
- For $A[1, \ldots, n]$, then $A[1, \ldots, n-1]$, then $A[1, \ldots, n-2]$,

Try it out

5	6	2	8	4	1	$(j = 1, i = 1)$
5	6	2	8	4	1	$(j = 2)$
5	2	6	8	4	1	$(j = 3)$
5	2	6	8	4	1	$(j = 4)$
5	2	6	4	8	1	$(j = 5)$
5	2	6	4	1	8	$(j = 1, i = 2)$
2	5	6	4	1	8	$(j = 2)$
2	5	6	4	1	8	$(j = 3)$
2	5	4	6	1	8	$(j = 4)$
2	5	4	1	6	8	$(j = 1, i = 3)$
2	5	4	1	6	8	$(j = 2)$
2	4	5	1	6	8	$(j = 3)$
2	4	1	5	6	8	$(j = 1, i = 4)$
2	4	1	5	6	8	$(j = 2)$
2	1	4	5	6	8	$(i = 1, j = 5)$
1	2	4	5	6	8	

- Apply the procedure iteratively.
- For $A[1, \ldots, n]$, then $A[1, \ldots, n-1]$, then $A[1, \ldots, n-2]$, etc.

Algorithm: Bubblesort

Input: Array $A = (A[1], \dots, A[n])$, $n \geq 0$.
Output: Sorted Array A

```
for i ← 1 to n − 1 do
    for j ← 1 to n − i do
        if A[j] > A[j + 1] then
            swap(A[j], A[j + 1]);
```

Analysis

Number key comparisons $\sum_{i=1}^{n-1}(n-i) = \frac{n(n-1)}{2} = \Theta(n^2)$.

Number swaps in the worst case: $\Theta(n^2)$

? What is the worst case?

Analysis

Number key comparisons $\sum_{i=1}^{n-1}(n-i) = \frac{n(n-1)}{2} = \Theta(n^2)$.

Number swaps in the worst case: $\Theta(n^2)$

? What is the worst case?

! If A is sorted in decreasing order.

Selection Sort

5 6 2 8 4 1 $(i = 1)$

- Selection of the smallest element by search in the unsorted part $A[i..n]$ of the array.

Selection Sort

5 6 2 8 4 1 $(i = 1)$

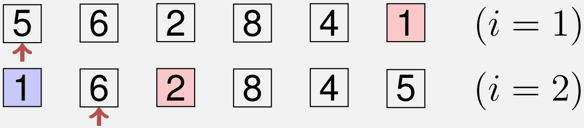
- Selection of the smallest element by search in the unsorted part $A[i..n]$ of the array.
- Swap the smallest element with the first element of the unsorted part.

Selection Sort

5	6	2	8	4	1	$(i = 1)$
1	6	2	8	4	5	$(i = 2)$

- Selection of the smallest element by search in the unsorted part $A[i..n]$ of the array.
- Swap the smallest element with the first element of the unsorted part.
- Unsorted part decreases in size by one element ($i \to i + 1$). Repeat until all is sorted. ($i = n$)

Selection Sort



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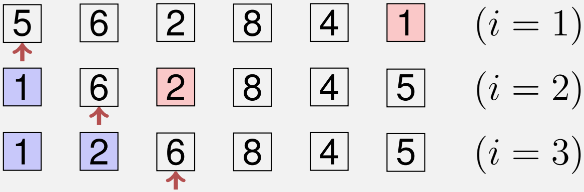

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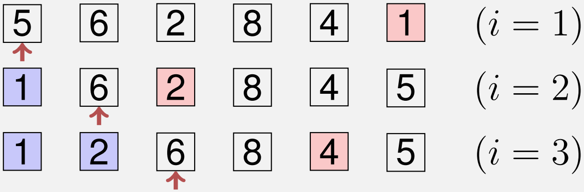

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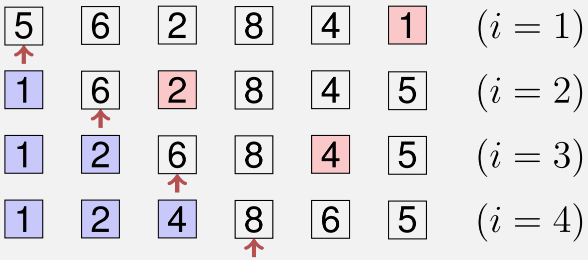

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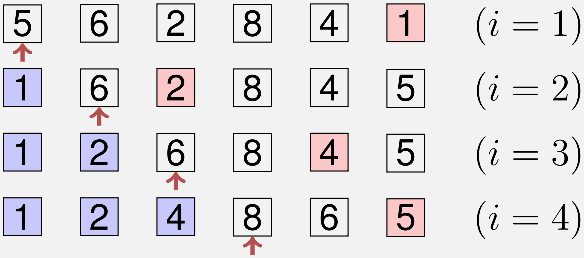

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Selection Sort

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1	6	2	8	4	5	$(i = 2)$
1	2	6	8	4	5	$(i = 3)$
1	2	4	8	6	5	$(i = 4)$
1	2	4	5	6	8	$(i = 5)$

- Selection of the smallest element by search in the unsorted part $A[i..n]$ of the array.
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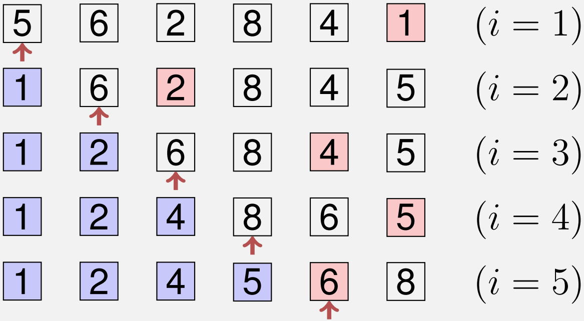

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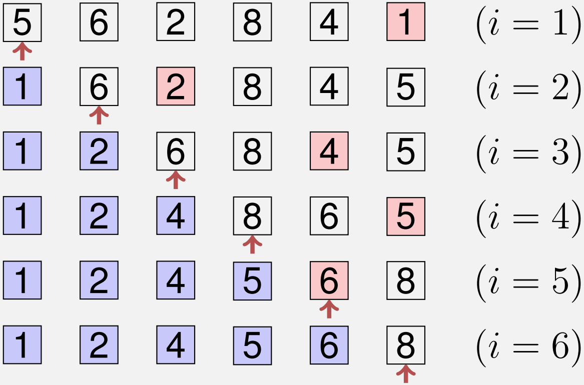
Selection Sort



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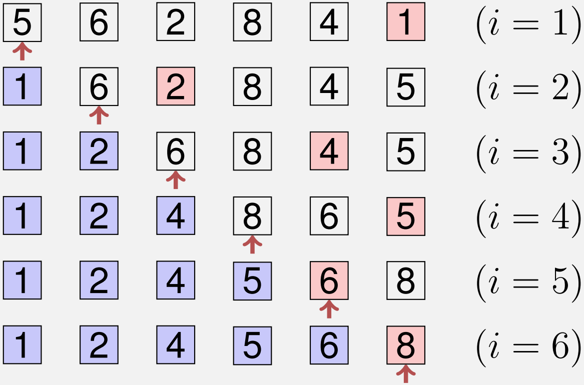
Selection Sort



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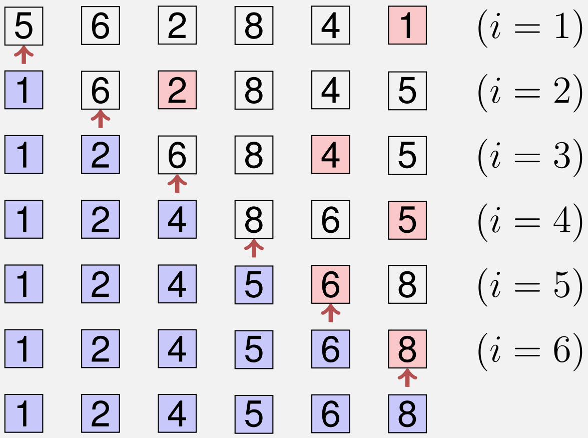

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- Swap the smallest element with the first element of the unsorted part.
- Unsorted part decreases in size by one element ($i \to i+1$). Repeat until all is sorted. ($i = n$)

Algorithm: Selection Sort

Input: Array $A = (A[1], \dots, A[n])$, $n \geq 0$.
Output: Sorted Array A

for $i \leftarrow 1$ **to** $n - 1$ **do**
    $p \leftarrow i$
    for $j \leftarrow i + 1$ **to** n **do**
        if $A[j] < A[p]$ **then**
            $p \leftarrow j$;
    swap$(A[i], A[p])$

Analysis

Number comparisons in worst case:

Analysis

Number comparisons in worst case: $\Theta(n^2)$.

Number swaps in the worst case:

Analysis

Number comparisons in worst case: $\Theta(n^2)$.

Number swaps in the worst case: $n - 1 = \Theta(n)$

Insertion Sort

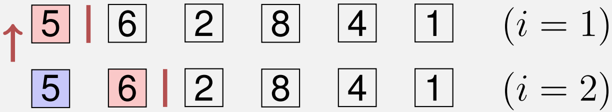
5 | 6 2 8 4 1 $(i = 1)$

Insertion Sort

↑ 5 | 6 2 8 4 1 $(i = 1)$

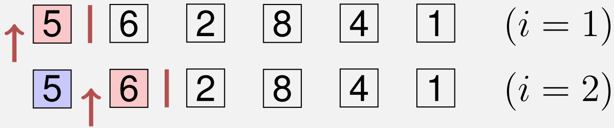
- Iterative procedure:
  $i = 1 ... n$

Insertion Sort



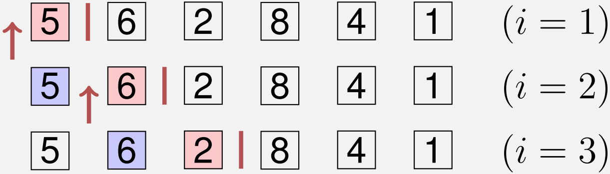
- Iterative procedure: $i = 1...n$
- Determine insertion position for element i.

Insertion Sort



- Iterative procedure: $i = 1...n$
- Determine insertion position for element i.
- Insert element i

Insertion Sort



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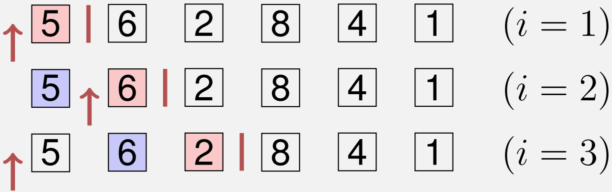
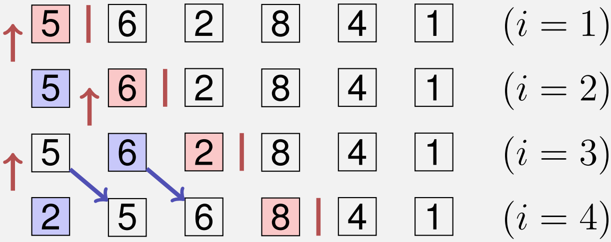


- Iterative procedure: $i = 1 \ldots n$
- Determine insertion position for element i.
- Insert element i

Insertion Sort



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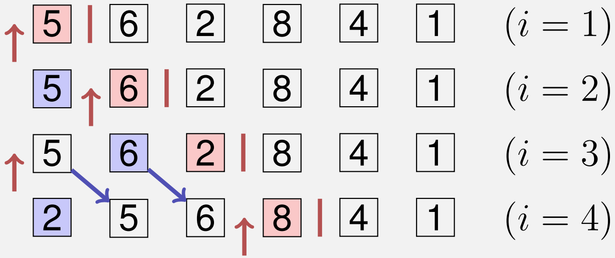
- Determine insertion position for element i.
- Insert element i array block movement potentially required

Insertion Sort



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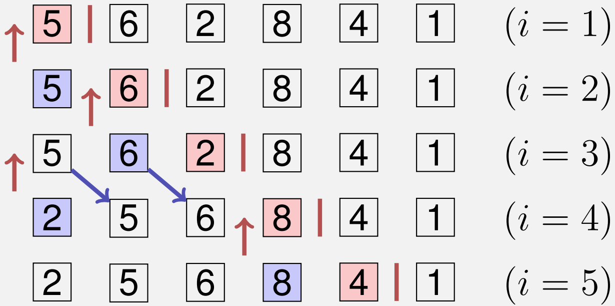

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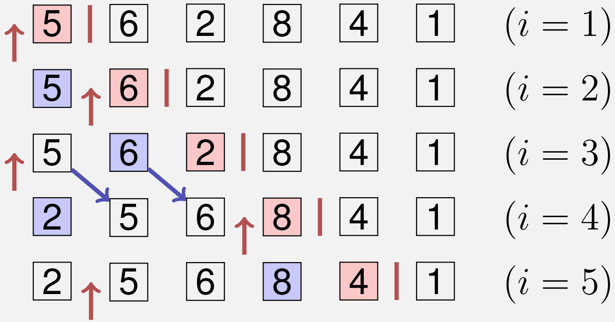

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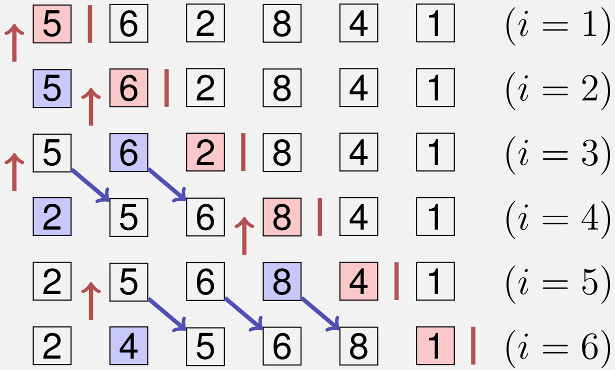

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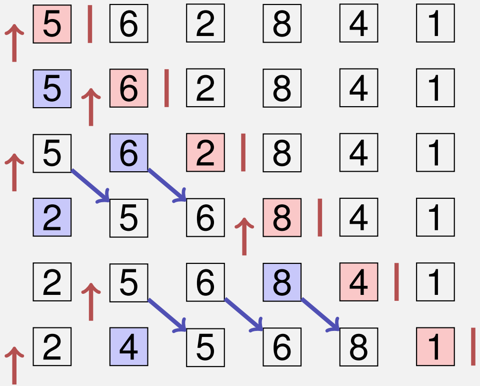
Insertion Sort



$(i = 1)$

$(i = 2)$

$(i = 3)$

$(i = 4)$

$(i = 5)$

$(i = 6)$

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- Determine insertion position for element i.
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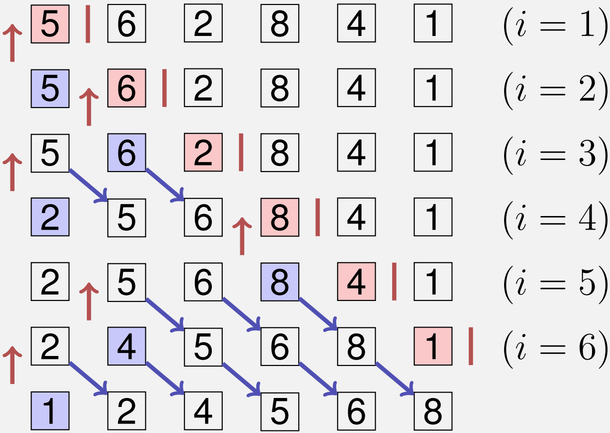
Insertion Sort



- Iterative procedure: $i = 1...n$
- Determine insertion position for element i.
- Insert element i array block movement potentially required

Insertion Sort

? What is the disadvantage of this algorithm compared to sorting by selection?

Insertion Sort

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! Many element movements in the worst case.

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Insertion Sort

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! Many element movements in the worst case.

? What is the advantage of this algorithm compared to selection sort?

! The search domain (insertion interval) is already sorted. Consequently: binary search possible.

Algorithm: Insertion Sort

Input: Array $A = (A[1], \ldots, A[n])$, $n \geq 0$.
Output: Sorted Array A

```
for i ← 2 to n do
    x ← A[i]
    p ← BinarySearch(A[1...i − 1], x); // Smallest p ∈ [1, i] with A[p] ≥ x
    for j ← i − 1 downto p do
        A[j + 1] ← A[j]
    A[p] ← x
```

Analysis

Number comparisons in the worst case:

Analysis

Number comparisons in the worst case:
$\sum_{k=1}^{n-1} a \cdot \log k = a \log((n-1)!) \in \mathcal{O}(n \log n)$.

Number swaps in the worst case

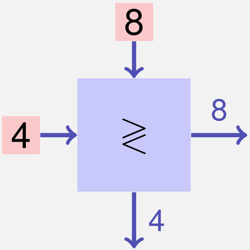
Analysis

Number comparisons in the worst case:
$\sum_{k=1}^{n-1} a \cdot \log k = a \log((n-1)!) \in \mathcal{O}(n \log n)$.

Number swaps in the worst case $\sum_{k=2}^{n} (k-1) \in \Theta(n^2)$

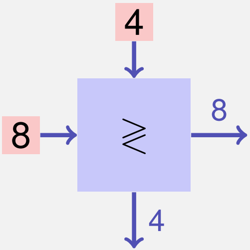
Different point of view

Sorting node:



Different point of view

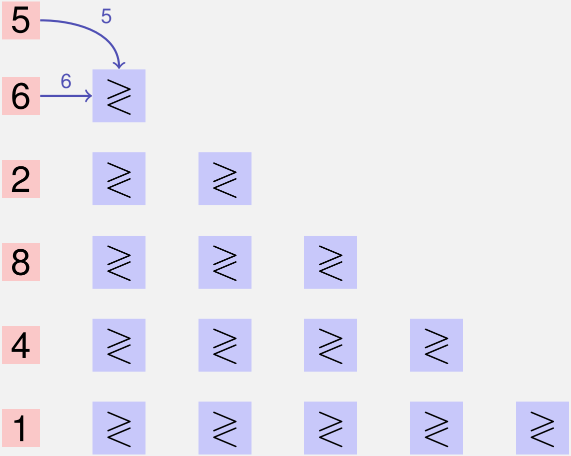
Sorting node:



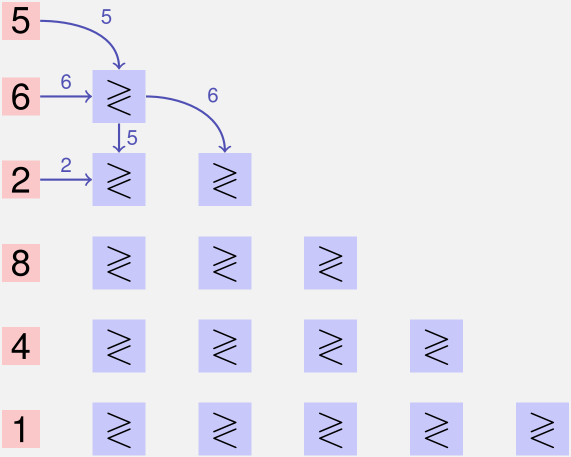
Different point of view

5					
6	$\gtrless$				
2	$\gtrless$	$\gtrless$			
8	$\gtrless$	$\gtrless$	$\gtrless$		
4	$\gtrless$	$\gtrless$	$\gtrless$	$\gtrless$	
1	$\gtrless$	$\gtrless$	$\gtrless$	$\gtrless$	$\gtrless$

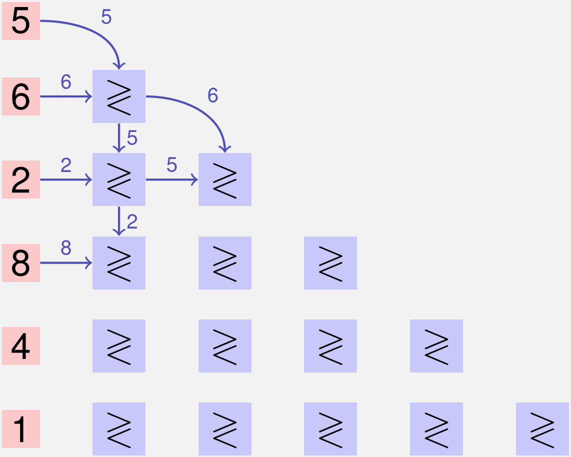
Different point of view



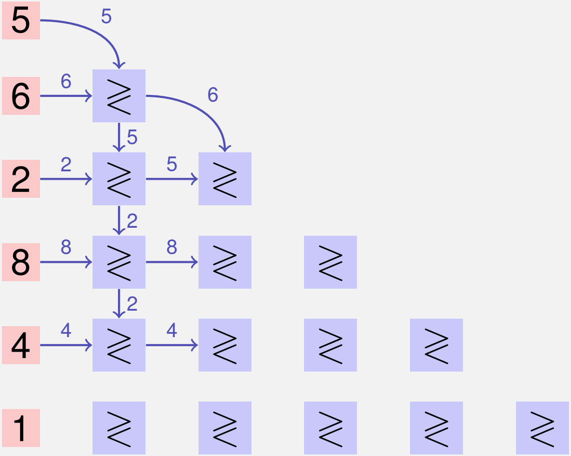
Different point of view



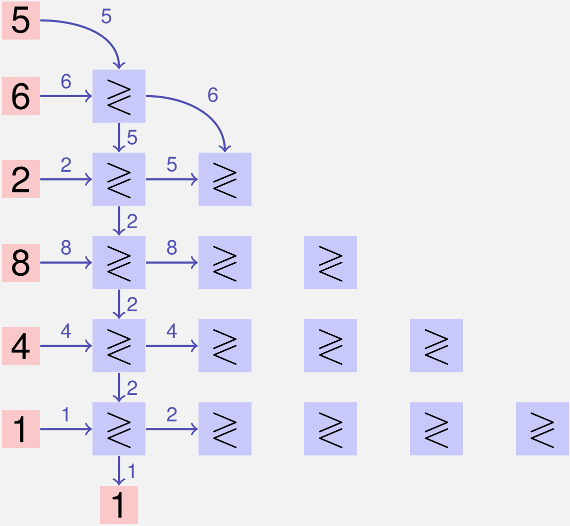
Different point of view



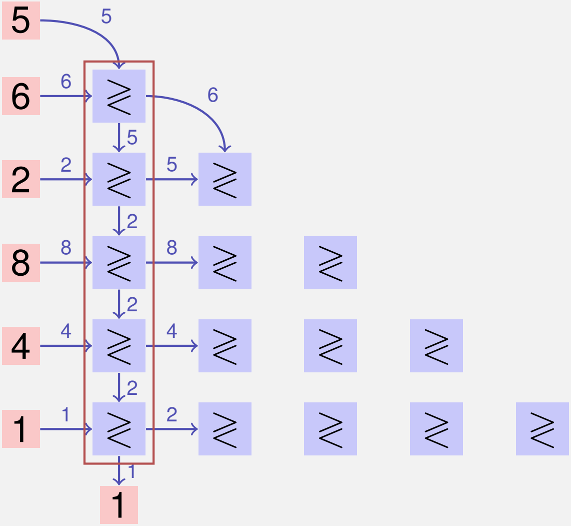
Different point of view



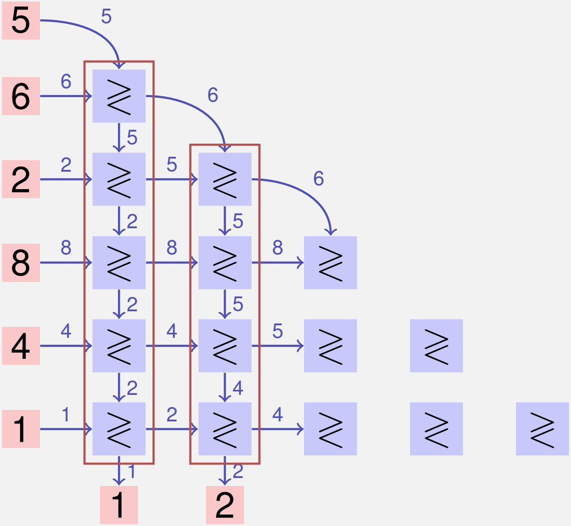
Different point of view



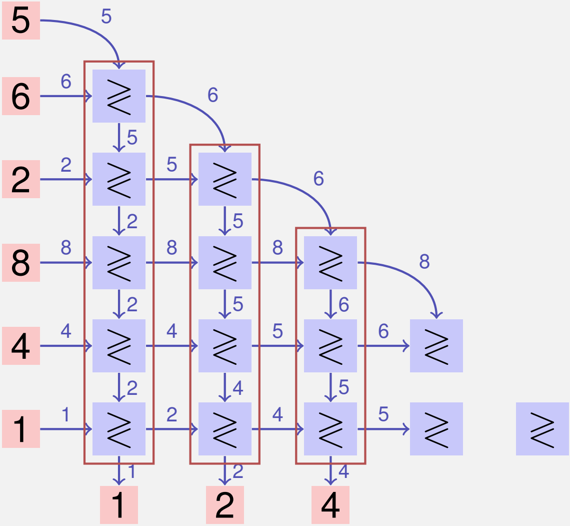
Different point of view



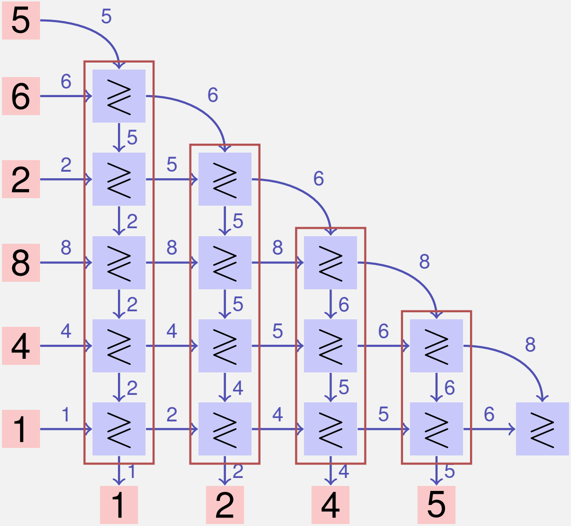
Different point of view



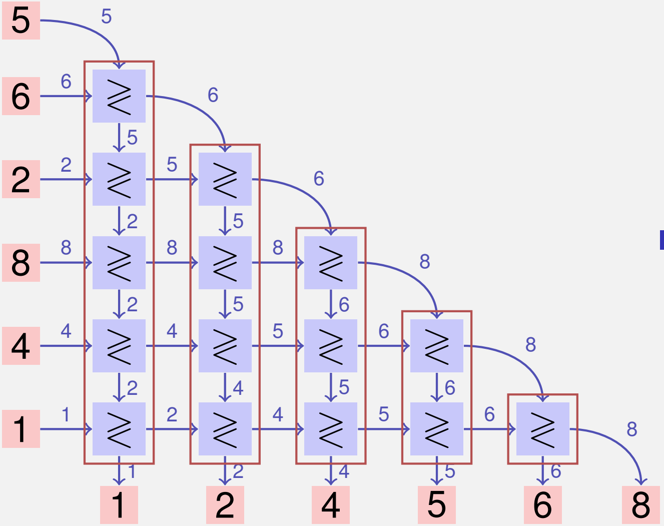
Different point of view



Different point of view

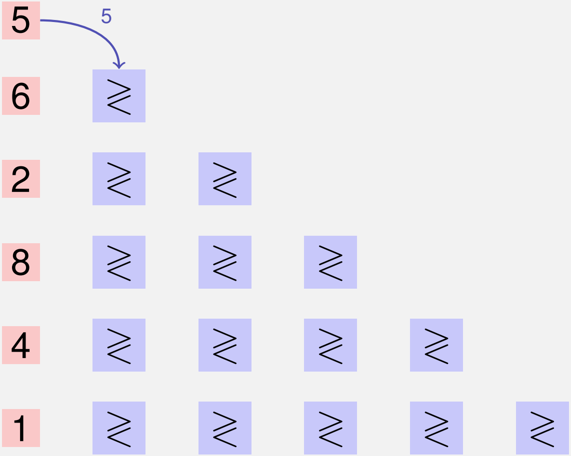


Different point of view

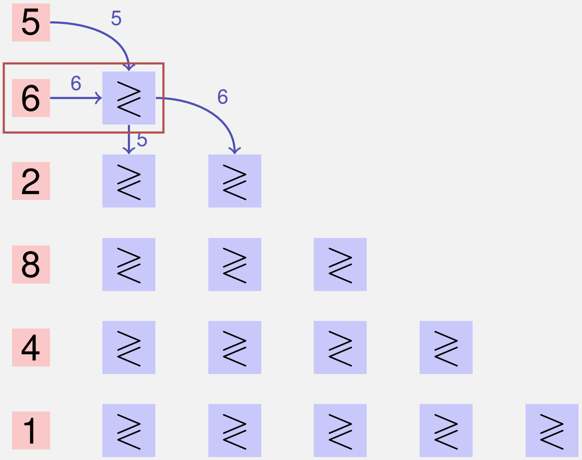


- Like selection sort
  [and like Bubblesort]

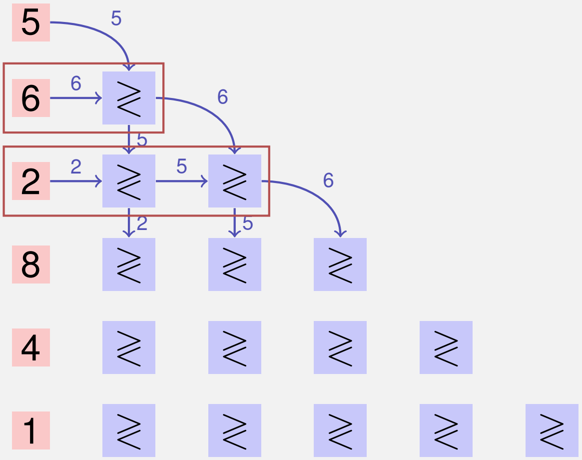
Different point of view



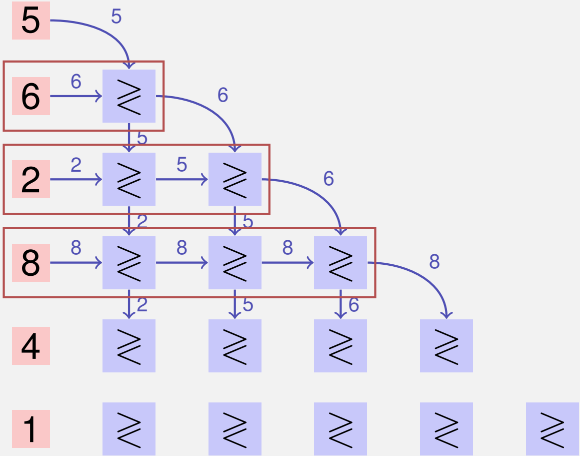
Different point of view



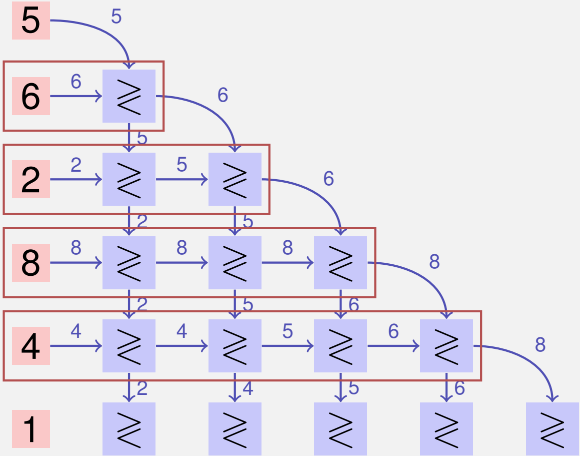
Different point of view



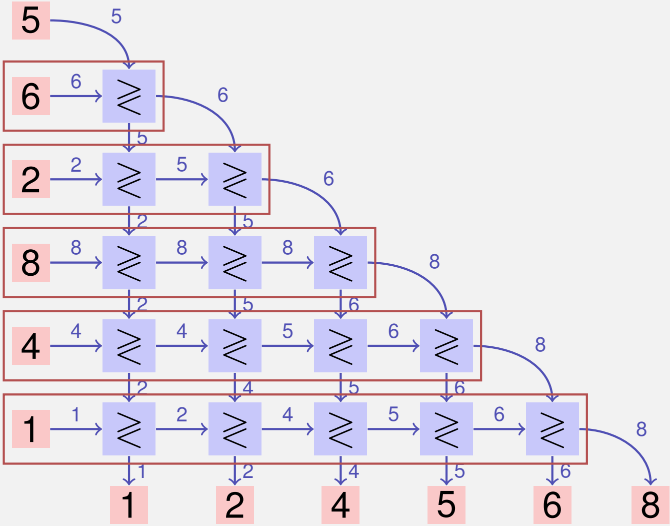
Different point of view



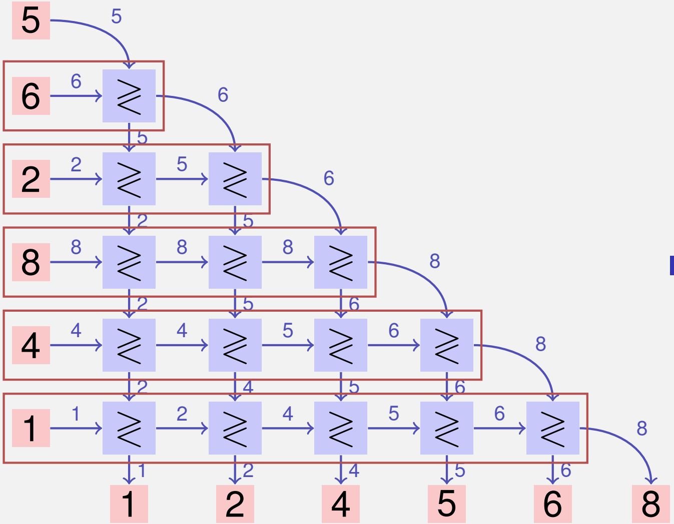
Different point of view



Different point of view



Different point of view



- Like insertion sort

Conclusion

In a certain sense, Selection Sort, Bubble Sort and Insertion Sort provide the same kind of sort strategy. Will be made more precise. [6]

[6] In the part about parallel sorting networks. For the sequential code of course the observations as described above still hold.

Shellsort (Donald Shell 1959)

Insertion sort on subsequences of the form $(A_{k\cdot i})$ ($i \in \mathbb{N}$) with decreasing distances k. Last considered distance must be $k = 1$.

Worst-case performance critically depends on the chosen subsequences

- Original concept with sequence $1, 2, 4, 8, ..., 2^k$. Running time: $\mathcal{O}(n^2)$
- Sequence $1, 3, 7, 15, ..., 2^{k-1}$ (Hibbard 1963). $\mathcal{O}(n^{3/2})$
- Sequence $1, 2, 3, 4, 6, 8, ..., 2^p 3^q$ (Pratt 1971). $\mathcal{O}(n \log^2 n)$

Shellsort

9 8 7 6 5 4 3 2 1 0

Shellsort

9	8	7	6	5	4	3	2	1	0	
1	8	7	6	5	4	3	2	9	0	insertion sort, $k = 4$

Shellsort

9	8	7	6	5	4	3	2	1	0	
1	8	7	6	5	4	3	2	9	0	insertion sort, $k = 4$
1	0	7	6	5	4	3	2	9	8	

Shellsort

9	8	7	6	5	4	3	2	1	0	
1	8	7	6	5	4	3	2	9	0	insertion sort, $k = 4$
1	0	7	6	5	4	3	2	9	8	
1	0	3	6	5	4	7	2	9	8	

Shellsort

9	8	7	6	5	4	3	2	1	0	
1	8	7	6	5	4	3	2	9	0	insertion sort, $k = 4$
1	0	7	6	5	4	3	2	9	8	
1	0	3	6	5	4	7	2	9	8	
1	0	3	2	5	4	7	6	9	8	

Shellsort

9	8	7	6	5	4	3	2	1	0	
1	8	7	6	5	4	3	2	9	0	insertion sort, $k = 4$
1	0	7	6	5	4	3	2	9	8	
1	0	3	6	5	4	7	2	9	8	
1	0	3	2	5	4	7	6	9	8	
1	0	3	2	5	4	7	6	9	8	insertion sort, $k = 2$

Shellsort

9	8	7	6	5	4	3	2	1	0	
1	8	7	6	5	4	3	2	9	0	insertion sort, $k = 4$
1	0	7	6	5	4	3	2	9	8	
1	0	3	6	5	4	7	2	9	8	
1	0	3	2	5	4	7	6	9	8	
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1	0	3	2	5	4	7	6	9	8	

Shellsort

9	8	7	6	5	4	3	2	1	0	
1	8	7	6	**5**	4	3	2	**9**	0	insertion sort, $k = 4$
1	**0**	7	6	5	**4**	3	2	9	**8**	
1	0	**3**	6	5	4	**7**	2	9	8	
1	0	3	**2**	5	4	7	**6**	9	8	
1	0	**3**	2	**5**	4	**7**	6	**9**	8	insertion sort, $k = 2$
1	**0**	3	**2**	5	**4**	7	**6**	9	**8**	
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	insertion sort, $k = 1$

8. Sorting II

Heapsort, Quicksort, Mergesort

8.1 Heapsort

[Ottman/Widmayer, Kap. 2.3, Cormen et al, Kap. 6]

Heapsort

Inspiration from selectsort: fast insertion

Inspiration from insertion sort: fast determination of position

? Can we have the best of both worlds?

Heapsort

Inspiration from selectsort: fast insertion

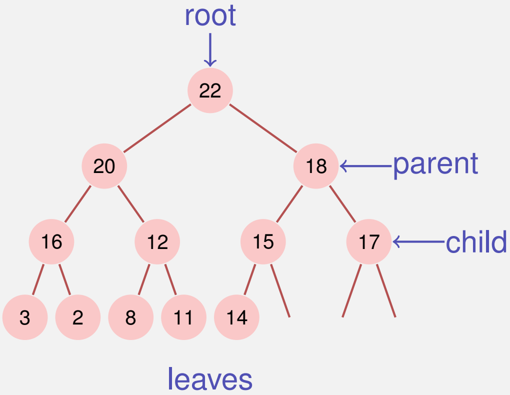
Inspiration from insertion sort: fast determination of position

? Can we have the best of both worlds?

! Yes, but it requires some more thinking...

[Max-]Heap[7]

Binary tree with the following properties

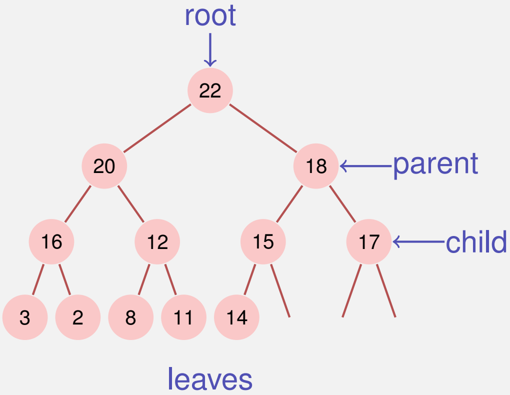


[7] Heap(data structure), not: as in "heap and stack" (memory allocation)

[Max-]Heap[7]

Binary tree with the following properties

1. complete up to the lowest level



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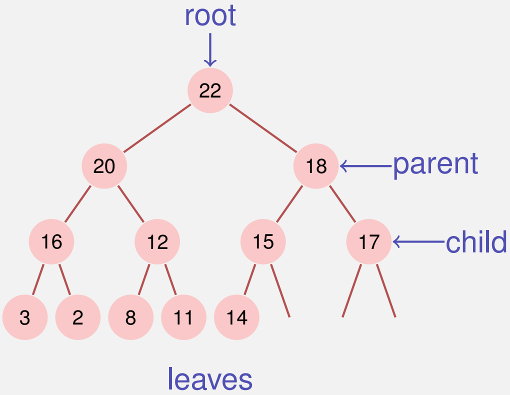
[Max-]Heap[7]

Binary tree with the following properties

1. complete up to the lowest level
2. Gaps (if any) of the tree in the last level to the right

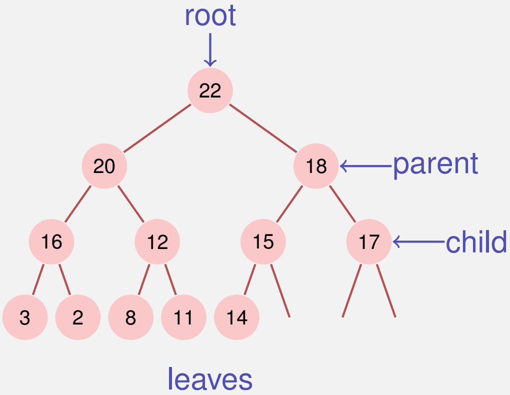


[7] Heap(data structure), not: as in “heap and stack” (memory allocation)

[Max-]Heap[7]

Binary tree with the following properties

1. complete up to the lowest level
2. Gaps (if any) of the tree in the last level to the right
3. *Heap-Condition:* Max-(Min-)Heap: key of a child smaller (greater) that that of the parent node

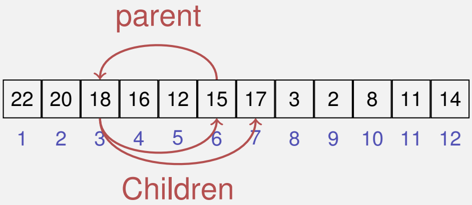


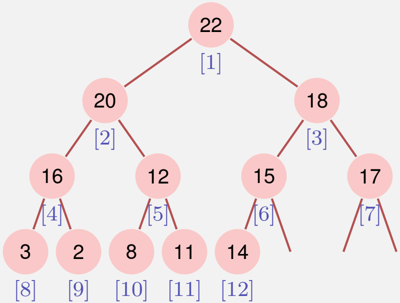
[7] Heap(data structure), not: as in “heap and stack” (memory allocation)

Heap as Array

Tree $\rightarrow$ Array:

- children$(i) = \{2i, 2i+1\}$
- parent$(i) = \lfloor i/2 \rfloor$

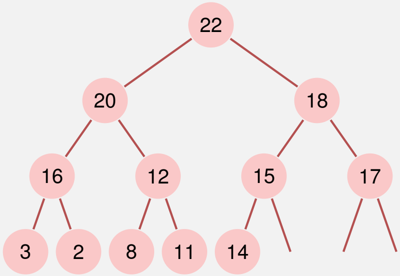




Depends on the starting index[8]

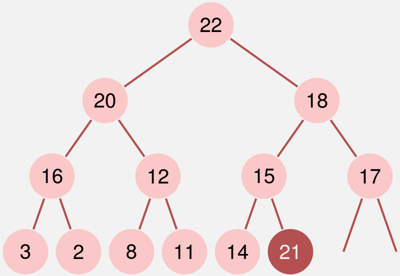
[8] For array that start at 0: $\{2i, 2i+1\} \rightarrow \{2i+1, 2i+2\}$, $\lfloor i/2 \rfloor \rightarrow \lfloor (i-1)/2 \rfloor$

Insert



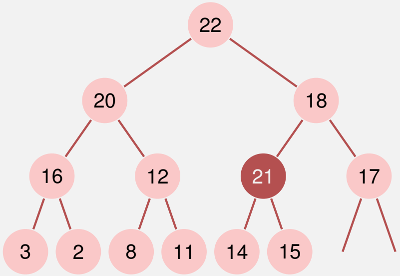
Insert

- Insert new element at the first free position. Potentially violates the heap property.



Insert

- Insert new element at the first free position. Potentially violates the heap property.
- Reestablish heap property: climb successively



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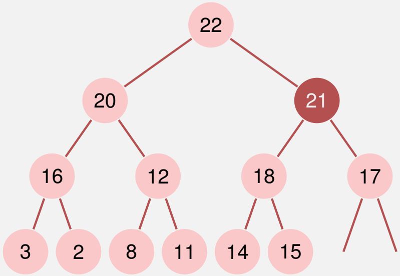
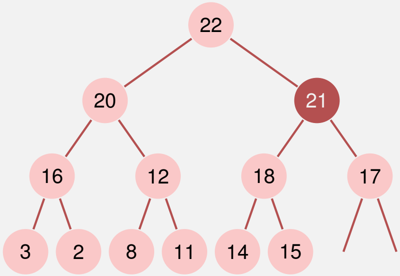


Insert

- Insert new element at the first free position. Potentially violates the heap property.
- Reestablish heap property: climb successively
- Worst case number of operations: $\mathcal{O}(\log n)$



Algorithm Sift-Up(A, m)

Input: Array A with at least $m+1$ and Max-Heap-Structure on $A[0,\dots,m-1]$

Output: Array A with Max-Heap-Structure on $A[0,\dots,m]$.

$v \leftarrow A[m]$ // value
$c \leftarrow m$ // current position
$p \leftarrow \lfloor (c-1)/2 \rfloor$ // parent node
while $c > 0$ and $v > A[p]$ **do**
    $A[c] \leftarrow A[p]$ // Value parent node $\rightarrow$ current node
    $c \leftarrow p$ // parent node $\rightarrow$ current node
    $p \leftarrow \lfloor (c-1)/2 \rfloor$
$A[c] \leftarrow v$ // value $\rightarrow$ current node

Height of a Heap

A complete binary tree with height[9] h provides

$$1 + 2 + 4 + 8 + ... + 2^{h-1} = \sum_{i=0}^{h-1} 2^i = 2^h - 1$$

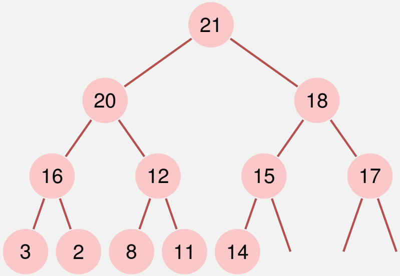
nodes. Thus for a heap with height h:

$$\begin{aligned} 2^{h-1} - 1 &< n \leq 2^h - 1 \\ \Leftrightarrow \quad 2^{h-1} &< n + 1 \leq 2^h \end{aligned}$$

Particularly $h(n) = \lceil \log_2(n+1) \rceil$ and $h(n) \in \Theta(\log n)$.

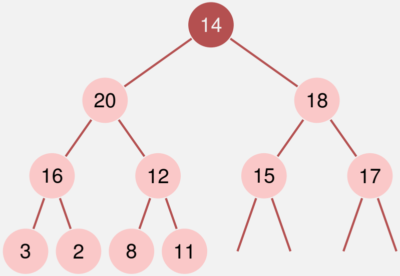
[9] here: number of edges from the root to a leaf

Remove the maximum



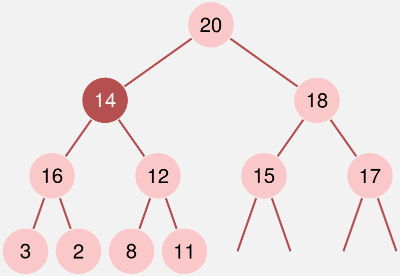
Remove the maximum

- Replace the maximum by the lower right element



Remove the maximum

- Replace the maximum by the lower right element
- Reestablish heap property: sink successively (in the direction of the greater child)



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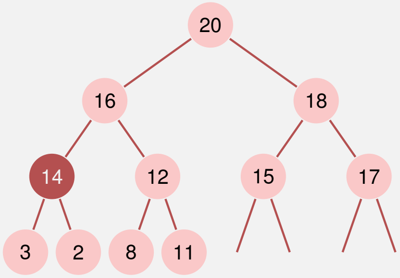

Remove the maximum

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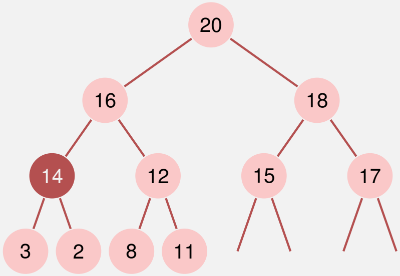
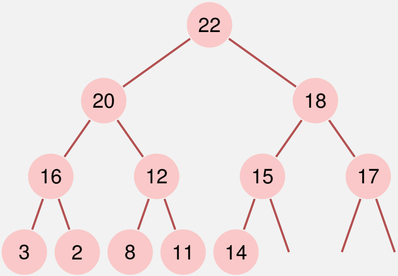
- Worst case number of operations: $\mathcal{O}(\log n)$



Why this is correct: Recursive heap structure

A heap consists of two heaps:



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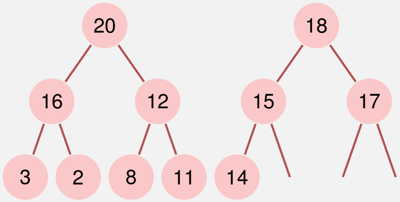


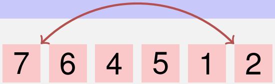
Algorithm SiftDown(A, i, m)

Input: Array A with heap structure for the children of i. Last element m.

Output: Array A with heap structure for i with last element m.

```
while 2i ≤ m do
    j ← 2i; // j left child
    if j < m and A[j] < A[j+1] then
        j ← j + 1; // j right child with greater key
    if A[i] < A[j] then
        swap(A[i], A[j])
        i ← j; // keep sinking down
    else
        i ← m; // sift down finished
```

Sort heap



$A[1, ..., n]$ is a Heap.
While $n > 1$

- swap($A[1]$, $A[n]$)
- SiftDown($A, 1, n-1$);
- $n \leftarrow n - 1$

Sort heap

	7	6	4	5	1	2
swap $\Rightarrow$	2	6	4	5	1	7

$A[1, ..., n]$ is a Heap.
While $n > 1$

- swap($A[1]$, $A[n]$)
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Sort heap

	7	6	4	5	1	2
swap $\Rightarrow$	2	6	4	5	1	7
siftDown $\Rightarrow$	6	5	4	2	1	7

$A[1, \dots, n]$ is a Heap.
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- swap($A[1]$, $A[n]$)
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Sort heap

$A[1, ..., n]$ is a Heap.
While $n > 1$

- swap($A[1]$, $A[n]$)
- SiftDown($A, 1, n-1$);
- $n \leftarrow n - 1$

		7	6	4	5	1	2
swap	$\Rightarrow$	2	6	4	5	1	7
siftDown	$\Rightarrow$	6	5	4	2	1	7
swap	$\Rightarrow$	1	5	4	2	6	7

Sort heap

$A[1, ..., n]$ is a Heap.
While $n > 1$

- swap($A[1]$, $A[n]$)
- SiftDown($A, 1, n-1$);
- $n \leftarrow n - 1$

		7	6	4	5	1	2
swap	$\Rightarrow$	2	6	4	5	1	7
siftDown	$\Rightarrow$	6	5	4	2	1	7
swap	$\Rightarrow$	1	5	4	2	6	7
siftDown	$\Rightarrow$	5	4	2	1	6	7
swap	$\Rightarrow$	1	4	2	5	6	7
siftDown	$\Rightarrow$	4	1	2	5	6	7
swap	$\Rightarrow$	2	1	4	5	6	7
siftDown	$\Rightarrow$	2	1	4	5	6	7
swap	$\Rightarrow$	1	2	4	5	6	7

Heap creation

Observation: Every leaf of a heap is trivially a correct heap.

Consequence:

Heap creation

Observation: Every leaf of a heap is trivially a correct heap.

Consequence: Induction from below!

Algorithm HeapSort(A, n)

Input: Array A with length n.
Output: A sorted.

```
// Build the heap.
for i ← n/2 downto 1 do
    SiftDown(A, i, n);
// Now A is a heap.
for i ← n downto 2 do
    swap(A[1], A[i])
    SiftDown(A, 1, i − 1)
// Now A is sorted.
```

Analysis: sorting a heap

SiftDown traverses at most $\log n$ nodes. For each node 2 key comparisons. $\Rightarrow$ sorting a heap costs in the worst case $2 \log n$ comparisons.

Number of memory movements of sorting a heap also $\mathcal{O}(n \log n)$.

Analysis: creating a heap

Calls to siftDown: $n/2$. Thus number of comparisons and movements: $v(n) \in \mathcal{O}(n \log n)$.

---

[10] $f(x) = \frac{1}{1-x} = 1 + x + x^2 \ldots \Rightarrow f'(x) = \frac{1}{(1-x)^2} = 1 + 2x + \ldots$

Analysis: creating a heap

Calls to siftDown: $n/2$. Thus number of comparisons and movements: $v(n) \in \mathcal{O}(n \log n)$.

But mean length of the sift-down paths is much smaller:

$$v(n) = \sum_{l=0}^{\lfloor \log n \rfloor} \underbrace{2^l}_{\text{number heaps on level l}} \cdot \underbrace{(\lfloor \log n \rfloor - l)}_{\text{height heaps on level l}} = \sum_{k=0}^{\lfloor \log n \rfloor} 2^{\lfloor \log n \rfloor - k} \cdot k$$

$$\leq \sum_{k=0}^{\lfloor \log n \rfloor} \frac{n}{2^k} \cdot k = n \cdot \sum_{k=0}^{\lfloor \log n \rfloor} \frac{k}{2^k} \in \mathcal{O}(\mathbf{n})$$

with $s(x) := \sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2} \quad (0 < x < 1)$ [10] and $s(\frac{1}{2}) = 2$

[10] $f(x) = \frac{1}{1-x} = 1 + x + x^2 ... \Rightarrow f'(x) = \frac{1}{(1-x)^2} = 1 + 2x + ...$

Intermediate result

Heapsort: $\mathcal{O}(n \log n)$ Comparisons and movements.

? Disadvantages of heapsort?

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- Missing locality: heapsort jumps around in the sorted array (negative cache effect).

Intermediate result

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Disadvantages of heapsort?

- Missing locality: heapsort jumps around in the sorted array (negative cache effect).
- Two comparisons required before each necessary memory movement.

8.2 Mergesort

[Ottman/Widmayer, Kap. 2.4, Cormen et al, Kap. 2.3],

Mergesort

Divide and Conquer!

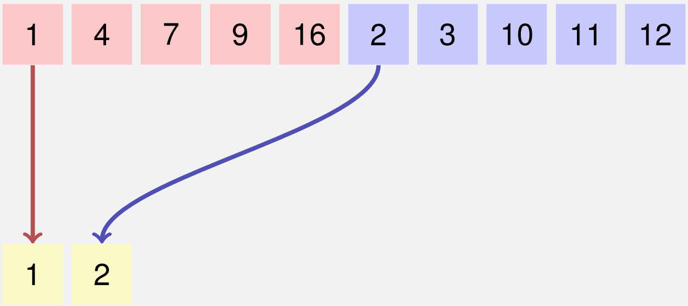
- Assumption: two halves of the array A are already sorted.
- Minimum of A can be evaluated with two comparisons.
- Iteratively: merge the two presorted halves of A in $\mathcal{O}(n)$.

Merge

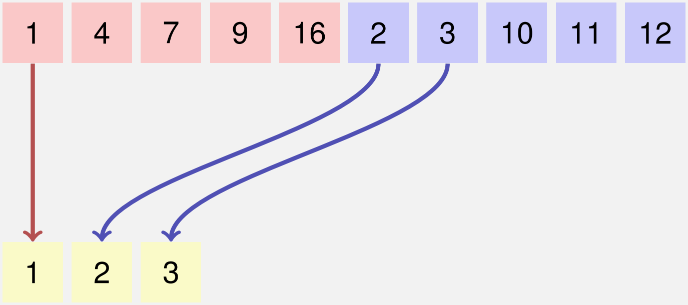
1	4	7	9	16	2	3	10	11	12

Merge

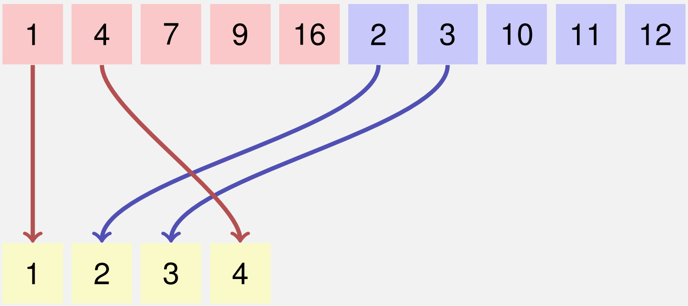
Merge



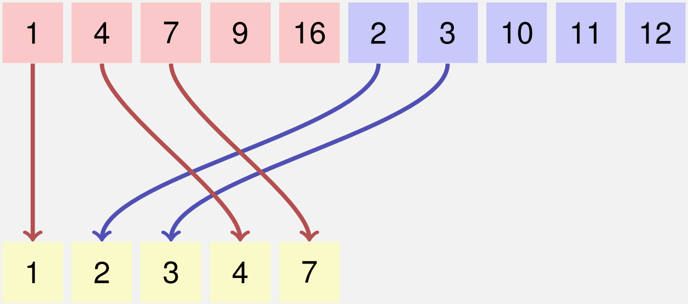
Merge



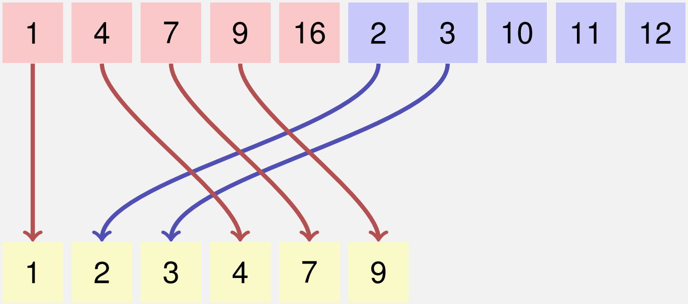
Merge



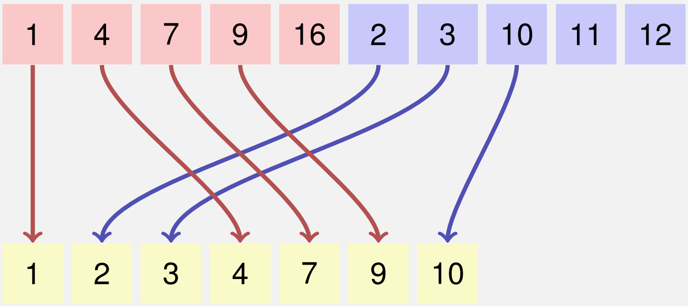
Merge



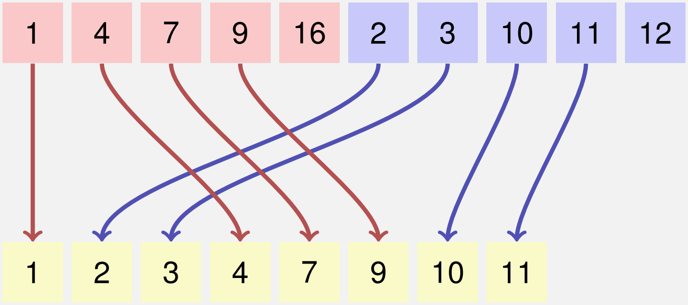
Merge



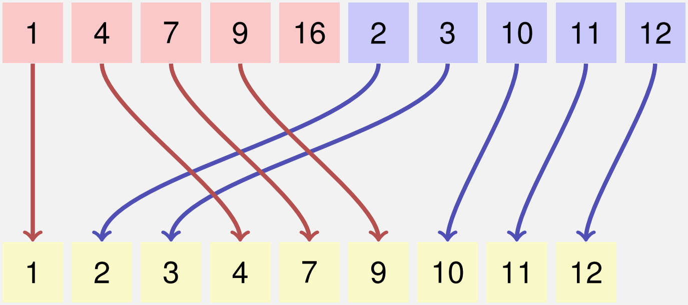
Merge



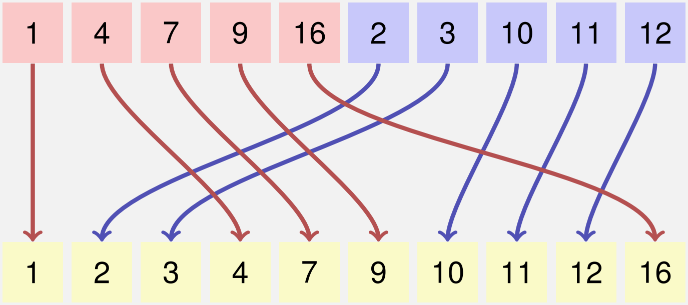
Merge



Merge



Merge



Algorithm Merge(A, l, m, r)

Input: Array A with length n, indexes $1 \leq l \leq m \leq r \leq n$.
$A[l, \dots, m]$, $A[m+1, \dots, r]$ sorted

Output: $A[l, \dots, r]$ sorted

$B \leftarrow$ new Array$(r - l + 1)$
$i \leftarrow l$; $j \leftarrow m + 1$; $k \leftarrow 1$
while $i \leq m$ and $j \leq r$ **do**
    if $A[i] \leq A[j]$ **then** $B[k] \leftarrow A[i]$; $i \leftarrow i + 1$
    else $B[k] \leftarrow A[j]$; $j \leftarrow j + 1$
    $k \leftarrow k + 1$;
while $i \leq m$ **do** $B[k] \leftarrow A[i]$; $i \leftarrow i + 1$; $k \leftarrow k + 1$
while $j \leq r$ **do** $B[k] \leftarrow A[j]$; $j \leftarrow j + 1$; $k \leftarrow k + 1$
for $k \leftarrow l$ **to** r **do** $A[k] \leftarrow B[k - l + 1]$

Correctness

Hypothesis: after k iterations of the loop in line 3 $B[1,\dots,k]$ is sorted and $B[k] \leq A[i]$, if $i \leq m$ and $B[k] \leq A[j]$ if $j \leq r$.

Proof by induction:
Base case: the empty array $B[1,\dots,0]$ is trivially sorted.
Induction step ($k \to k+1$):

- wlog $A[i] \leq A[j]$, $i \leq m, j \leq r$.
- $B[1,\dots,k]$ is sorted by hypothesis and $B[k] \leq A[i]$.
- After $B[k+1] \leftarrow A[i]$ $B[1,\dots,k+1]$ is sorted.
- $B[k+1] = A[i] \leq A[i+1]$ (if $i+1 \leq m$) and $B[k+1] \leq A[j]$ if $j \leq r$.
- $k \leftarrow k+1, i \leftarrow i+1$: Statement holds again.

Analysis (Merge)

Lemma

If: array A with length n, indexes $1 \leq l < r \leq n$. $m = \lfloor (l+r)/2 \rfloor$ and $A[l, \dots, m]$, $A[m+1, \dots, r]$ sorted.
Then: in the call of Merge(A, l, m, r) a number of $\Theta(r - l)$ key movements and comparisons are executed.

Proof: straightforward(Inspect the algorithm and count the operations.)

Mergesort

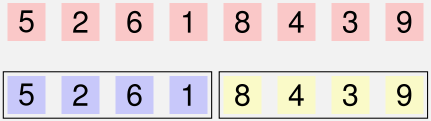
5	2	6	1	8	4	3	9

Mergesort

5	2	6	1	8	4	3	9

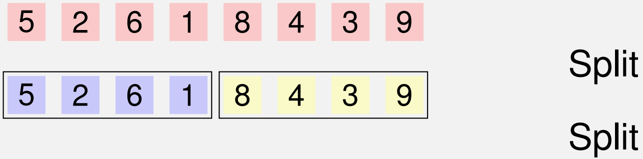
Split

Mergesort

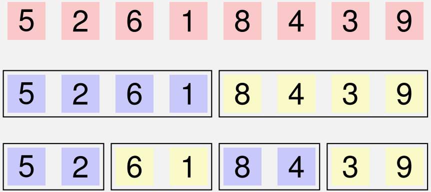


Split

Mergesort



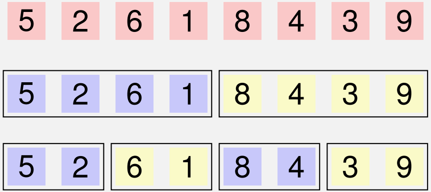
Mergesort



Split

Split

Mergesort

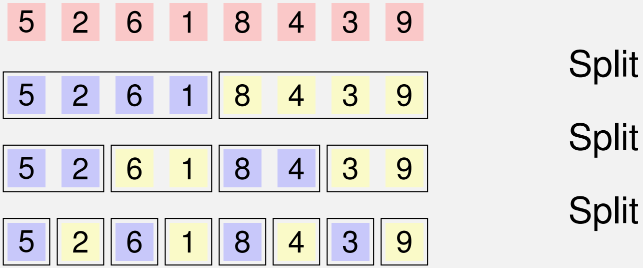


Split

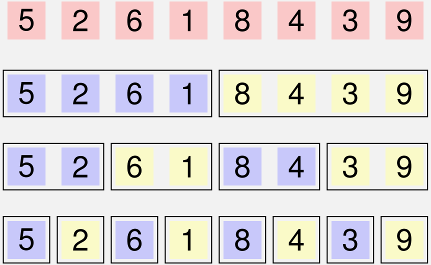
Split

Split

Mergesort



Mergesort



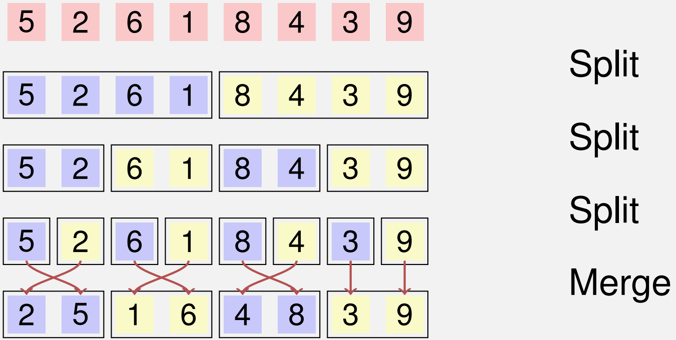
Split

Split

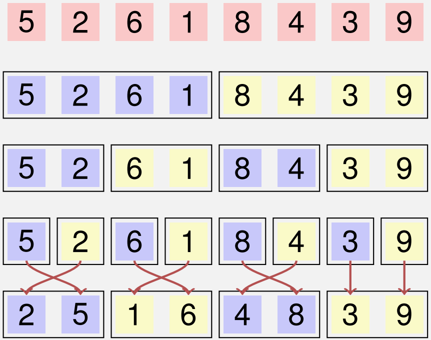
Split

Merge

Mergesort


5 2 6 1 8 4 3 9
5 2 6 1 | 8 4 3 9
5 2 | 6 1 | 8 4 | 3 9
5 | 2 | 6 | 1 | 8 | 4 | 3 | 9
2 5 | 1 6 | 4 8 | 3 9
Split
Split
Split
Merge

Mergesort



Split

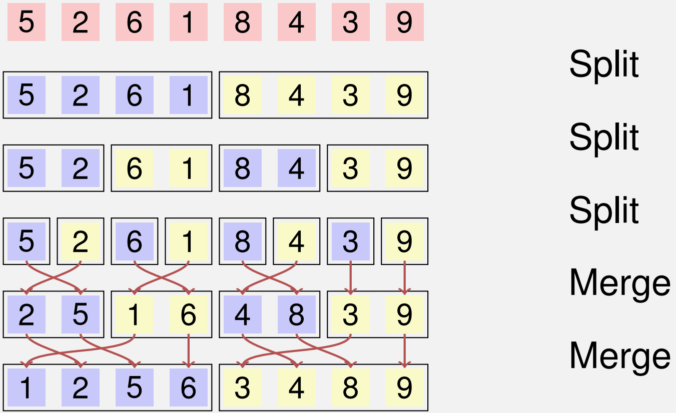
Split

Split

Merge

Merge

Mergesort


5 2 6 1 8 4 3 9
5 2 6 1 8 4 3 9
5 2 6 1 8 4 3 9
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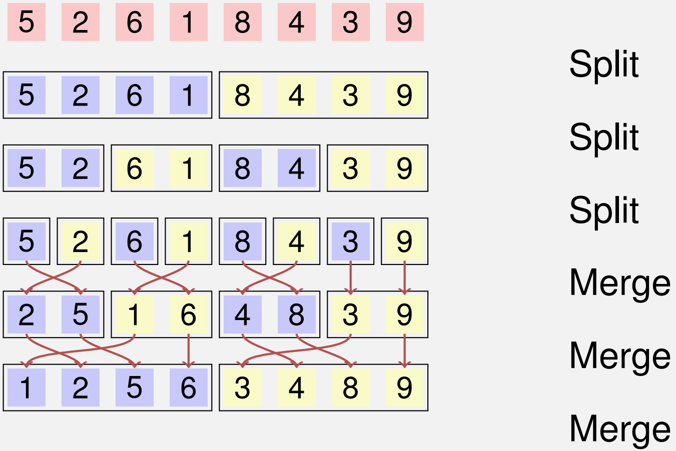
1 2 5 6 3 4 8 9
Split
Split
Split
Merge
Merge

Mergesort


5 2 6 1 8 4 3 9
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5 2 6 1 8 4 3 9
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2 5 1 6 4 8 3 9
1 2 5 6 3 4 8 9
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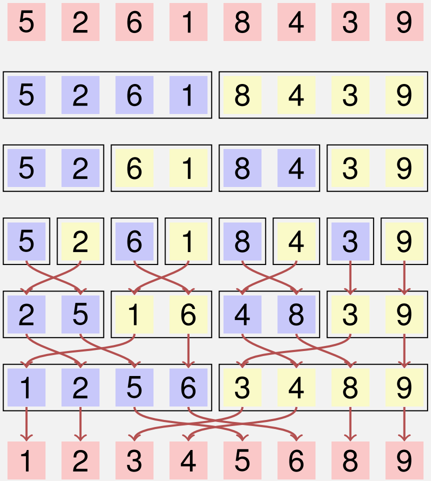
Merge

Mergesort



Split

Split

Split

Merge

Merge

Merge

Algorithm (recursive 2-way) Mergesort(A, l, r)

Input: Array A with length n. $1 \leq l \leq r \leq n$
Output: Array $A[l, \dots, r]$ sorted.

if $l < r$ **then**
    $m \leftarrow \lfloor (l+r)/2 \rfloor$ // middle position
    Mergesort(A, l, m)   // sort lower half
    Mergesort($A, m+1, r$)   // sort higher half
    Merge(A, l, m, r)   // Merge subsequences

Analysis

Recursion equation for the number of comparisons and key movements:

$$T(n) = T(\left\lceil \frac{n}{2} \right\rceil) + T(\left\lfloor \frac{n}{2} \right\rfloor) + \Theta(n)$$

Analysis

Recursion equation for the number of comparisons and key movements:

$$T(n) = T\left(\left\lceil\frac{n}{2}\right\rceil\right) + T\left(\left\lfloor\frac{n}{2}\right\rfloor\right) + \Theta(n) \in \Theta(n \log n)$$

Algorithm StraightMergesort(A)

Avoid recursion: merge sequences of length $1, 2, 4, \ldots$ directly

```
Input:   Array A with length n
Output:  Array A sorted
length ← 1
while length < n do                      // Iterate over lengths n
    r ← 0
    while r + length < n do              // Iterate over subsequences
        l ← r + 1
        m ← l + length − 1
        r ← min(m + length, n)
        Merge(A, l, m, r)
    length ← length · 2
```

Analysis

Like the recursive variant, the straight 2-way mergesort always executes a number of $\Theta(n \log n)$ key comparisons and key movements.

Natural 2-way mergesort

Observation: the variants above do not make use of any presorting and always execute $\Theta(n \log n)$ memory movements.

? How can partially presorted arrays be sorted better?

Natural 2-way mergesort

Observation: the variants above do not make use of any presorting and always execute $\Theta(n \log n)$ memory movements.

? How can partially presorted arrays be sorted better?

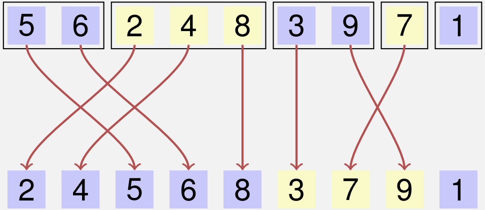
! Recursive merging of previously sorted parts (*runs*) *of* A.

Natural 2-way mergesort

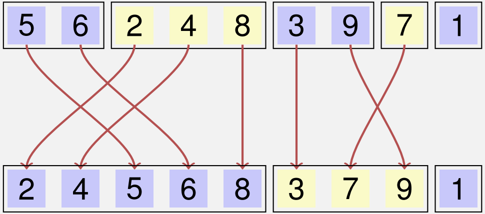
5	6	2	4	8	3	9	7	1

Natural 2-way mergesort

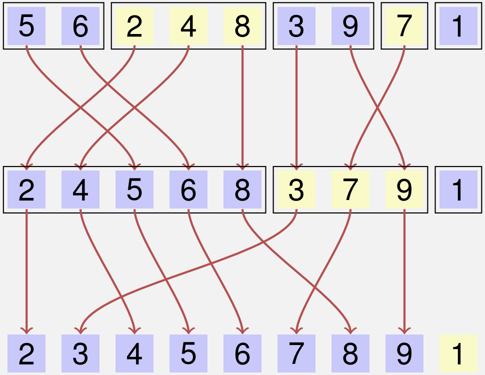
Natural 2-way mergesort



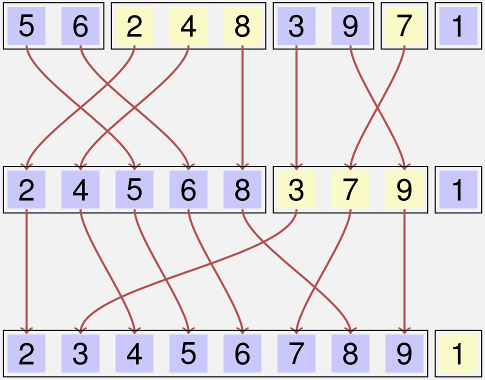
Natural 2-way mergesort



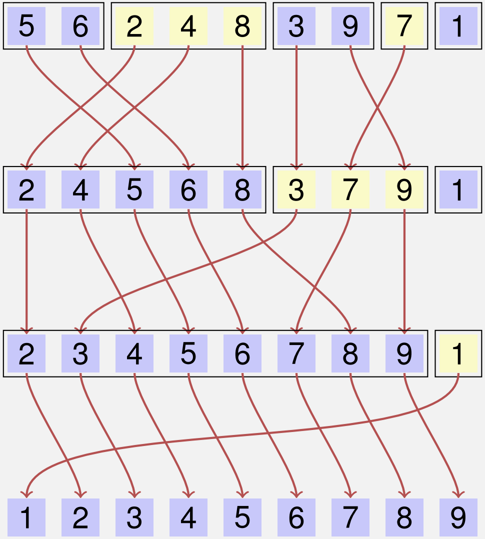
Natural 2-way mergesort



Natural 2-way mergesort



Natural 2-way mergesort



Algorithm NaturalMergesort(A)

Input: Array A with length $n > 0$
Output: Array A sorted

```
repeat
    r ← 0
    while r < n do
        l ← r + 1
        m ← l; while m < n and A[m + 1] ≥ A[m] do m ← m + 1
        if m < n then
            r ← m + 1; while r < n and A[r + 1] ≥ A[r] do r ← r + 1
            Merge(A, l, m, r);
        else
            r ← n
until l = 1
```

Analysis

? Is it also asymptotically better than StraightMergesort on average?

Analysis

? Is it also asymptotically better than StraightMergesort on average?

! No. Given the assumption of pairwise distinct keys, on average there are $n/2$ positions i with $k_i > k_{i+1}$, i.e. $n/2$ runs. Only one iteration is saved on average.

Natural mergesort executes in the worst case and on average a number of $\Theta(n \log n)$ comparisons and memory movements.

8.3 Quicksort

[Ottman/Widmayer, Kap. 2.2, Cormen et al, Kap. 7]

Quicksort

? What is the disadvantage of Mergesort?

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! Make sure that the left part contains only smaller elements than the right part.

? How?

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! Requires additional $\Theta(n)$ storage for merging.

? How could we reduce the merge costs?

! Make sure that the left part contains only smaller elements than the right part.

? How?

! Pivot and Partition!

Use a pivot

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1. Choose a (an arbitrary) *pivot* p

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2. Partition A in two parts, one part L with the elements with $A[i] \leq p$ and another part R with $A[i] > p$

p	$>$	$\leq$	$\leq$	$>$	$>$	$\leq$	$\leq$	$>$	$\leq$

Use a pivot

1. Choose a (an arbitrary) *pivot* p
2. Partition A in two parts, one part L with the elements with $A[i] \leq p$ and another part R with $A[i] > p$
3. Quicksort: Recursion on parts L and R

p	$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	$>$	$>$	$>$	$>$

Use a pivot

1. Choose a (an arbitrary) *pivot* p
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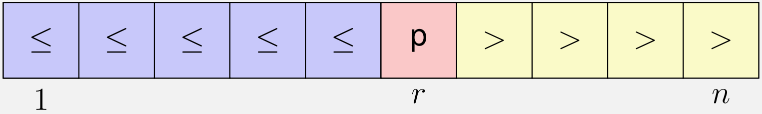

Algorithm Partition($A[l..r], p$)

Input: Array A, that contains the pivot p in the interval $[l, r]$ at least once.
Output: Array A partitioned in $[l..r]$ around p. Returns position of p.

```
while l ≤ r do
    while A[l] < p do
        l ← l + 1
    while A[r] > p do
        r ← r − 1
    swap(A[l], A[r])
    if A[l] = A[r] then
        l ← l + 1
return l-1
```

Algorithm Quicksort($A[l, \dots, r]$)

Input: Array A with length n. $1 \leq l \leq r \leq n$.
Output: Array A, sorted between l and r.

if $l < r$ **then**

- Choose pivot $p \in A[l, \dots, r]$
- $k \leftarrow$ Partition($A[l, \dots, r], p$)
- Quicksort($A[l, \dots, k-1]$)
- Quicksort($A[k+1, \dots, r]$)

Quicksort (arbitrary pivot)

2	4	5	6	8	3	7	9	1

Quicksort (arbitrary pivot)

2	4	5	6	8	3	7	9	1

Quicksort (arbitrary pivot)

2	4	5	6	8	3	7	9	1
2	1	3	6	8	5	7	9	4

Quicksort (arbitrary pivot)

2 4 5 6 8 3 7 9 1

2 1 3 6 8 5 7 9 4

Quicksort (arbitrary pivot)

2	4	5	6	8	3	7	9	1
2	1	3	6	8	5	7	9	4
1	2	3	4	5	8	7	9	6

Quicksort (arbitrary pivot)

2	4	5	6	8	3	7	9	1
2	1	3	6	8	5	7	9	4
1	2	3	4	5	8	7	9	6

Quicksort (arbitrary pivot)

2	4	5	6	8	3	7	9	1
2	1	3	6	8	5	7	9	4
1	2	3	4	5	8	7	9	6
1	2	3	4	5	6	7	9	8

Quicksort (arbitrary pivot)

2	4	5	6	8	3	7	9	1
2	1	3	6	8	5	7	9	4
1	2	3	4	5	8	7	9	6
1	2	3	4	5	6	7	9	8

Quicksort (arbitrary pivot)

2	4	5	6	8	3	7	9	1
2	1	3	6	8	5	7	9	4
1	2	3	4	5	8	7	9	6
1	2	3	4	5	6	7	9	8
1	2	3	4	5	6	7	8	9

Quicksort (arbitrary pivot)

2	4	5	6	8	3	7	9	1
2	1	3	6	8	5	7	9	4
1	2	3	4	5	8	7	9	6
1	2	3	4	5	6	7	9	8
1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9

Analysis: number comparisons

Worst case.

Analysis: number comparisons

Worst case. Pivot = min or max; number comparisons:

$$T(n) = T(n-1) + c \cdot n,\ T(1) = 0 \quad \Rightarrow \quad T(n) \in \Theta(n^2)$$

Analysis: number swaps

Result of a call to partition (pivot 3):

2	1	3	6	8	5	7	9	4

? How many swaps have taken place?

Analysis: number swaps

Result of a call to partition (pivot 3):

2	1	3	6	8	5	7	9	4

? How many swaps have taken place?

! 2. The maximum number of swaps is given by the number of keys in the smaller part.

Analysis: number swaps

Thought experiment

Analysis: number swaps

Thought experiment

- Each key from the smaller part pays a coin when it is being swapped.

Analysis: number swaps

Thought experiment

- Each key from the smaller part pays a coin when it is being swapped.
- After a key has paid a coin the domain containing the key decreases to half its previous size.

Analysis: number swaps

Thought experiment

- Each key from the smaller part pays a coin when it is being swapped.
- After a key has paid a coin the domain containing the key decreases to half its previous size.
- Every key needs to pay at most $\log n$ coins. But there are only n keys.

Analysis: number swaps

Thought experiment

- Each key from the smaller part pays a coin when it is being swapped.
- After a key has paid a coin the domain containing the key decreases to half its previous size.
- Every key needs to pay at most $\log n$ coins. But there are only n keys.

Consequence: there are $\mathcal{O}(n \log n)$ key swaps in the worst case.

Randomized Quicksort

Despite the worst case running time of $\Theta(n^2)$, quicksort is used practically very often.

Reason: quadratic running time unlikely provided that the choice of the pivot and the pre-sorting are not very disadvantageous.

Avoidance: randomly choose pivot. Draw uniformly from $[l, r]$.

Analysis (randomized quicksort)

Expected number of compared keys with input length n:

$$T(n) = (n-1) + \frac{1}{n}\sum_{k=1}^{n}\left(T(k-1) + T(n-k)\right), \; T(0) = T(1) = 0$$

Claim $T(n) \leq 4n\log n$.

Proof by induction:

Base case straightforward for $n = 0$ (with $0\log 0 := 0$) and for $n = 1$.

Hypothesis: $T(n) \leq 4n\log n$ for some n.

Induction step: $(n-1 \rightarrow n)$

Analysis (randomized quicksort)

$$
\begin{aligned}
T(n) &= n-1+\frac{2}{n}\sum_{k=0}^{n-1} T(k) \overset{\text{H}}{\leq} n-1+\frac{2}{n}\sum_{k=0}^{n-1} 4k\log k \\
&= n-1+\sum_{k=1}^{n/2} 4k \underbrace{\log k}_{\leq \log n - 1} + \sum_{k=n/2+1}^{n-1} 4k \underbrace{\log k}_{\leq \log n} \\
&\leq n-1+\frac{8}{n}\left((\log n - 1)\sum_{k=1}^{n/2} k + \log n \sum_{k=n/2+1}^{n-1} k\right) \\
&= n-1+\frac{8}{n}\left((\log n)\cdot\frac{n(n-1)}{2} - \frac{n}{4}\left(\frac{n}{2}+1\right)\right) \\
&= 4n\log n - 4\log n - 3 \leq 4n\log n
\end{aligned}
$$

Analysis (randomized quicksort)

Theorem

On average randomized quicksort requires $\mathcal{O}(n \cdot \log n)$ *comparisons.*

Practical Considerations

Worst case recursion depth $n-1$[11]. Then also a memory consumption of $\mathcal{O}(n)$.

Can be avoided: recursion only on the smaller part. Then guaranteed $\mathcal{O}(\log n)$ worst case recursion depth and memory consumption.

[11] stack overflow possible!

Quicksort with logarithmic memory consumption

Input: Array A with length n. $1 \leq l \leq r \leq n$.
Output: Array A, sorted between l and r.

while $l < r$ **do**
- Choose pivot $p \in A[l, \ldots, r]$
- $k \leftarrow$ Partition$(A[l, \ldots, r], p)$
- **if** $k - l < r - k$ **then**
  - Quicksort$(A[l, \ldots, k-1])$
  - $l \leftarrow k + 1$
- **else**
  - Quicksort$(A[k+1, \ldots, r])$
  - $r \leftarrow k - 1$

The call of Quicksort$(A[l, \ldots, r])$ in the original algorithm has moved to iteration (tail recursion!): the if-statement became a while-statement.

Practical Considerations.

- Practically the pivot is often the median of three elements. For example: Median3($A[l], A[r], A[\lfloor l + r/2 \rfloor]$).
- There is a variant of quicksort that requires only constant storage. Idea: store the old pivot at the position of the new pivot.
- Complex divide-and-conquer algorithms often use a trivial ($\Theta(n^2)$) algorithm as base case to deal with small problem sizes.

8.4 Appendix

Derivation of some mathematical formulas

$\log n! \in \Theta(n \log n)$

$$\log n! = \sum_{i=1}^{n} \log i \leq \sum_{i=1}^{n} \log n = n \log n$$

$$\begin{aligned}
\sum_{i=1}^{n} \log i &= \sum_{i=1}^{\lfloor n/2 \rfloor} \log i + \sum_{\lfloor n/2 \rfloor + 1}^{n} \log i \\
&\geq \sum_{i=2}^{\lfloor n/2 \rfloor} \log 2 + \sum_{\lfloor n/2 \rfloor + 1}^{n} \log \frac{n}{2} \\
&= (\underbrace{\lfloor n/2 \rfloor}_{> n/2 - 1} - 2 + 1) + (\underbrace{n - \lfloor n/2 \rfloor}_{\geq n/2})(\log n - 1) \\
&> \frac{n}{2} \log n - 2.
\end{aligned}$$

$[n! \in o(n^n)]$

$$
\begin{aligned}
n \log n &\geq \sum_{i=1}^{\lfloor n/2 \rfloor} \log 2i + \sum_{i=\lfloor n/2 \rfloor + 1}^{n} \log i \\
&= \sum_{i=1}^{n} \log i + \left\lfloor \frac{n}{2} \right\rfloor \log 2 \\
&> \sum_{i=1}^{n} \log i + n/2 - 1 = \log n! + n/2 - 1
\end{aligned}
$$

$$
n^n = 2^{n \log_2 n} \geq 2^{\log_2 n!} \cdot 2^{n/2} \cdot 2^{-1} = n! \cdot 2^{n/2-1}
$$

$$
\Rightarrow \frac{n!}{n^n} \leq 2^{-n/2+1} \xrightarrow{n \to \infty} 0 \Rightarrow n! \in o(n^n) = \mathcal{O}(n^n) \backslash \Omega(n^n)
$$

[Even $n! \in o((n/c)^n)\ \forall\, 0 < c < e$]

Konvergenz oder Divergenz von $f_n = \frac{n!}{(n/c)^n}$.

Ratio Test

$$\frac{f_{n+1}}{f_n} = \frac{(n+1)!}{\left(\frac{n+1}{c}\right)^{n+1}} \cdot \frac{\left(\frac{n}{c}\right)^n}{n!} = c \cdot \left(\frac{n}{n+1}\right)^n \longrightarrow c \cdot \frac{1}{e} \lessgtr 1 \text{ if } c \lessgtr e$$

because $\left(1 + \frac{1}{n}\right)^n \to e$. Even the series $\sum_{i=1}^n f_n$ converges / diverges for $c \lessgtr e$.

f_n diverges for $c = e$, because (Stirling): $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

[Ratio Test]

Ratio test for a sequence $(f_n)_{n\in\mathbb{N}}$: If $\frac{f_{n+1}}{f_n} \xrightarrow[n\to\infty]{} \lambda$, then the sequence f_n and the series $\sum_{i=1}^{n} f_i$

- converge, if $\lambda < 1$ and
- diverge, if $\lambda > 1$.

[Ratio Test Derivation]

Ratio test is implied by Geometric Series

$$S_n(r) := \sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r}.$$

converges for $n \to \infty$ if and only if $-1 < r < 1$.

Let $0 \leq \lambda < 1$:

$$\begin{aligned} &\forall \varepsilon > 0\, \exists n_0 : f_{n+1}/f_n < \lambda + \varepsilon\, \forall n \geq n_0 \\ \Rightarrow &\exists \varepsilon > 0, \exists n_0 : f_{n+1}/f_n \leq \mu < 1\, \forall n \geq n_0 \end{aligned}$$

Thus

$$\sum_{n=n_0}^{\infty} f_n \leq f_{n_0} \cdot \sum_{n=n_0}^{\infty} \cdot \mu^{n-n_0} \quad \text{konvergiert.}$$

(Analogously for divergence)