20. Dynamic Programming II

Subset sum problem, knapsack problem, greedy algorithm vs dynamic programming [Ottman/Widmayer, Kap. 7.2, 7.3, 5.7, Cormen et al, Kap. 15,35.5]

Task

Partition the set of the “item” above into two set such that both sets have the same value.

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A solution:

Subset Sum Problem

Consider $n \in \mathbb{N}$ numbers $a_1, \ldots, a_n \in \mathbb{N}$.

Goal: decide if a selection $I \subseteq \{1, \ldots, n\}$ exists such that

$$\sum_{i \in I} a_i = \sum_{i \in \{1,\ldots,n\} \setminus I} a_i .$$

Naive Algorithm

Check for each bit vector $b = (b_1, \ldots, b_n) \in \{0, 1\}^n$, if

$$\sum_{i=1}^{n} b_i a_i \overset{?}{=} \sum_{i=1}^{n} (1 - b_i) a_i$$

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Worst case: n steps for each of the 2^n bit vectors b. Number of steps: $\mathcal{O}(n \cdot 2^n)$.

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    - If $S_i^1 + S_j^2 > h$ then $j \leftarrow j - 1$
    - If $S_i^1 + S_j^2 < h$ then $i \leftarrow i + 1$

Example

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0	1	6	7	0	2	3	4	5	6	7	9

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$\boxed{0}$	1	6	7	0	2	3	4	5	6	7	$\boxed{9}$

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$\Leftrightarrow$ One possible solution: $\{1, 3, 4\}$

Analysis

- Generate partial sums for each part: $\mathcal{O}(2^{n/2} \cdot n)$.
- Each sorting: $\mathcal{O}(2^{n/2} \log(2^{n/2})) = \mathcal{O}(n2^{n/2})$.
- Merge: $\mathcal{O}(2^{n/2})$

Overal running time

$$\mathcal{O}\left(n \cdot 2^{n/2}\right) = \mathcal{O}\left(n \left(\sqrt{2}\right)^{n}\right).$$

Substantial improvement over the naive method – but still exponential!

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DP-table: $[0, \ldots, n] \times [0, \ldots, z]$-table T with boolean entries. $T[k, s]$ specifies if there is a selection $I_k \subset \{1, \ldots, k\}$ such that $\sum_{i \in I_k} a_i = s$.

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Computation:

$$T[k, s] \leftarrow \begin{cases} T[k-1, s] & \text{if } s < a_k \\ T[k-1, s] \vee T[k-1, s-a_k] & \text{if } s \geq a_k \end{cases}$$

for increasing k and then within k increasing s.

Example

$\{1, 6, 2, 5\}$

summe s →

0 1 2 3 4 5 6 7 8 9 10 11 12 13 14

k ↓

Example

$\{1, 6, 2, 5\}$

$\xrightarrow{\text{summe } s}$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	●	·	·	·	·	·	·	·	·	·	·	·	·	·	·

$k \big\downarrow$

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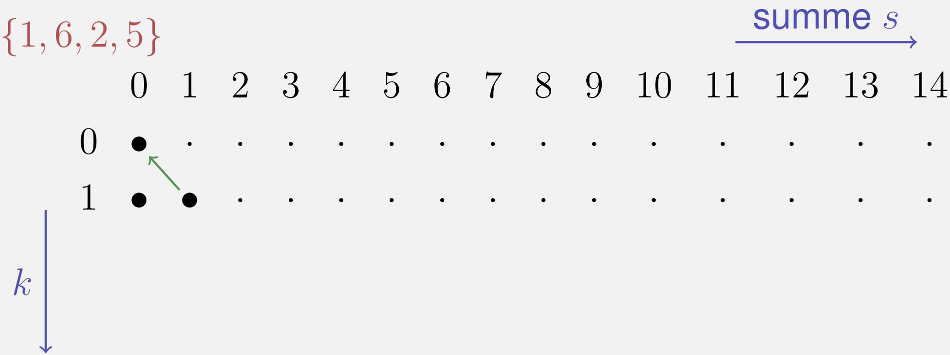

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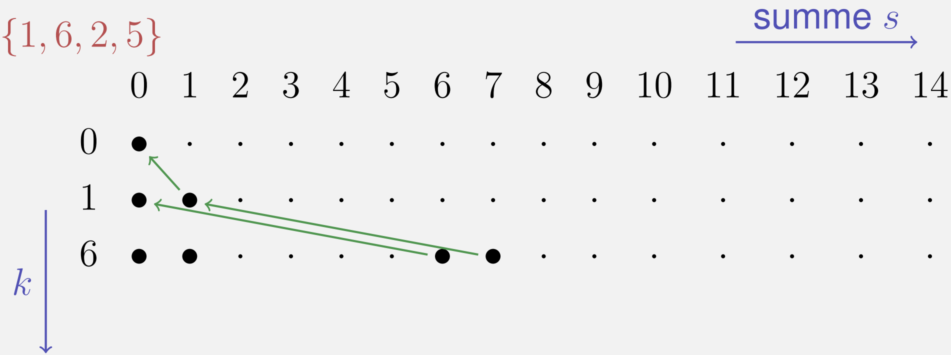

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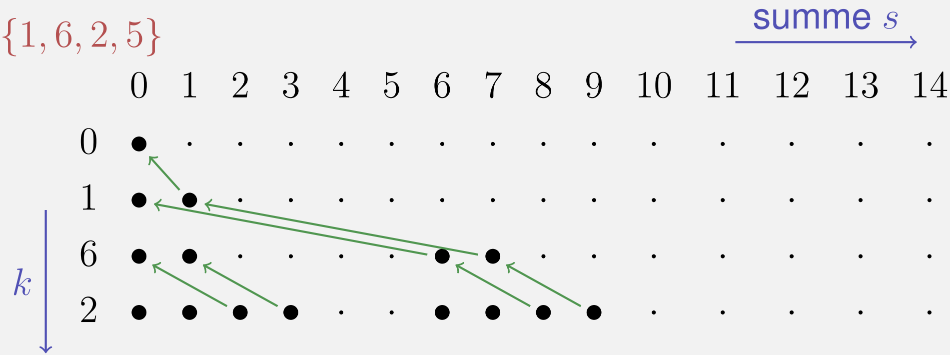

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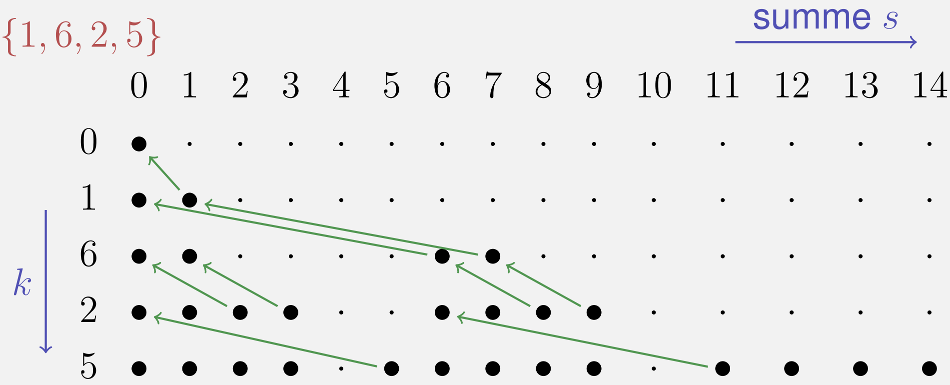

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1	●	●	·	·	·	·	·	·	·	·	·	·	·	·	·
6	●	●	·	·	·	·	●	●	·	·	·	·	·	·	·
2	●	●	●	●	·	·	●	●	●	●	·	·	·	·	·
5	●	●	●	●	·	●	●	●	●	●	·	●	●	●	●

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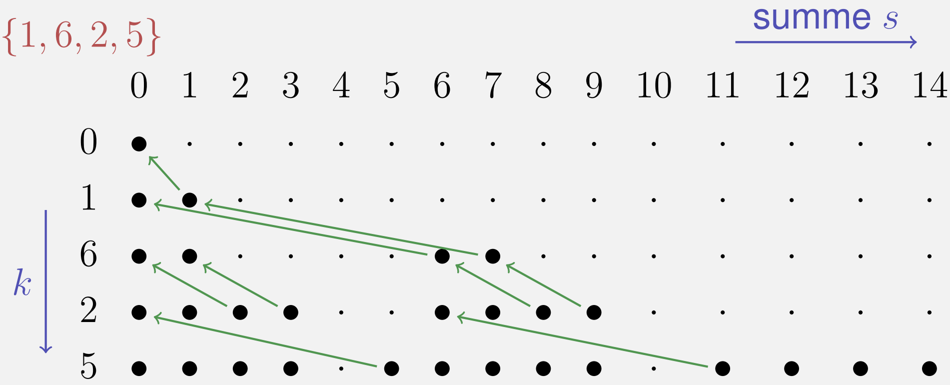

Determination of the solution: if $T[k, s] = T[k-1, s]$ then a_k unused and continue with $T[k-1, s]$, otherwise a_k used and continue with $T[k-1, s-a_k]$.

That is mysterious

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If, however, z is polynomial in n then the algorithm has polynomial run time in n. This is called *pseudo-polynomial*.

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Implications:

- NP contains P.
- Problems can be *verified* in polynomial time.
- Under the not (yet?) proven assumption[41] that NP $\neq$ P, there is *no algorithm with polynomial run time* for the problem considered above.

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The knapsack problem

We pack our suitcase with ...

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Aim to take as much as possible with us. But some things are more valuable than others!

Knapsack problem

Given:

- set of $n \in \mathbb{N}$ items $\{1, \ldots, n\}$.
- Each item i has value $v_i \in \mathbb{N}$ and weight $w_i \in \mathbb{N}$.
- Maximum weight $W \in \mathbb{N}$.
- Input is denoted as $E = (v_i, w_i)_{i=1,\ldots,n}$.

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- Maximum weight $W \in \mathbb{N}$.
- Input is denoted as $E = (v_i, w_i)_{i=1,\dots,n}$.

Wanted:

a selection $I \subseteq \{1, \dots, n\}$ that maximises $\sum_{i \in I} v_i$ under $\sum_{i \in I} w_i \leq W$.

Greedy heuristics

Sort the items decreasingly by value per weight v_i/w_i: Permutation p with $v_{p_i}/w_{p_i} \geq v_{p_{i+1}}/w_{p_{i+1}}$

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That is fast: $\Theta(n \log n)$ for sorting and $\Theta(n)$ for the selection. But is it good?

Counterexample

$$
\begin{array}{lll}
v_1 = 1 & w_1 = 1 & v_1/w_1 = 1 \\
v_2 = W - 1 & w_2 = W & v_2/w_2 = \frac{W-1}{W}
\end{array}
$$

Counterexample

$$\begin{array}{lll} v_1 = 1 & w_1 = 1 & v_1/w_1 = 1 \\ v_2 = W - 1 & w_2 = W & v_2/w_2 = \frac{W-1}{W} \end{array}$$

Greed algorithm chooses $\{v_1\}$ with value 1.
Best selection: $\{v_2\}$ with value $W - 1$ and weight W.

Greedy heuristics can be arbitrarily bad.

Dynamic Programming

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Three dimensional table $m[i, w, v]$ (“doable”) of boolean values.

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Three dimensional table $m[i, w, v]$ (“doable”) of boolean values.

$m[i, w, v] =$ true if and only if

- A selection of the first i parts exists $(0 \leq i \leq n)$
- with overal weight w $(0 \leq w \leq W)$ and
- a value of at least v $(0 \leq v \leq \sum_{i=1}^{n} v_i)$.

Computation of the DP table

Initially

- $m[i, w, 0] \leftarrow$ true für alle $i \geq 0$ und alle $w \geq 0$.
- $m[0, w, v] \leftarrow$ false für alle $w \geq 0$ und alle $v > 0$.

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Computation

$$m[i, w, v] \leftarrow \begin{cases} m[i-1, w, v] \vee m[i-1, w-w_i, v-v_i] & \text{if } w \geq w_i \text{ und } v \geq v_i \\ m[i-1, w, v] & \text{otherwise.} \end{cases}$$

increasing in i and for each i increasing in w and for fixed i and w increasing by v.

Solution: largest v, such that $m[i, w, v] =$ true for some i and w.

Observation

The definition of the problem obviously implies that

- for $m[i, w, v] =$ true it holds:
  $m[i', w, v] =$ true $\forall i' \geq i$,
  $m[i, w', v] =$ true $\forall w' \geq w$,
  $m[i, w, v'] =$ true $\forall v' \leq v$.
- fpr $m[i, w, v] =$ false it holds:
  $m[i', w, v] =$ false $\forall i' \leq i$,
  $m[i, w', v] =$ false $\forall w' \leq w$,
  $m[i, w, v'] =$ false $\forall v' \geq v$.

This strongly suggests that we do not need a 3d table!

2d DP table

Table entry $t[i, w]$ contains, instead of boolean values, the largest v, that can be achieved[42] with

- items $1, \ldots, i$ $(0 \leq i \leq n)$
- at maximum weight w $(0 \leq w \leq W)$.

---

[42] We could have followed a similar idea in order to reduce the size of the sparse table.

Computation

Initially

- $t[0, w] \leftarrow 0$ for all $w \geq 0$.

We compute

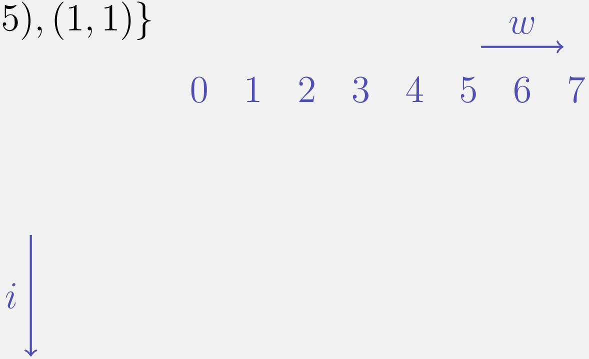
$$t[i,w] \leftarrow \begin{cases} t[i-1,w] & \text{if } w < w_i \\ \max\{t[i-1,w], t[i-1,w-w_i]+v_i\} & \text{otherwise.} \end{cases}$$

increasing by i and for fixed i increasing by w.

Solution is located in $t[n, w]$

Example

$E = \{(2,3),(4,5),(1,1)\}$



$\xrightarrow{w}$

0 1 2 3 4 5 6 7

$i \downarrow$

Example

$E = \{(2, 3), (4, 5), (1, 1)\}$

$\xrightarrow{w}$

	0	1	2	3	4	5	6	7
$\emptyset$	0	0	0	0	0	0	0	0

$i \downarrow$

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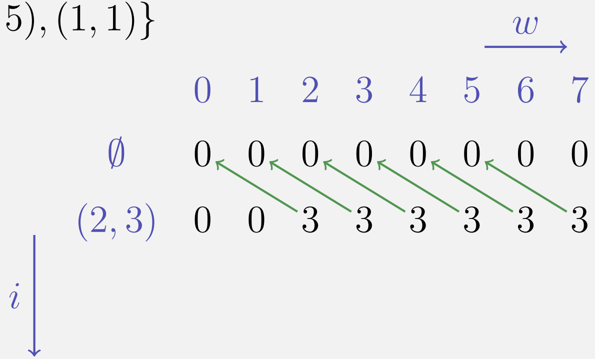

	0	1	2	3	4	5	6	7
$\emptyset$	0	0	0	0	0	0	0	0
$(2,3)$	0	0	3	3	3	3	3	3

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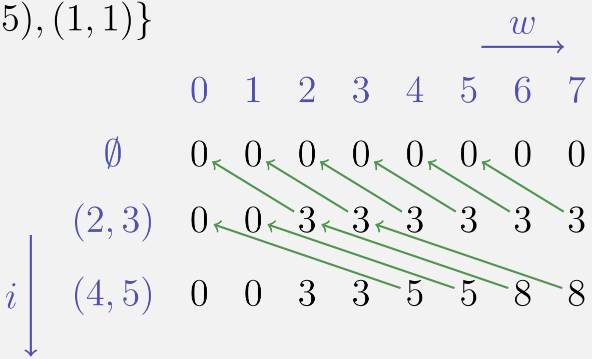

	0	1	2	3	4	5	6	7
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$(2,3)$	0	0	3	3	3	3	3	3
$(4,5)$	0	0	3	3	5	5	8	8

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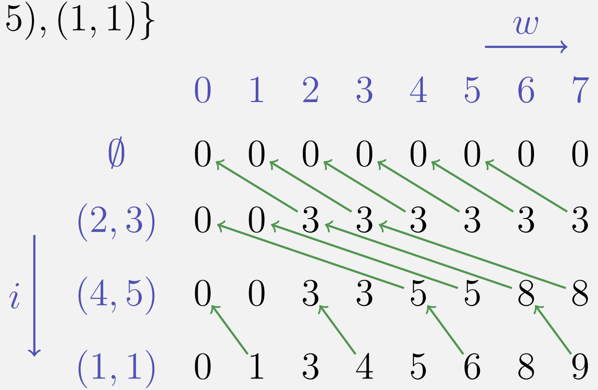

	0	1	2	3	4	5	6	7
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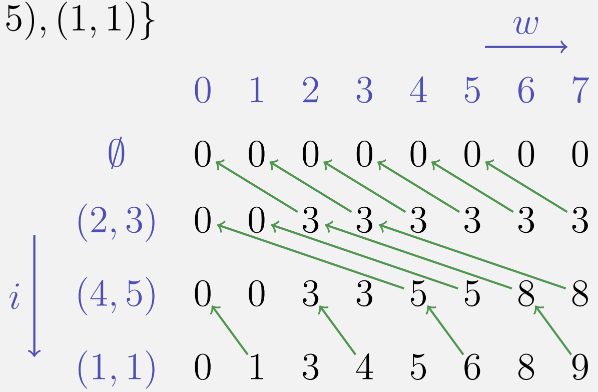
$(2,3)$	0	0	3	3	3	3	3	3
$(4,5)$	0	0	3	3	5	5	8	8
$(1,1)$	0	1	3	4	5	6	8	9

Example

$E = \{(2,3),(4,5),(1,1)\}$



	0	1	2	3	4	5	6	7
$\emptyset$	0	0	0	0	0	0	0	0
$(2,3)$	0	0	3	3	3	3	3	3
$(4,5)$	0	0	3	3	5	5	8	8
$(1,1)$	0	1	3	4	5	6	8	9

Reading out the solution: if $t[i,w] = t[i-1,w]$ then item i unused and continue with $t[i-1,w]$ otherwise used and continue with $t[i-1,s-w_i]$.

Analysis

The two algorithms for the knapsack problem provide a run time in $\Theta(n \cdot W \cdot \sum_{i=1}^{n} v_i)$ (3d-table) and $\Theta(n \cdot W)$ (2d-table) and are thus both pseudo-polynomial, but they deliver the best possible result.

The greedy algorithm is very fast butmight deliver an arbitrarily bad result.

Now we consider a solution between the two extremes.

21. Dynamic Programming III

FPTAS [Ottman/Widmayer, Kap. 7.2, 7.3, Cormen et al, Kap. 15,35.5]

Approximation

Let $\varepsilon \in (0,1)$ given. Let I_{opt} an optimal selection.

No try to find a valid selection I with

$$\sum_{i \in I} v_i \geq (1-\varepsilon) \sum_{i \in I_{\text{opt}}} v_i .$$

Sum of weights may not violate the weight limit.

Different formulation of the algorithm

Before: weight limit $w \to$ maximal value v

Reversed: value $v \to$ minimal weight w

$\Rightarrow$ **alternative table** $g[i, v]$ provides the minimum weight with

- a selection of the first i items ($0 \leq i \leq n$) that
- provide a value of exactly v ($0 \leq v \leq \sum_{i=1}^{n} v_i$).

Computation

Initially

- $g[0,0] \leftarrow 0$
- $g[0,v] \leftarrow \infty$ (Value v cannot be achieved with 0 items.).

Computation

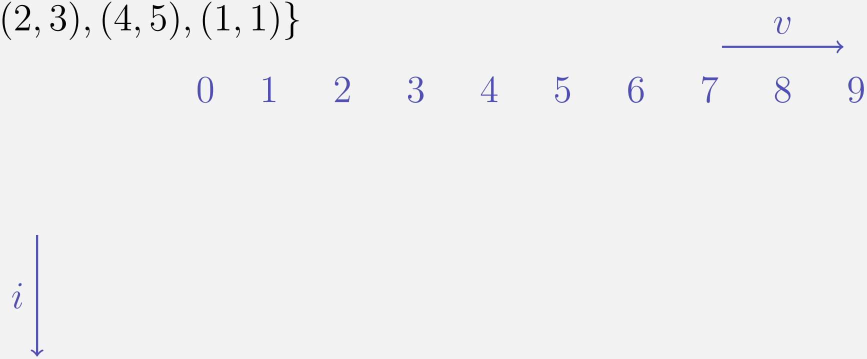
$$g[i,v] \leftarrow \begin{cases} g[i-1,v] & \text{falls } v < v_i \\ \min\{g[i-1,v], g[i-1,v-v_i]+w_i\} & \text{sonst.} \end{cases}$$

incrementally in i and for fixed i increasing in v.

Solution can be found at largest index v with $g[n,v] \leq w$.

Example

$E = \{(2,3),(4,5),(1,1)\}$



Example

$E = \{(2,3),(4,5),(1,1)\}$

$\xrightarrow{v}$

	0	1	2	3	4	5	6	7	8	9
$\emptyset$	0	∞	∞	∞	∞	∞	∞	∞	∞	∞

$i \downarrow$

Example

$E = \{(2,3),(4,5),(1,1)\}$

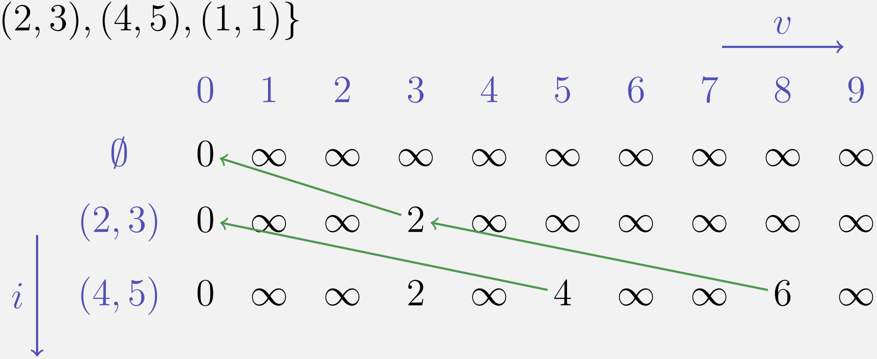
$$\xrightarrow{v}$$

$$\begin{array}{c|cccccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\emptyset & 0 & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty & \infty \\
(2,3) & 0 & \infty & \infty & 2 & \infty & \infty & \infty & \infty & \infty & \infty
\end{array}$$

$i \big\downarrow$

Example

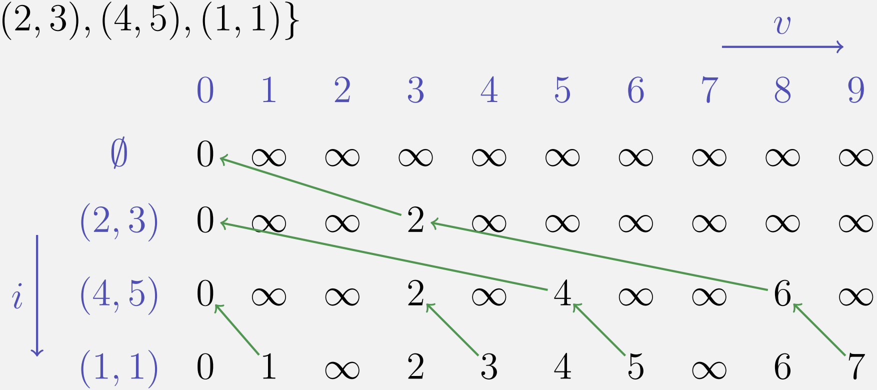
$E = \{(2,3), (4,5), (1,1)\}$



	0	1	2	3	4	5	6	7	8	9
$\emptyset$	0	∞	∞	∞	∞	∞	∞	∞	∞	∞
$(2,3)$	0	∞	∞	2	∞	∞	∞	∞	∞	∞
$(4,5)$	0	∞	∞	2	∞	4	∞	∞	6	∞

Example

$E = \{(2,3), (4,5), (1,1)\}$



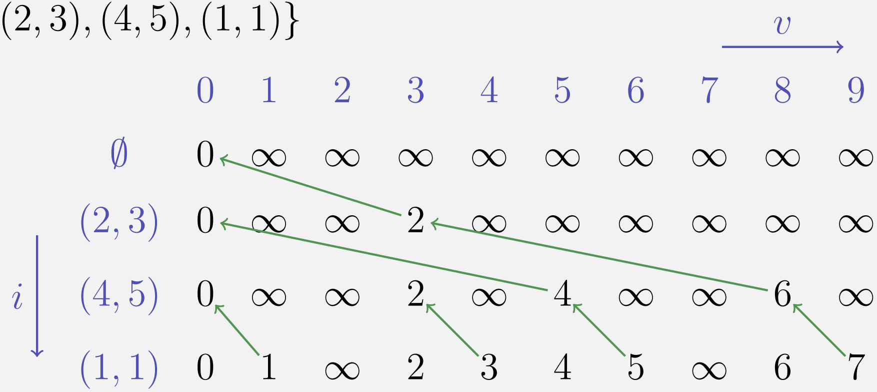
$v \longrightarrow$

$i \downarrow$

	0	1	2	3	4	5	6	7	8	9
$\emptyset$	0	∞	∞	∞	∞	∞	∞	∞	∞	∞
$(2,3)$	0	∞	∞	2	∞	∞	∞	∞	∞	∞
$(4,5)$	0	∞	∞	2	∞	4	∞	∞	6	∞
$(1,1)$	0	1	∞	2	3	4	5	∞	6	7

Example

$E = \{(2,3),(4,5),(1,1)\}$



	0	1	2	3	4	5	6	7	8	9
$\emptyset$	0	∞	∞	∞	∞	∞	∞	∞	∞	∞
$(2,3)$	0	∞	∞	2	∞	∞	∞	∞	∞	∞
$(4,5)$	0	∞	∞	2	∞	4	∞	∞	6	∞
$(1,1)$	0	1	∞	2	3	4	5	∞	6	7

Read out the solution: if $g[i,v] = g[i-1,v]$ then item i unused and continue with $g[i-1,v]$ otherwise used and continue with $g[i-1,b-v_i]$.

The approximation trick

Pseuduopolynomial run time gets polynmial if the number of occuring values can be bounded by a polynom of the input length.

Let $K > 0$ be chosen *appropriately*. Replace values v_i by “rounded values” $\tilde{v}_i = \lfloor v_i/K \rfloor$ delivering a new input $E' = (w_i, \tilde{v}_i)_{i=1\dots n}$.

Apply the algorithm on the input E' with the same weight limit W.

Idea

Example $K = 5$

Values

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots, 98, 99, 100$$
$$\rightarrow$$
$$0, 0, 0, 0, 1, 1, 1, 1, 1, 2, \ldots, 19, 19, 20$$

Obviously less different values

Properties of the new algorithm

- Selection of items in E' is also admissible in E. Weight remains unchanged!
- Run time of the algorithm is bounded by $\mathcal{O}(n^2 \cdot v_{\max}/K)$ $(v_{\max} := \max\{v_i | 1 \leq i \leq n\})$

How good is the approximation?

It holds that

$$v_i - K \leq K \cdot \left\lfloor \frac{v_i}{K} \right\rfloor = K \cdot \tilde{v}_i \leq v_i$$

Let I'_{opt} be an optimal solution of E'. Then

$$\left(\sum_{i \in I_{\text{opt}}} v_i\right) - n \cdot K \overset{|I_{\text{opt}}| \leq n}{\leq} \sum_{i \in I_{\text{opt}}} (v_i - K) \leq \sum_{i \in I_{\text{opt}}} (K \cdot \tilde{v}_i) = K \sum_{i \in I_{\text{opt}}} \tilde{v}_i$$

$$\underset{I'_{\text{opt}}\text{optimal}}{\leq} K \sum_{i \in I'_{\text{opt}}} \tilde{v}_i = \sum_{i \in I'_{\text{opt}}} K \cdot \tilde{v}_i \leq \sum_{i \in I'_{\text{opt}}} v_i.$$

Choice of K

Requirement:

$$\sum_{i \in I'} v_i \geq (1 - \varepsilon) \sum_{i \in I_{\text{opt}}} v_i.$$

Inequality from above:

$$\sum_{i \in I'_{\text{opt}}} v_i \geq \left(\sum_{i \in I_{\text{opt}}} v_i \right) - n \cdot K$$

thus: $K = \varepsilon \frac{\sum_{i \in I_{\text{opt}}} v_i}{n}$.

Choice of K

Choose $K = \varepsilon \frac{\sum_{i \in I_{\text{opt}}} v_i}{n}$. The optimal sum is unknown. Therefore we choose $K' = \varepsilon \frac{v_{\max}}{n}$.[43]

It holds that $v_{\max} \leq \sum_{i \in I_{\text{opt}}} v_i$ and thus $K' \leq K$ and the approximation is even slightly better.

The run time of the algorithm is bounded by

$$\mathcal{O}(n^2 \cdot v_{\max}/K') = \mathcal{O}(n^2 \cdot v_{\max}/(\varepsilon \cdot v_{\max}/n)) = \mathcal{O}(n^3/\varepsilon).$$

---

[43] We can assume that items i with $w_i > W$ have been removed in the first place.

FPTAS

Such a family of algorithms is called an *approximation scheme*: the choice of ε controls both running time and approximation quality.

The runtime $\mathcal{O}(n^3/\varepsilon)$ is a polynom in n and in $\frac{1}{\varepsilon}$. The scheme is therefore also called a *FPTAS - Fully Polynomial Time Approximation Scheme*

21. Dynamic Programming III

Optimal Search Tree [Ottman/Widmayer, Kap. 5.7]

Optimal binary Search Trees

Given: search probabilities p_i for each key k_i ($i = 1, \ldots, n$) and q_i of each interval d_i ($i = 0, \ldots, n$) between search keys of a binary search tree. $\sum_{i=1}^{n} p_i + \sum_{i=0}^{n} q_i = 1$.

Optimal binary Search Trees

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Wanted: optimal search tree T with key depths $\mathrm{depth}(\cdot)$, that minimizes the expected search costs

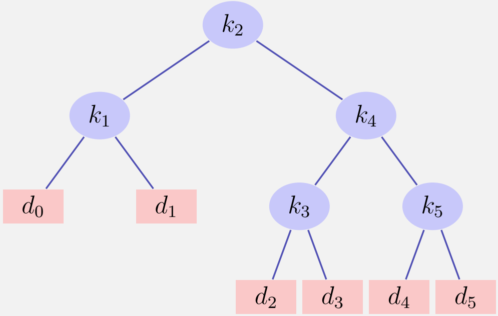
$$\begin{aligned} C(T) &= \sum_{i=1}^{n} p_i \cdot (\mathrm{depth}(k_i) + 1) + \sum_{i=0}^{n} q_i \cdot (\mathrm{depth}(d_i) + 1) \\ &= 1 + \sum_{i=1}^{n} p_i \cdot \mathrm{depth}(k_i) + \sum_{i=0}^{n} q_i \cdot \mathrm{depth}(d_i) \end{aligned}$$

Example

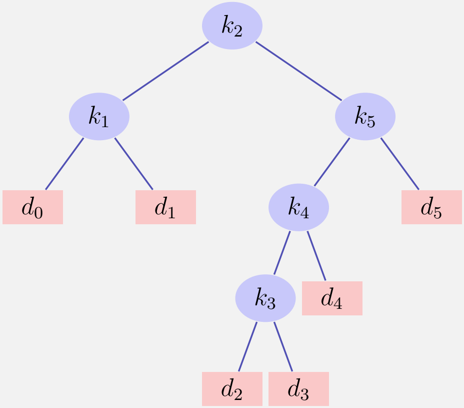
Expected Frequencies

i	0	1	2	3	4	5
p_i		0.15	0.10	0.05	0.10	0.20
q_i	0.05	0.10	0.05	0.05	0.05	0.10

Example



Search tree with expected costs 2.8



Search tree with expected costs 2.75

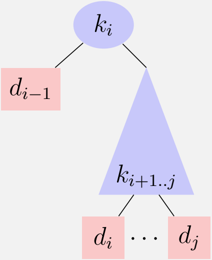
Structure of a optimal binary search tree

- Subtree with keys $k_i, \dots, k_j$ and intervals $d_{i-1}, \dots, d_j$ must be optimal for the respective sub-problem.[44]
- Consider all subtrees with roots k_r and optimal subtrees for keys $k_i, \dots, k_{r-1}$ and $k_{r+1}, \dots, k_j$

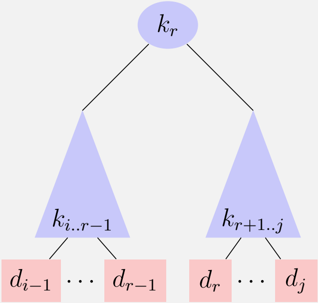
---

[44] The usual argument: if it was not optimal, it could be replaced by a better solution improving the overal solution.

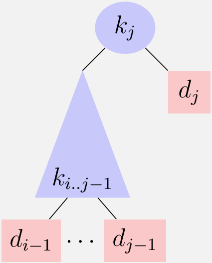
Sub-trees for Searching



empty left subtree



non-empty left and right subtrees



empty right subtree

Expected Search Costs

Let $\text{depth}_T(k)$ be the depth of a node k in the sub-tree T. Let k be the root of subtrees T_r and T_{L_r} and T_{R_r} be the left and right sub-tree of T_r. Then

$$\begin{aligned}\text{depth}_T(k_i) &= \text{depth}_{T_{L_r}}(k_i) + 1,\ (i < r)\\ \text{depth}_T(k_i) &= \text{depth}_{T_{R_r}}(k_i) + 1,\ (i > r)\end{aligned}$$

Expected Search Costs

Let $e[i, j]$ be the costs of an optimal search tree with nodes $k_i, \dots, k_j$.

Base case $e[i, i-1]$, expected costs d_{i-1}

Let $w(i, j) = \sum_{l=i}^{j} p_l + \sum_{l=i-1}^{j} q_l$.

If k_r is the root of an optimal search tree with keys $k_i, \dots, k_j$, then

$$e[i, j] = p_r + (e[i, r-1] + w(i, r-1)) + (e[r+1, j] + w(r+1, j))$$

with $w(i, j) = w(i, r-1) + p_r + w(r+1, j)$:

$$e[i, j] = e[i, r-1] + e[r+1, j] + w(i, j).$$

Dynamic Programming

$$e[i,j] = \begin{cases} q_{i-1} & \text{if } j = i-1, \\ \min_{i \le r \le j}\{e[i,r-1] + e[r+1,j] + w[i,j]\} & \text{if } i \le j \end{cases}$$

Computation

Tables $e[1 \dots n+1, 0 \dots n], w[1 \dots n+1, 0 \dots m], r[1 \dots n, 1 \dots n]$
Initially

- $e[i, i-1] \leftarrow q_{i-1}, w[i, i-1] \leftarrow q_{i-1}$ for all $1 \leq i \leq n+1$.

We compute

$$
\begin{aligned}
w[i,j] &= w[i,j-1] + p_j + q_j \\
e[i,j] &= \min_{i \leq r \leq j} \{e[i,r-1] + e[r+1,j] + w[i,j]\} \\
r[i,j] &= \arg\min_{i \leq r \leq j} \{e[i,r-1] + e[r+1,j] + w[i,j]\}
\end{aligned}
$$

for intervals $[i,j]$ with increasing lengths $l = 1, \dots, n$, each for $i = 1, \dots, n-l+1$. Result in $e[1,n]$, reconstruction via r. Runtime $\Theta(n^3)$.

Example

i	0	1	2	3	4	5
p_i		0.15	0.10	0.05	0.10	0.20
q_i	0.05	0.10	0.05	0.05	0.05	0.10

e

j							
0	0.05						
1	0.45	0.10					
2	0.90	0.40	0.05				
3	1.25	0.70	0.25	0.05			
4	1.75	1.20	0.60	0.30	0.05		
5	2.75	2.00	1.30	0.90	0.50	0.10	
	1	2	3	4	5	6	i

w

j							
0	0.05						
1	0.30	0.10					
2	0.45	0.25	0.05				
3	0.55	0.35	0.15	0.05			
4	0.70	0.50	0.30	0.20	0.05		
5	1.00	0.80	0.60	0.50	0.35	0.10	
	1	2	3	4	5	6	i

r

j						
1	1					
2	1	2				
3	2	2	3			
4	2	2	4	4		
5	2	4	5	5	5	
	1	2	3	4	5	i