19. Dynamic Programming I

Memoization, Optimal Substructure, Overlapping Sub-Problems, Dependencies, General Procedure. Examples: Fibonacci, Rod Cutting, Longest Ascending Subsequence, Longest Common Subsequence, Edit Distance, Matrix Chain Multiplication (Strassen)

[Ottman/Widmayer, Kap. 1.2.3, 7.1, 7.4, Cormen et al, Kap. 15]

Fibonacci Numbers

 (again)

$$F_n := \begin{cases} n & \text{if } n < 2 \\ F_{n-1} + F_{n-2} & \text{if } n \geq 2. \end{cases}$$

Analysis: why ist the recursive algorithm so slow?

Algorithm FibonacciRecursive(n)

Input: $n \geq 0$
Output: n-th Fibonacci number

if $n < 2$ **then**
    $f \leftarrow n$
else
    $f \leftarrow \text{FibonacciRecursive}(n-1) + \text{FibonacciRecursive}(n-2)$
return f

Analysis

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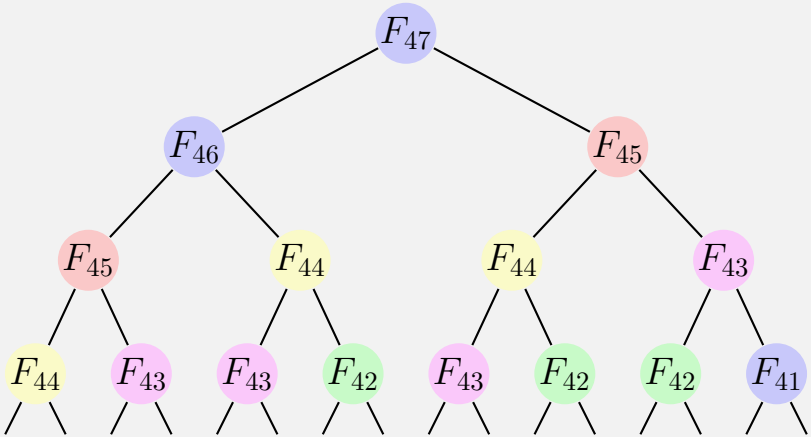
Algorithm is *exponential* in n.

Reason (visual)



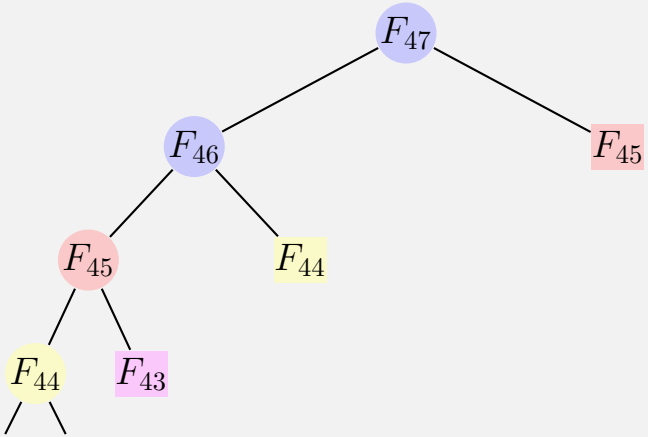
Nodes with same values are evaluated (too) often.

Memoization

Memoization (sic) saving intermediate results.

- Before a subproblem is solved, the existence of the corresponding intermediate result is checked.
- If an intermediate result exists then it is used.
- Otherwise the algorithm is executed and the result is saved accordingly.

Memoization with Fibonacci



Rechteckige Knoten wurden bereits ausgewertet.

Algorithm FibonacciMemoization(n)

Input: $n \geq 0$
Output: n-th Fibonacci number

if $n \leq 2$ **then**
  $f \leftarrow 1$
else if $\exists$memo$[n]$ **then**
  $f \leftarrow$ memo$[n]$
else
  $f \leftarrow$ FibonacciMemoization$(n-1)$ + FibonacciMemoization$(n-2)$
  memo$[n] \leftarrow f$
return f

Analysis

Computational complexity:

$$T(n) = T(n-1) + c = ... = \mathcal{O}(n).$$

because after the call to $f(n-1)$, $f(n-2)$ has already been computed.

A different argument: $f(n)$ is computed exactly once recursively for each n. Runtime costs: n calls with $\Theta(1)$ costs per call $n \cdot c \in \Theta(n)$. The recursion vanishes from the running time computation.

Algorithm requires $\Theta(n)$ memory.[38]

[38] But the naive recursive algorithm also requires $\Theta(n)$ memory implicitly.

Looking closer ...

... the algorithm computes the values of F_1, F_2, F_3,... in the *top-down* approach of the recursion.

Can write the algorithm *bottom-up*. This is characteristic for *dynamic programming*.

Algorithm FibonacciBottomUp(n)

Input: $n \geq 0$
Output: n-th Fibonacci number

$F[1] \leftarrow 1$
$F[2] \leftarrow 1$
for $i \leftarrow 3, \ldots, n$ **do**
  $F[i] \leftarrow F[i-1] + F[i-2]$
return $F[n]$

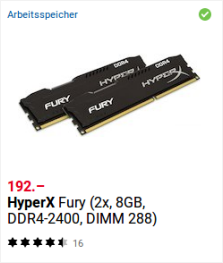
Dynamic Programming: Idea

- Divide a complex problem into a reasonable number of sub-problems
- The solution of the sub-problems will be used to solve the more complex problem
- Identical problems will be computed only once

Dynamic Programming Consequence

Identical problems will be computed only once

$\Rightarrow$ Results are saved



We trade spee against memory consumption

Dynamic Programming: Description

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Runtime (typical) = number entries of the table times required operations per entry.

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   F_n ist die n-te Fibonacci-Zahl.

Dynamic Programming = Divide-And-Conquer ?

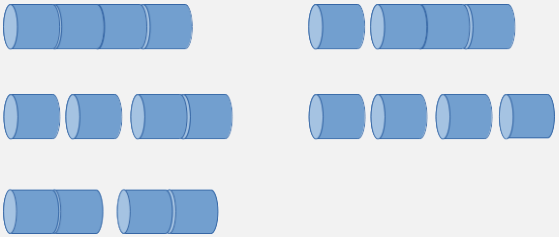
- In both cases the original problem can be solved (more easily) by utilizing the solutions of sub-problems. The problem provides *optimal substructure*.
- Divide-And-Conquer algorithms (such as Mergesort): sub-problems are independent; their solutions are required only once in the algorithm.
- DP: sub-problems are dependent. The problem is said to have *overlapping sub-problems* that are required multiple-times in the algorithm.
- In order to avoid redundant computations, results are tabulated. For *sub-problems there must not be any circular dependencies*.

Rod Cutting

- Rods (metal sticks) are cut and sold.
- Rods of length $n \in \mathbb{N}$ are available. A cut does not provide any costs.
- For each length $l \in \mathbb{N}$, $l \leq n$ known is the value $v_l \in \mathbb{R}^+$
- Goal: cut the rods such (into $k \in \mathbb{N}$ pieces) that

$$\sum_{i=1}^{k} v_{l_i} \text{ is maximized subject to } \sum_{i=1}^{k} l_i = n.$$

Rod Cutting: Example



Possibilities to cut a rod of length 4 (without permutations)

Length	0	1	2	3	4
Price	0	2	3	8	9

$\Rightarrow$ Best cut: 3 + 1 with value 10.

Wie findet man den DP Algorithms

0. Exact formulation of the wanted solution
1. Define sub-problems (and compute the cardinality)
2. Guess / Enumerate (and determine the running time for guessing)
3. Recursion: relate sub-problems
4. Memoize / Tabularize. Determine the dependencies of the sub-problems
5. Solve the problem
   Running time = #sub-problems $\times$ time/sub-problem

Structure of the problem

0. *Wanted:* r_n = maximal value of rod (cut or as a whole) with length n.
1. *sub-problems*: maximal value r_k for each $0 \leq k < n$
2. *Guess* the length of the first piece
3. *Recursion*

$$r_k = \max\{v_i + r_{k-i} : 0 < i \leq k\}, \quad k > 0$$
$$r_0 = 0$$

4. *Dependency:* r_k depends (only) on values v_i, $1 \leq i \leq k$ and the optimal cuts r_i, $i < k$
5. *Solution* in r_n

Algorithm RodCut(v,n)

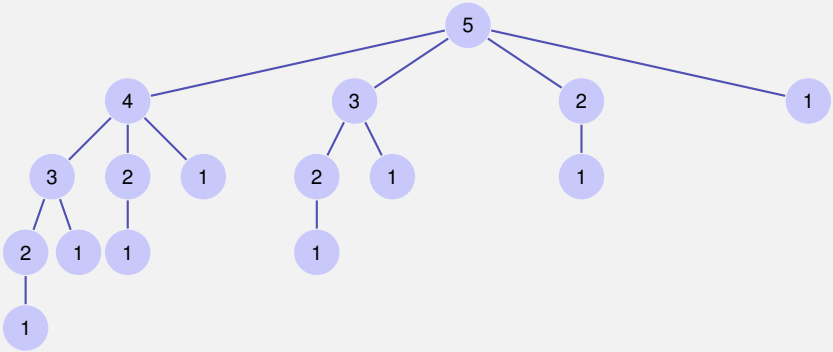
Input: $n \geq 0$, Prices v
Output: best value

$q \leftarrow 0$
if $n > 0$ **then**
  for $i \leftarrow 1, \ldots, n$ **do**
    $q \leftarrow \max\{q, v_i + \mathsf{RodCut}(v, n-i)\}$;
return q

Running time $T(n) = \sum_{i=0}^{n-1} T(i) + c \quad \Rightarrow^{39} \quad T(n) \in \Theta(2^n)$

[39] $T(n) = T(n-1) + \sum_{i=0}^{n-2} T(i) + c = T(n-1) + (T(n-1) - c) + c = 2T(n-1) \quad (n > 0)$

Recursion Tree



Algorithm RodCutMemoized(m, v, n)

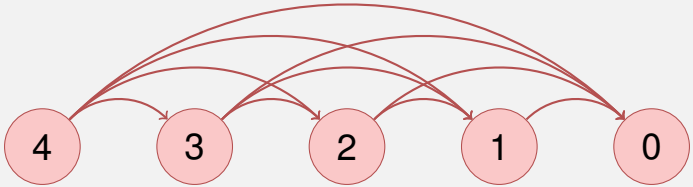
Input: $n \geq 0$, Prices v, Memoization Table m
Output: best value

$q \leftarrow 0$
if $n > 0$ **then**
  if $\exists\ m[n]$ **then**
    $q \leftarrow m[n]$
  else
    for $i \leftarrow 1, \ldots, n$ **do**
      $q \leftarrow \max\{q, v_i + \text{RodCutMemoized}(m, v, n - i)\}$;
    $m[n] \leftarrow q$
return q

Running time $\sum_{i=1}^{n} i = \Theta(n^2)$

Subproblem-Graph

Describes the mutual dependencies of the subproblems



and must not contain cycles

Construction of the Optimal Cut

- During the (recursive) computation of the optimal solution for each $k \leq n$ the recursive algorithm determines the optimal length of the first rod
- Store the lenght of the first rod in a separate table of length n

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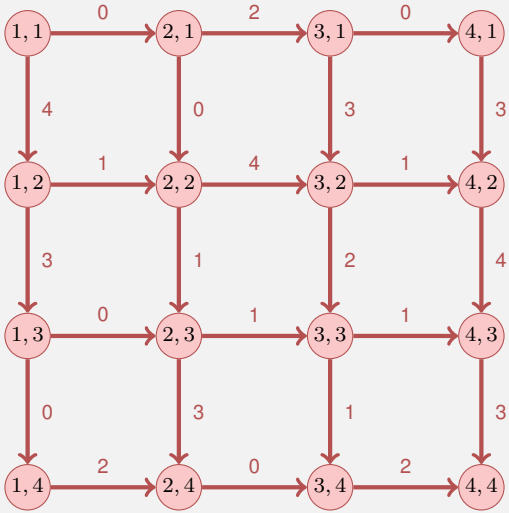
   r_n is the best value for the rod of length n.

Rabbit!

A rabbit sits on cite $(1, 1)$ of an $n \times n$ grid. It can only move to east or south. On each pathway there is a number of carrots. How many carrots does the rabbit collect maximally?

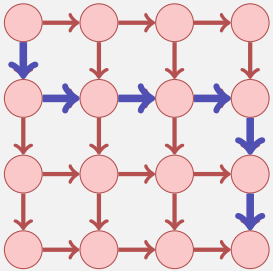


Rabbit!

Number of possible paths?

- Choice of $n-1$ ways to south out of $2n-2$ ways overal.

$\Rightarrow$ No chance for a naive algorithm



The path 100011
(1:to south, 0: to east)

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- $$\binom{2n-2}{n-1} \in \Omega(2^n)$$

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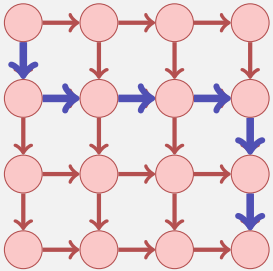

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Recursion

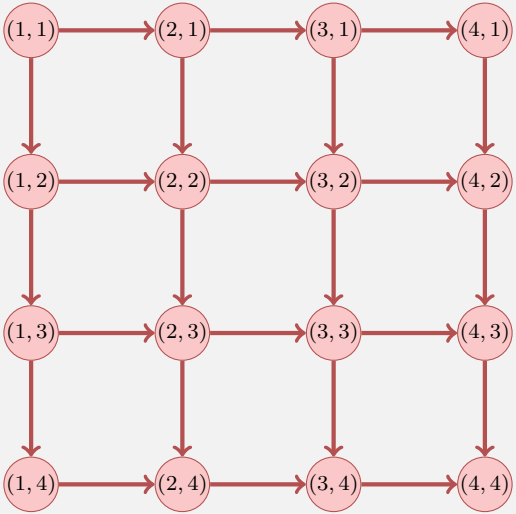
Wanted: $T_{0,0}$ = *maximal number carrots from* $(0,0)$ *to* (n,n).

Let $w_{(i,j)-(i',j')}$ number of carrots on egde from (i,j) to (i',j').

Recursion (maximal number of carrots from (i,j) to (n,n)

$$T_{ij} = \begin{cases} \max\{w_{(i,j)-(i,j+1)} + T_{i,j+1}, w_{(i,j)-(i+1,j)} + T_{i+1,j}\}, & i < n, j < n \\ w_{(i,j)-(i,j+1)} + T_{i,j+1}, & i = n, j < n \\ w_{(i,j)-(i+1,j)} + T_{i+1,j}, & i < n, j = n \\ 0 & i = j = n \end{cases}$$

Graph of Subproblem Dependencies



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   $T_{1,1}$ provides the maximal number of carrots.

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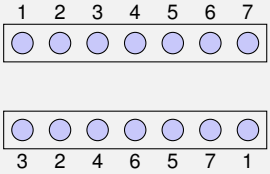

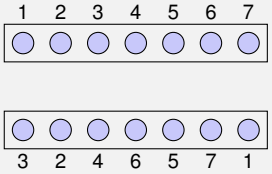

Connect as many as possible fitting ports without lines crossing.

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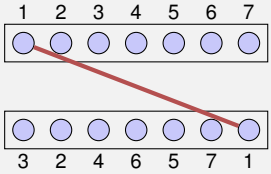

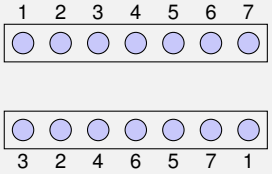

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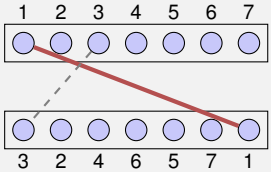

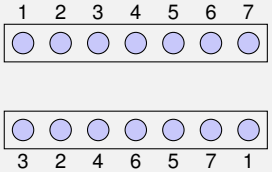

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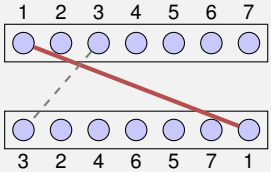

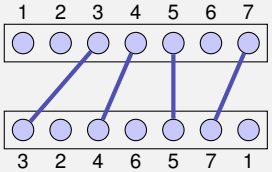

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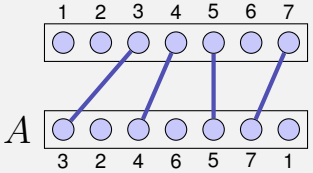
Formally

- Consider Sequence $A_n = (a_1, \ldots, a_n)$.
- Search for a longest increasing subsequence of A_n.
- Examples of increasing subsequences: $(3, 4, 5)$, $(2, 4, 5, 7)$, $(3, 4, 5, 7)$, $(3, 7)$.



Formally

- Consider Sequence $A_n = (a_1, \ldots, a_n)$.
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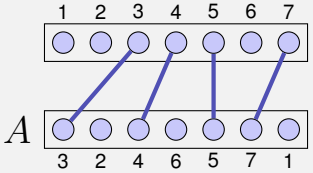
- Examples of increasing subsequences: $(3, 4, 5)$, $(2, 4, 5, 7)$, $(3, 4, 5, 7)$, $(3, 7)$.



Generalization: allow any numbers, even with duplicates (still only strictly increasing subsequences permitted). Example: $(2, 3, 3, 3, 5, 1)$ with increasing subsequence $(2, 3, 5)$.

First idea

Let L_i = *longest ascending subsequence of* A_i $(1 \leq i \leq n)$

Assumption: LAS L_k of A_k known for Now want to compute L_{k+1} for A_{k+1} .

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It does not work this way, we cannot infer L_{k+1} from L_k.

Second idea.

Let L_i = *longest ascending subsequence of* A_i $(1 \leq i \leq n)$

Assumption: a LAS L_j is known for each $j \leq k$. Now compute LAS L_{k+1} for $k+1$.

Second idea.

Let L_i = *longest ascending subsequence of* A_i $(1 \leq i \leq n)$

Assumption: a LAS L_j is known for each $j \leq k$. Now compute LAS L_{k+1} for $k+1$.

Look at all fitting $L_{k+1} = L_j \oplus a_{k+1}$ $(j \leq k)$ and choose a longest sequence.

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Counterexample: $A_5 = (1, 2, 5, 3, 4)$. Let $A_4 = (1, 2, 5, 3)$ with $L_1 = (1)$, $L_2 = (1, 2)$, $L_3 = (1, 2, 5)$, $L_4 = (1, 2, 5)$. Determine L_5 from $L_1, \ldots, L_4$?

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Look at all fitting $L_{k+1} = L_j \oplus a_{k+1}$ $(j \leq k)$ and choose a longest sequence.

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That does not work either: cannot infer L_{k+1} from only *an arbitrary solution* L_j. We need to consider all LAS. Too many.

Third approach

Let $M_{n,i}$ = *longest ascending subsequence of* A_i $(1 \leq i \leq n)$

Assumption: the LAS M_j for A_k, *that end with smallest element* are known for each of the lengths $1 \leq j \leq k$.

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Assumption: the LAS M_j for A_k, *that end with smallest element* are known for each of the lengths $1 \leq j \leq k$.

Consider all fitting $M_{k,j} \oplus a_{k+1}$ $(j \leq k)$ and update the table of the LAS,that end with smallest possible element.

Third approach Example

Example: $A = (1, 1000, 1001, 4, 5, 2, 6, 7)$

A	LAT $M_{k,\cdot}$
1	(1)

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$+\,4$	$(1), (1, 4), (1, 1000, 1001)$
$+\,5$	$(1), (1, 4), (1, 4, 5)$

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- Idea: save the last element of the increasing sequence $M_{k,j}$ at slot j.

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- Solution: second table with the predecessors.

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Index	0	1	2	3	4	...
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Index	1	2	3	4	5	6
Wert	3	2	5	1	6	4
Predecessor	$-\infty$	$-\infty$	2	$-\infty$	5	1

Index	0	1	2	3	4	...
$(L_j)_j$	-∞	1	4	6	∞	

Dynamic Programming Algorithm LAS

1. Table dimension? Semantics?

Dynamic Programming Algorithm LAS

Table dimension? Semantics?

Two tables $T[0, \ldots, n]$ and $V[1, \ldots, n]$.

1. $T[j]$: last Element of the increasing subequence $M_{n,j}$
   $V[j]$: Value of the predecessor of a_j.
   Start with $T[0] \leftarrow -\infty$, $T[i] \leftarrow \infty \; \forall i > 1$

Dynamic Programming Algorithm LAS

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Computation of an entry

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   Start with $T[0] \leftarrow -\infty$, $T[i] \leftarrow \infty \; \forall i > 1$

Computation of an entry

2. Entries in T sorted in ascending order. For each new entry a_{k+1} binary search for l, such that $T[l] < a_k < T[l+1]$. Set $T[l+1] \leftarrow a_{k+1}$. Set $V[k] = T[l]$.

Dynamic Programming algorithm LAS

3 Computation order

Dynamic Programming algorithm LAS

3 **Computation order**

Traverse the list anc compute $T[k]$ and $V[k]$ with ascending k

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How can the solution be determined from the table?

4

Dynamic Programming algorithm LAS

3 **Computation order**

Traverse the list anc compute $T[k]$ and $V[k]$ with ascending k

How can the solution be determined from the table?

4 Search the largest l with $T[l] < \infty$. l is the last index of the LAS. Starting at l search for the index $i < l$ such that $V[l] = a_i$, i is the predecessor of l. Repeat with $l \leftarrow i$ until $T[l] = -\infty$

Analysis

- Computation of the table:
  - Initialization: $\Theta(n)$ Operations
  - Computation of the kth entry: binary search on positions $\{1, \ldots, k\}$ plus constant number of assignments.

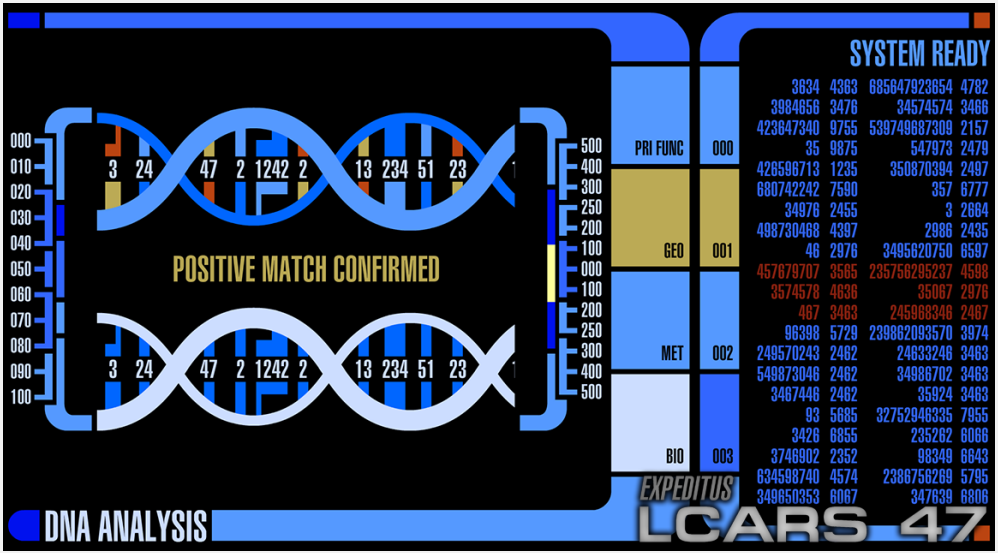
$$\sum_{k=1}^{n} (\log k + \mathcal{O}(1)) = \mathcal{O}(n) + \sum_{k=1}^{n} \log(k) = \Theta(n \log n).$$

- Reconstruction: traverse A from right to left: $\mathcal{O}(n)$.

Overal runtime:

$$\Theta(n \log n).$$

DNA - Comparison (Star Trek)



DNA - Comparison

- DNA consists of sequences of four different nucleotides **A**denine **G**uanine **T**hymine **C**ytosine
- DNA sequences (genes) thus can be described with strings of A, G, T and C.
- Possible comparison of two genes: determine the **longest common subsequence**

The longest common subsequence problem is a special case of the minimal edit distance problem.

Minimal Editing Distance

Editing distance of two sequences $A_n = (a_1, \ldots, a_m)$, $B_m = (b_1, \ldots, b_m)$.

Editing operations:

- Insertion of a character
- Deletion of a character
- Replacement of a character

Question: how many editing operations at least required in order to transform string A into string B.

TIGER ZIGER ZIEGER ZIEGE

Minimal Editing Distance

Wanted: cheapest character-wise transformation $A_n \to B_m$ with costs

operation	Levenshtein	LCS[40]	general
Insert c	1	1	ins(c)
Delete c	1	1	del(c)
Replace $c \to c'$	$\mathbb{1}(c \neq c')$	$\infty \cdot \mathbb{1}(c \neq c')$	repl(c, c')

Beispiel

T	I	G	E	R		T	I	_	G	E	R		T→Z	+E	-R
Z	I	E	G	E		Z	I	E	G	E	_		Z→T	-E	+R

[40] Longest common subsequence – A special case of an editing problem

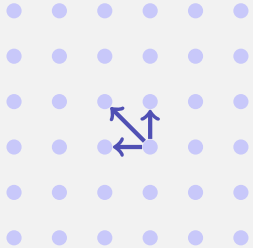
DP

0. $E(n, m)$ = mimimum number edit operations (ED cost) $a_{1...n} \to b_{1...m}$
1. Subproblems $E(i, j)$ = ED von $a_{1...i}$. $b_{1...j}$. #SP $= n \cdot m$
2. Guess Costs$\Theta(1)$
   - $a_{1..i} \to a_{1...i-1}$ (delete)
   - $a_{1..i} \to a_{1...i}b_j$ (insert)
   - $a_{1..i} \to a_{1...i_1}b_j$ (replace)
3. Rekursion

$$E(i,j) = \min \begin{cases} \text{del}(a_i) + E(i-1, j), \\ \text{ins}(b_j) + E(i, j-1), \\ \text{repl}(a_i, b_j) + E(i-1, j-1) \end{cases}$$

DP

4. Dependencies



$\Rightarrow$ Computation from left top to bottom right. Row- or column-wise.

5. Solution in $E(n, m)$

Example (Levenshtein Distance)

$$E[i,j] \leftarrow \min\left\{E[i-1,j]+1, E[i,j-1]+1, E[i-1,j-1]+\mathbb{1}(a_i \neq b_j)\right\}$$

	∅	Z	I	E	G	E
∅	0	1	2	3	4	5
T	1	1	2	3	4	5
I	2	2	1	2	3	4
G	3	3	2	2	2	3
E	4	4	3	2	3	2
R	5	5	4	3	3	3

Editing steps: from bottom right to top left, following the recursion.
Bottom-Up description of the algorithm: exercise

Bottom-Up DP algorithm ED]

1. **Dimension of the table? Semantics?**

Bottom-Up DP algorithm ED]

Dimension of the table? Semantics?

1. Table $E[0, \ldots, m][0, \ldots, n]$. $E[i, j]$: minimal edit distance of the strings $(a_1, \ldots, a_i)$ and $(b_1, \ldots, b_j)$

Bottom-Up DP algorithm ED]

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Bottom-Up DP algorithm ED]

Dimension of the table? Semantics?

1. Table $E[0,\ldots,m][0,\ldots,n]$. $E[i,j]$: minimal edit distance of the strings $(a_1,\ldots,a_i)$ and $(b_1,\ldots,b_j)$

Computation of an entry

2. $E[0,i] \leftarrow i\ \forall 0 \leq i \leq m$, $E[j,0] \leftarrow i\ \forall 0 \leq j \leq n$. Computation of $E[i,j]$ otherwise via $E[i,j] =$
$\min\{\mathsf{del}(a_i) + E(i-1,j), \mathsf{ins}(b_j) + E(i,j-1), \mathsf{repl}(a_i,b_j) + E(i-1,j-1)\}$

Bottom-Up DP algorithm ED

3 Computation order

Bottom-Up DP algorithm ED

3 **Computation order**

Rows increasing and within columns increasing (or the other way round).

Bottom-Up DP algorithm ED

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Rows increasing and within columns increasing (or the other way round).

Reconstruct solution?

4

Bottom-Up DP algorithm ED

3 **Computation order**

Rows increasing and within columns increasing (or the other way round).

4 **Reconstruct solution?**

Start with $j = m$, $i = n$. If $E[i,j] = \text{repl}(a_i, b_j) + E(i-1, j-1)$ then output $a_i \to b_j$ and continue with $(j,i) \leftarrow (j-1, i-1)$; otherwise, if $E[i,j] = \text{del}(a_i) + E(i-1,j)$ output $\text{del}(a_i)$ and continue with $j \leftarrow j-1$ otherwise, if $E[i,j] = \text{ins}(b_j) + E(i,j-1)$, continue with $i \leftarrow i-1$. Terminate for $i = 0$ and $j = 0$.

Matrix-Chain-Multiplication

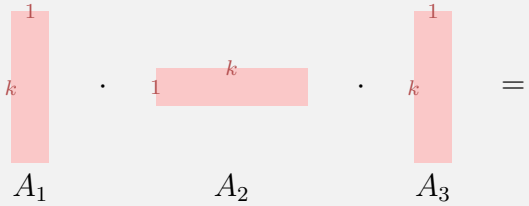
Task: Computation of the product $A_1 \cdot A_2 \cdot ... \cdot A_n$ of matrices $A_1, \ldots, A_n$.

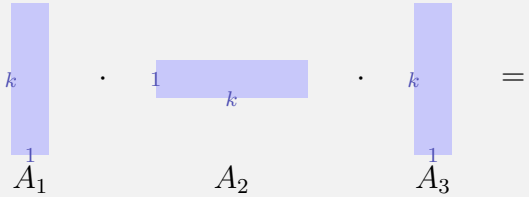
Matrix multiplication is associative, i.e. the order of evalution can be chosen arbitrarily

Goal: efficient computation of the product.

Assumption: multiplicaiton of an $(r \times s)$-matrix with an $(s \times u)$-matrix provides costs $r \cdot s \cdot u$.

Does it matter?





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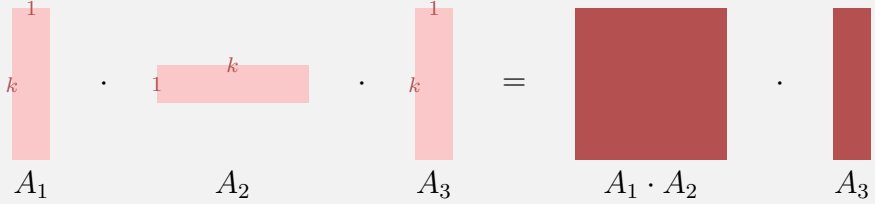

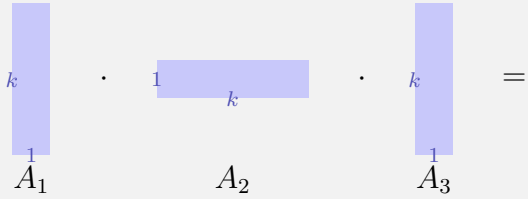

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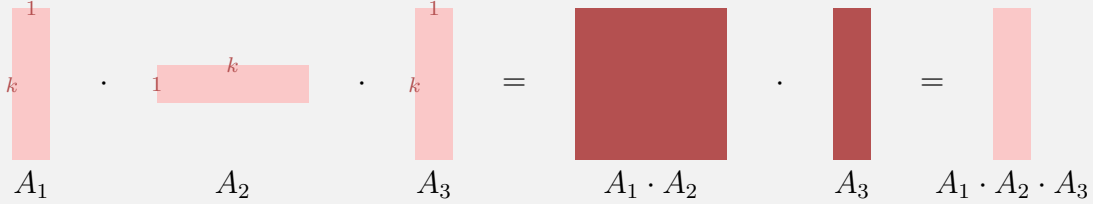

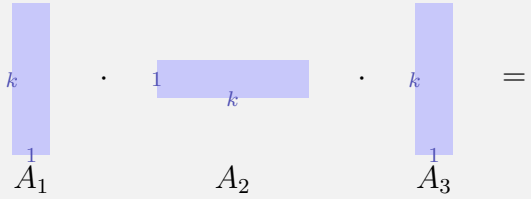

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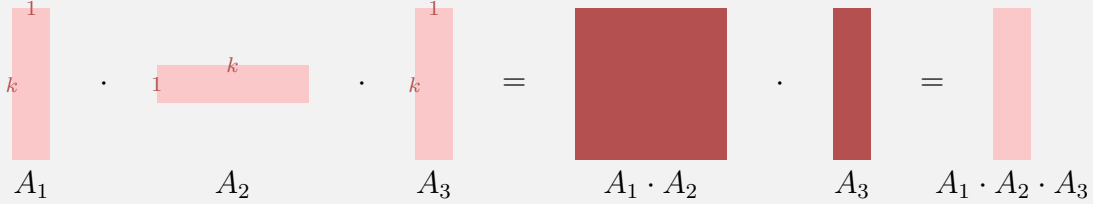

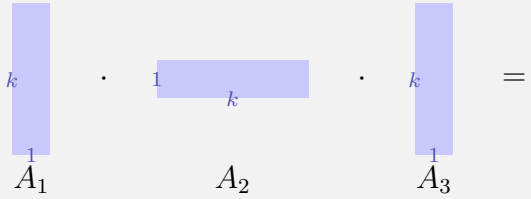

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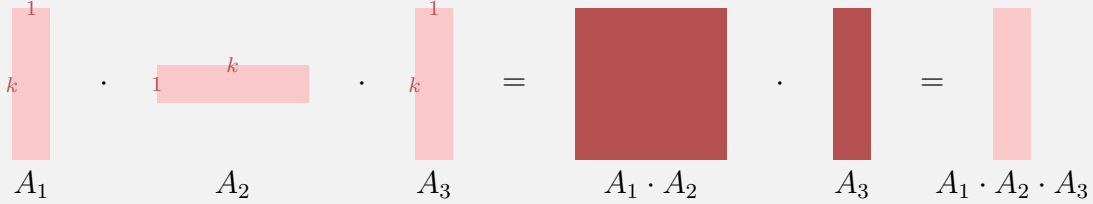
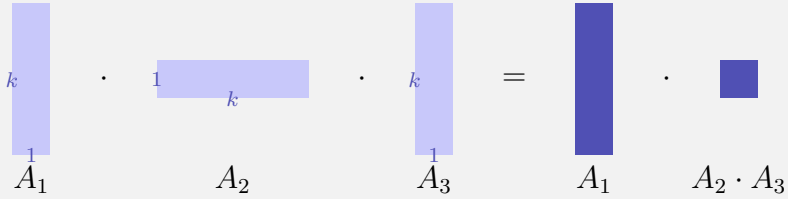
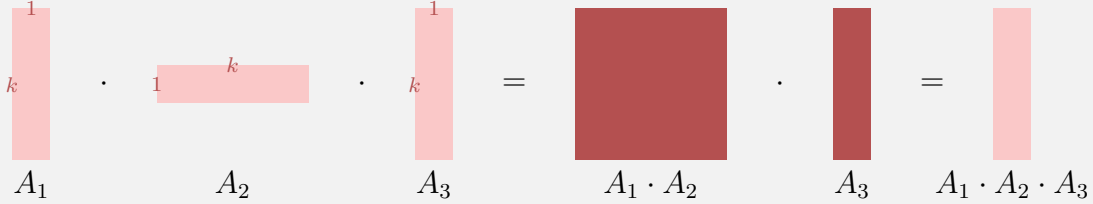
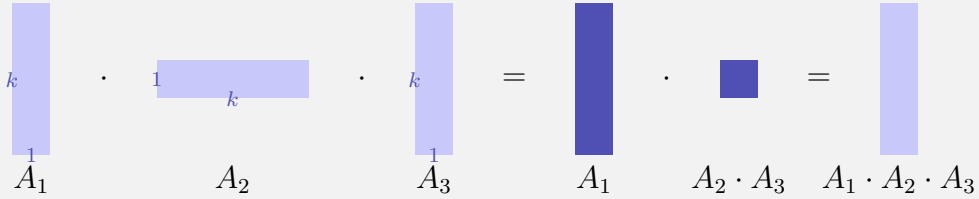




Does it matter?





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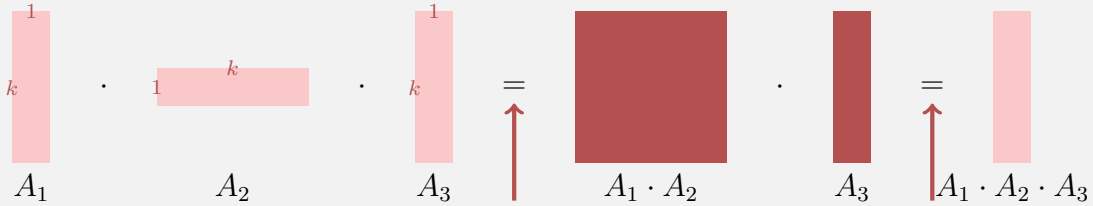
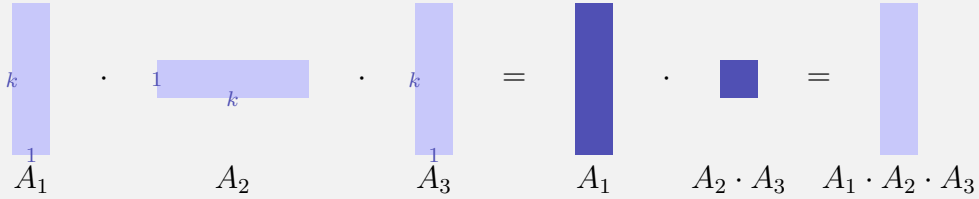
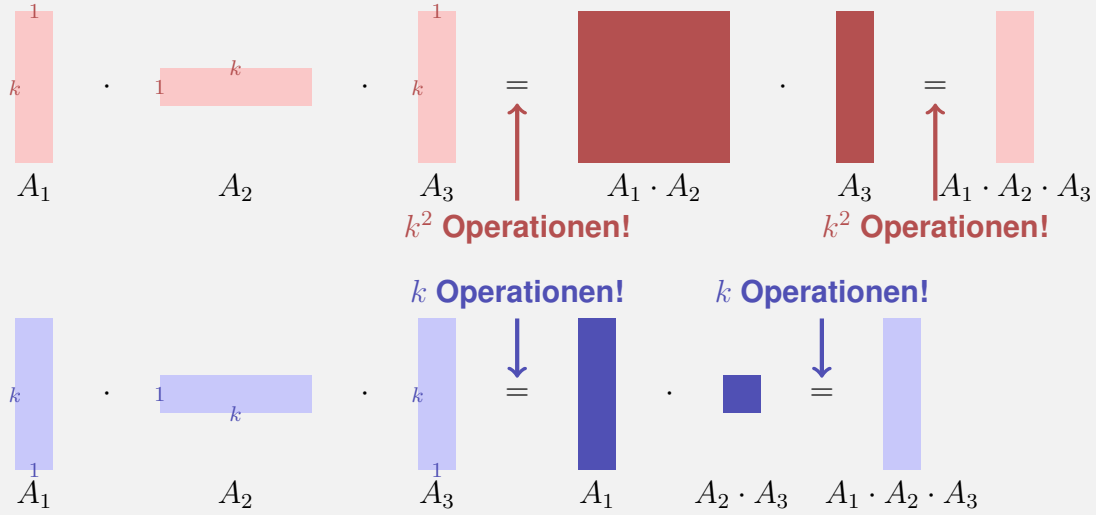




Does it matter?



Recursion

- Assume that the best possible computation of $(A_1 \cdot A_2 \cdots A_i)$ and $(A_{i+1} \cdot A_{i+2} \cdots A_n)$ is known for each i.
- Compute best i, done.

$n \times n$-table M. entry $M[p, q]$ provides costs of the best possible bracketing $(A_p \cdot A_{p+1} \cdots A_q)$.

$$M[p, q] \leftarrow \min_{p \leq i < q} (M[p, i] + M[i + 1, q] + \text{costs of the last multiplication})$$

Computation of the DP-table

- Base cases $M[p,p] \leftarrow 0$ for all $1 \leq p \leq n$.
- Computation of $M[p,q]$ depends on $M[i,j]$ with $p \leq i \leq j \leq q$, $(i,j) \neq (p,q)$.
  In particular $M[p,q]$ depends at most from entries $M[i,j]$ with $i - j < q - p$.
  Consequence: fill the table from the diagonal.

Analysis

DP-table has n^2 entries. Computation of an entry requires considering up to $n-1$ other entries.

Overal runtime $\mathcal{O}(n^3)$.

Readout the order from M: exercise!

Digression: matrix multiplication

Consider the mutliplicaiton of two $n \times n$ matrices.

Let

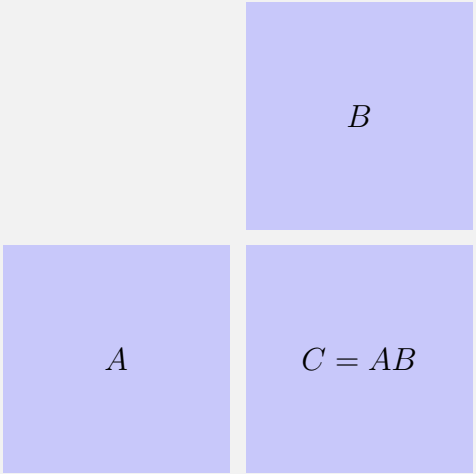
$$A = (a_{ij})_{1 \leq i,j \leq n}, B = (b_{ij})_{1 \leq i,j \leq n}, C = (c_{ij})_{1 \leq i,j \leq n},$$
$$C = A \cdot B$$

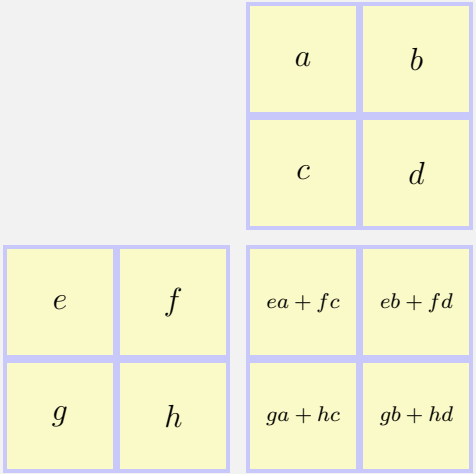
then

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Naive algorithm requires $\Theta(n^3)$ elementary multiplications.

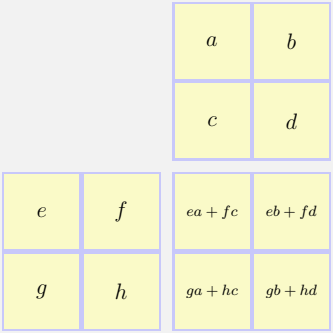
Divide and Conquer





Divide and Conquer

- Assumption $n = 2^k$.
- Number of elementary multiplications: $M(n) = 8M(n/2)$, $M(1) = 1$.
- yields $M(n) = 8^{\log_2 n} = n^{\log_2 8} = n^3$. No advantage ☹



Strassen's Matrix Multiplication

- Nontrivial observation by Strassen (1969):

  It suffices to compute the seven products

  $A = (e+h)\cdot(a+d)$, $B = (g+h)\cdot a$,

  $C = e\cdot(b-d)$, $D = h\cdot(c-a)$, $E = (e+f)\cdot d$,

  $F = (g-e)\cdot(a+b)$, $G = (f-h)\cdot(c+d)$. Denn:

  $ea+fc = A+D-E+G$, $eb+fd = C+E$,

  $ga+hc = B+D$, $gb+hd = A-B+C+F$.

- This yields $M'(n) = 7M(n/2), M'(1) = 1$.
  Thus $M'(n) = 7^{\log_2 n} = n^{\log_2 7} \approx n^{2.807}$.
- Fastest currently known algorithm:
  $\mathcal{O}(n^{2.37})$