Data Structures and Algorithms

Course at D-MATH (CSE) of ETH Zurich

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1. Introduction

Overview, Algorithms and Data Structures, Correctness, First Example

Goals of the course

- Understand the design and analysis of fundamental algorithms and data structures.
- An advanced insight into a modern programming model (with C++).
- Knowledge about chances, problems and limits of the parallel and concurrent computing.

Contents

data structures / algorithms

The notion invariant, cost model, Landau notation

algorithms design, induction

searching, selection and sorting

amortized analysis

dynamic programming

dictionaries: hashing and search trees

Fundamental algorithms on graphs,

shortest paths, Max-Flow

Minimum Spanning Trees, Fibonacci Heaps

van-Emde Boas Trees, Splay-Trees

prorgamming with C++

RAII, Move Konstruktion, Smart Pointers,

Templates and generic programming

Exceptions

functors and lambdas

threads, mutex and monitors

promises and futures

parallel programming

parallelism vs. concurrency, speedup (Amdahl/-Gustavson), races, memory reordering, atomir registers, RMW (CAS,TAS), deadlock/starvation

1.2 Algorithms

[Cormen et al, Kap. 1;Ottman/Widmayer, Kap. 1.1]

Algorithm

Algorithm: well defined computing procedure to compute *output* data from *input* data

example problem

Input: A sequence of n numbers $(a_1, a_2, \ldots, a_n)$

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Possible input

$(1, 7, 3)$, $(15, 13, 12, -0.5)$, (1) $\dots$

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$(1, 7, 3)$, $(15, 13, 12, -0.5)$, (1) ...

Every example represents a *problem instance*

The performance (speed) of an algorithm usually depends on the problem instance. Often there are “good” and “bad” instances.

Examples for algorithmic problems

- Tables and statistis: sorting, selection and searching

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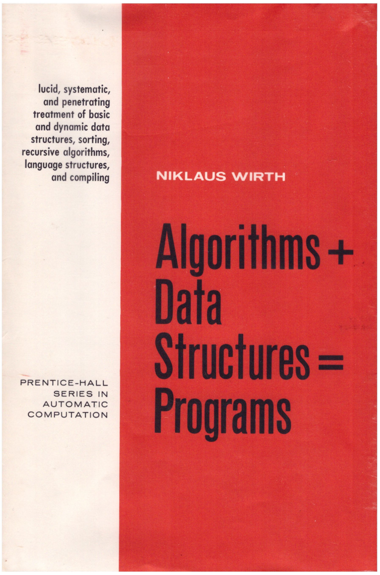
- Fast Lookup : Hash-Tables
- The travelling Salesman: Dynamic Programming, Minimum Spanning Tree, Simulated Annealing

Characteristics

- Extremely large number of potential solutions
- Practical applicability

Data Structures

- A data structure is a particular way of *organizing data* in a computer so that they can be *used efficiently* (in the algorithms operating on them).
- Programs = algorithms + data structures.



Efficiency

Illusion:

- If computers were infinitely fast and had an infinite amount of memory ...
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Reality: resources are bounded and not free:

- Computing time $\rightarrow$ Efficiency
- Storage space $\rightarrow$ Efficiency

Actually, this course is nearly only about efficiency.

Hard problems.

- NP-complete problems: no known efficient solution (the existence of such a solution is very improbable – but it has not yet been proven that there is none!)
- Example: travelling salesman problem

This course is *mostly* about problems that can be solved efficiently (in polynomial time).

2. Efficiency of algorithms

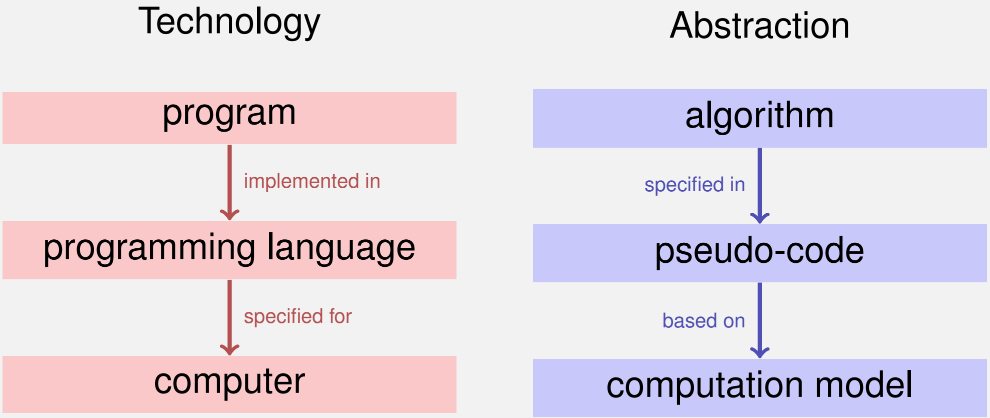
Efficiency of Algorithms, Random Access Machine Model, Function Growth, Asymptotics [Cormen et al, Kap. 2.2,3,4.2-4.4 | Ottman/Widmayer, Kap. 1.1]

Efficiency of Algorithms

Goals

- Quantify the runtime behavior of an algorithm independent of the machine.
- Compare efficiency of algorithms.
- Understand dependece on the input size.

Programs and Algorithms



Technology Model

Random Access Machine (RAM)

- Execution model: instructions are executed one after the other (on one processor core).

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- Unit cost model: fundamental operations provide a cost of 1.
- Data types: fundamental types like size-limited integer or floating point number.

Size of the Input Data

Typical: number of input objects (of fundamental type).

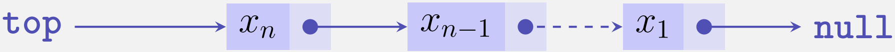
Sometimes: number bits for a *reasonable / cost-effective* representation of the data.

fundamental types fit into word of size : $w \geq \log(\text{sizeof(mem)})$ bits.

Pointer Machine Model

We assume

- Objects bounded in size can be dynamically allocated in constant time
- Fields (with word-size) of the objects can be accessed in constant time 1.



Asymptotic behavior

An exact running time of an algorithm can normally not be predicted even for small input data.

- We consider the asymptotic behavior of the algorithm.
- And ignore all constant factors.

Example

An operation with cost 20 is no worse than one with cost 1
Linear growth with gradient 5 is as good as linear growth with gradient 1.

2.2 Function growth

$\mathcal{O}$, Θ, Ω [Cormen et al, Kap. 3; Ottman/Widmayer, Kap. 1.1]

Superficially

Use the asymptotic notation to specify the execution time of algorithms.

We write $\Theta(n^2)$ and mean that the algorithm behaves for large n like n^2: when the problem size is doubled, the execution time multiplies by four.

More precise: asymptotic upper bound

provided: a function $g : \mathbb{N} \to \mathbb{R}$.

Definition:[1]

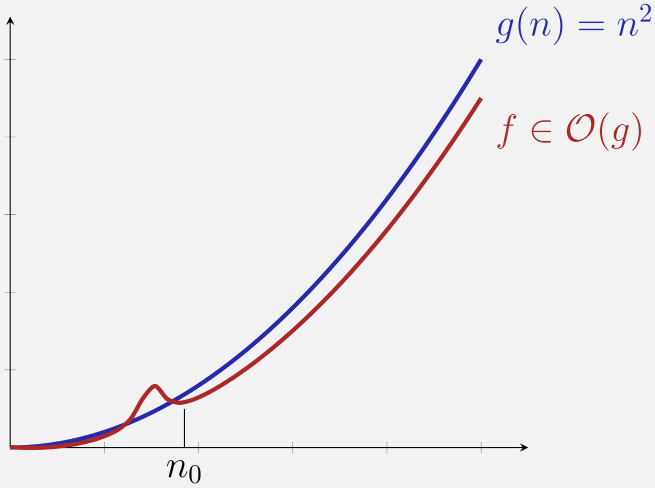
$$\begin{aligned}\mathcal{O}(g) = \{f : \mathbb{N} \to \mathbb{R} \mid \\ \exists\, c > 0, \exists n_0 \in \mathbb{N} : \\ \forall\, n \geq n_0 : 0 \leq f(n) \leq c \cdot g(n)\}\end{aligned}$$

Notation:

$$\mathcal{O}(g(n)) := \mathcal{O}(g(\cdot)) = \mathcal{O}(g).$$

[1] Ausgesprochen: Set of all functions $f : \mathbb{N} \to \mathbb{R}$ that satisfy: there is some (real valued) $c > 0$ and some $n_0 \in \mathbb{N}$ such that $0 \leq f(n) \leq n \cdot g(n)$ for all $n \geq n_0$.

Graphic



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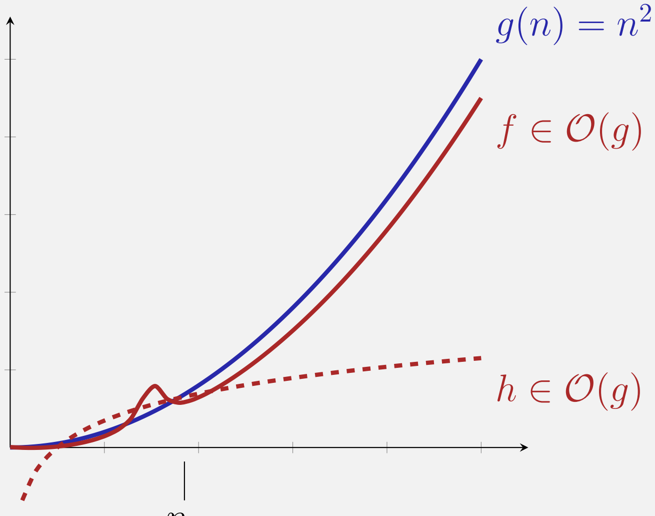

Examples

$$\mathcal{O}(g) = \{f : \mathbb{N} \to \mathbb{R} |\ \exists c > 0, \exists n_0 \in \mathbb{N} : \forall n \geq n_0 : 0 \leq f(n) \leq c \cdot g(n)\}$$

$f(n)$	$f \in \mathcal{O}(?)$	Example
$3n + 4$		
$2n$		
$n^2 + 100n$		
$n + \sqrt{n}$		

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$n + \sqrt{n}$		

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$n^2 + 100n$	$\mathcal{O}(n^2)$	$c = 2, n_0 = 100$
$n + \sqrt{n}$		

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$n^2 + 100n$	$\mathcal{O}(n^2)$	$c = 2, n_0 = 100$
$n + \sqrt{n}$	$\mathcal{O}(n)$	$c = 2, n_0 = 1$

Property

$$f_1 \in \mathcal{O}(g), f_2 \in \mathcal{O}(g) \Rightarrow f_1 + f_2 \in \mathcal{O}(g)$$

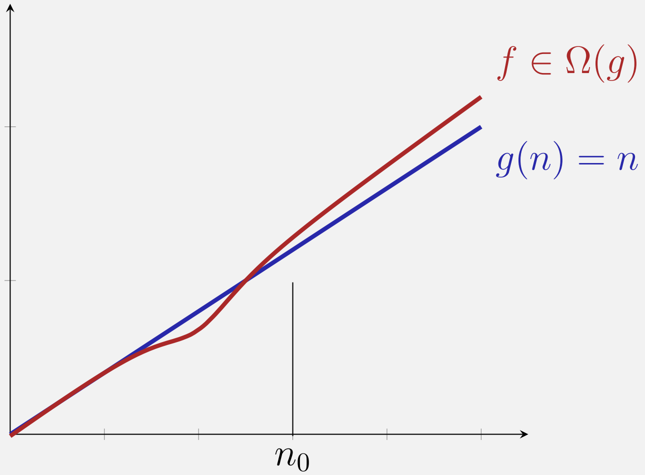
Converse: asymptotic lower bound

Given: a function $g : \mathbb{N} \to \mathbb{R}$.

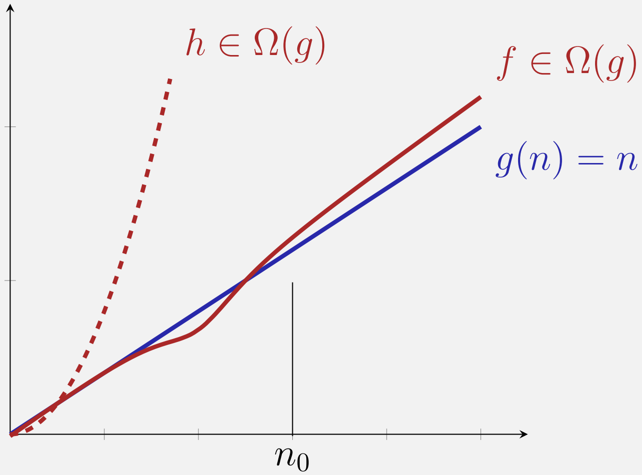
Definition:

$$
\begin{aligned}
\Omega(g) = \{f : \mathbb{N} \to \mathbb{R} | \\
& \exists\, c > 0, \exists n_0 \in \mathbb{N} : \\
& \forall\, n \geq n_0 : 0 \leq c \cdot g(n) \leq f(n)\}
\end{aligned}
$$

Example



Example



Asymptotic tight bound

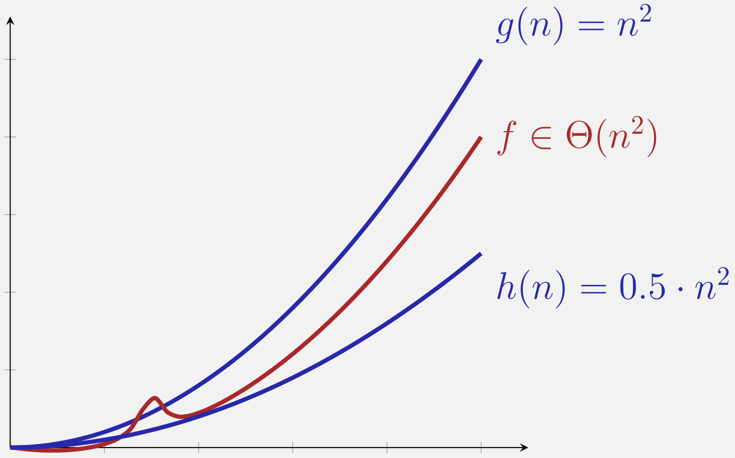
Given: function $g : \mathbb{N} \to \mathbb{R}$.

Definition:

$$\Theta(g) := \Omega(g) \cap \mathcal{O}(g).$$

Simple, closed form: exercise.

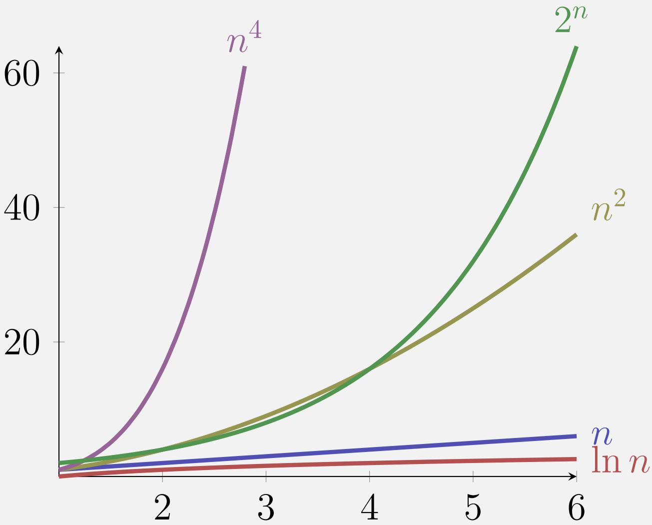
Example



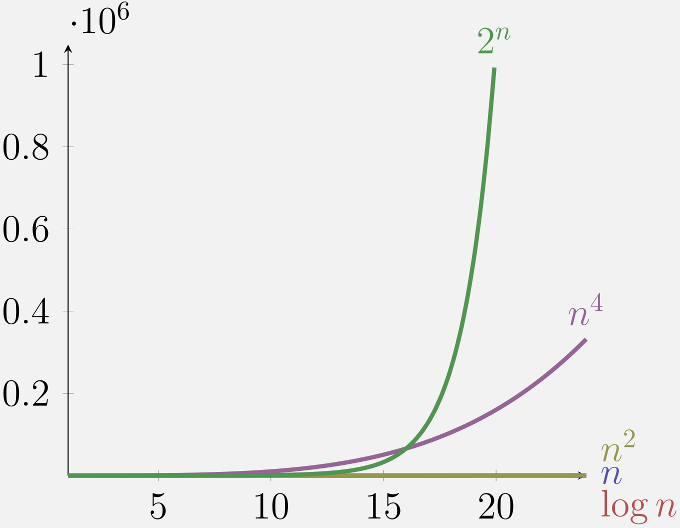
Notions of Growth

$\mathcal{O}(1)$	bounded	array access
$\mathcal{O}(\log \log n)$	double logarithmic	interpolated binary sorted sort
$\mathcal{O}(\log n)$	logarithmic	binary sorted search
$\mathcal{O}(\sqrt{n})$	like the square root	naive prime number test
$\mathcal{O}(n)$	linear	unsorted naive search
$\mathcal{O}(n \log n)$	superlinear / loglinear	good sorting algorithms
$\mathcal{O}(n^2)$	quadratic	simple sort algorithms
$\mathcal{O}(n^c)$	polynomial	matrix multiply
$\mathcal{O}(2^n)$	exponential	Travelling Salesman Dynamic Programming
$\mathcal{O}(n!)$	factorial	Travelling Salesman naively

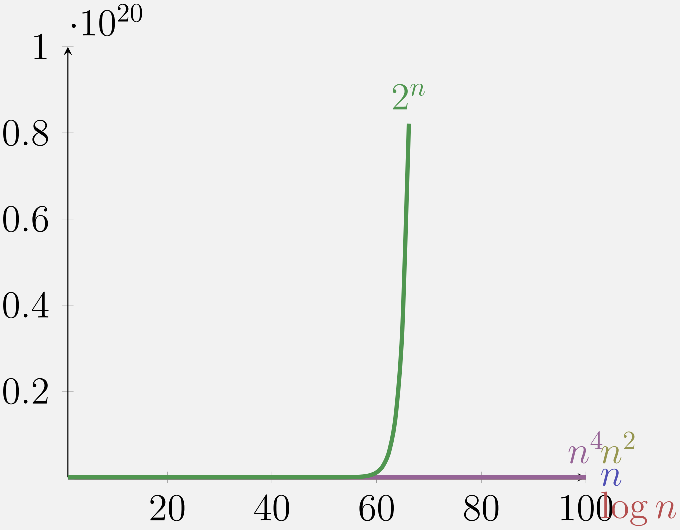
Small n



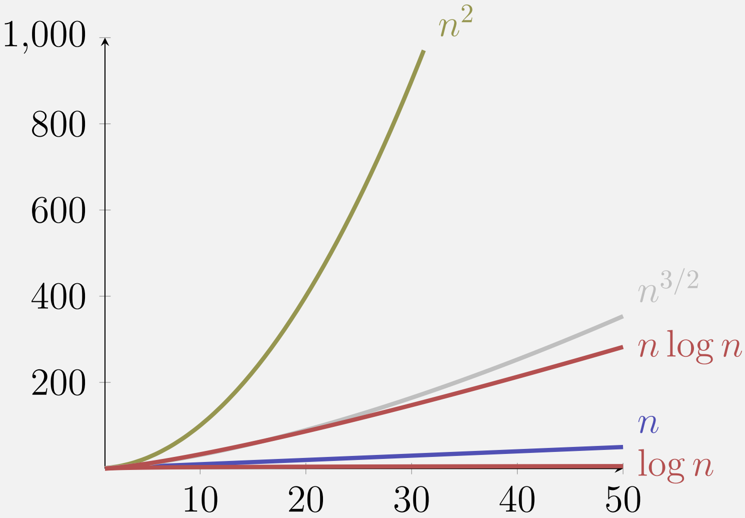
Larger n



“Large” n



Logarithms



Time Consumption

Assumption 1 Operation = $1\mu s$.

problem size	1	100	10000	10^6	10^9
$\log_2 n$	$1\mu s$				
n	$1\mu s$				
$n \log_2 n$	$1\mu s$				
n^2	$1\mu s$				
2^n	$1\mu s$				

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n	$1\mu s$	$100\mu s$	$1/100s$	$1s$	17 minutes
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n^2	$1\mu s$	$1/100s$	1.7 minutes	11.5 days	317 centuries
2^n	$1\mu s$				

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problem size	1	100	10000	10^6	10^9
$\log_2 n$	$1\mu s$	$7\mu s$	$13\mu s$	$20\mu s$	$30\mu s$
n	$1\mu s$	$100\mu s$	$1/100s$	$1s$	17 minutes
$n \log_2 n$	$1\mu s$				
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$n \log_2 n$	$1\mu s$	$700\mu s$	$13/100\mu s$	$20s$	8.5 hours
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n^2	$1\mu s$	$1/100s$	1.7 minutes	11.5 days	317 centuries
2^n	$1\mu s$	10^{14} centuries	$\approx \infty$	$\approx \infty$	$\approx \infty$

About the Notation

Common casual notation

$$f = \mathcal{O}(g)$$

should be read as $f \in \mathcal{O}(g)$.

Clearly it holds that

$$f_1 = \mathcal{O}(g), f_2 = \mathcal{O}(g) \not\Rightarrow f_1 = f_2!$$

Beispiel

$n = \mathcal{O}(n^2), n^2 = \mathcal{O}(n^2)$ but naturally $n \neq n^2$.

We avoid this notation where it could lead to ambiguities.

Reminder: Efficiency: Arrays vs. Linked Lists

- Memory: our `avec` requires roughly n `int`s (vector size n), our `llvec` roughly $3n$ `int`s (a pointer typically requires 8 byte)
- Runtime (with `avec` = `std::vector`, `llvec` = `std::list`):

```
prepending (insert at front) [100,000x]:
    ► avec:     675 ms
    ► llvec:     10 ms
appending (insert at back) [100,000x]:
    ► avec:       2 ms
    ► llvec:      9 ms
removing first [100,000x]:
    ► avec:     675 ms
    ► llvec:      4 ms
removing last [100,000x]:
    ► avec:       0 ms
    ► llvec:      4 ms
removing randomly [10,000x]:
    ► avec:       3 ms
    ► llvec:    113 ms
inserting randomly [10,000x]:
    ► avec:      16 ms
    ► llvec:    117 ms
fully iterate sequentially (5000 elements) [5,000x]:
    ► avec:     354 ms
    ► llvec:    525 ms
```

Asymptotic Runtimes

With our new language ($\Omega, \mathcal{O}, \Theta$), we can now *state the behavior of the data structures and their algorithms more precisely*

Typical asymptotic running times (Anticipation!)

Data structure	Random Access	Insert	Next	Insert After Element	Search
`std::vector`	$\Theta(1)$	$\Theta(1)\ A$	$\Theta(1)$	$\Theta(n)$	$\Theta(n)$
`std::list`	$\Theta(n)$	$\Theta(1)$	$\Theta(1)$	$\Theta(1)$	$\Theta(n)$
`std::set`	–	$\Theta(\log n)$	$\Theta(\log n)$	–	$\Theta(\log n)$
`std::unordered_set`	–	$\Theta(1)\ P$	–	–	$\Theta(1)\ P$

A = amortized, P=expected, otherwise worst case

Complexity

Complexity of a problem P: minimal (asymptotic) costs over all algorithms A that solve P.

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Complexity of the single-digit multiplication of two numbers with n digits is $\Omega(n)$ and $\mathcal{O}(n^{\log_3 2})$ (Karatsuba Ofman).

Complexity

Example:

Problem	Complexity	$\mathcal{O}(n)$	$\mathcal{O}(n)$	$\mathcal{O}(n^2)$
		$\uparrow$	$\uparrow$	$\uparrow$
Algorithm	Costs[2]	$3n-4$	$\mathcal{O}(n)$	$\Theta(n^2)$
		$\downarrow$	$\updownarrow$	$\updownarrow$
Program	Execution time	$\Theta(n)$	$\mathcal{O}(n)$	$\Theta(n^2)$

[2]Number funamental operations