10. Sorting III

Lower bounds for the comparison based sorting, radix- and bucket-sort

10.1 Lower bounds for comparison based sorting

[Ottman/Widmayer, Kap. 2.8, Cormen et al, Kap. 8.1]

Lower bound for sorting

Up to here: worst case sorting takes $\Omega(n \log n)$ steps.

Is there a better way?

Lower bound for sorting

Up to here: worst case sorting takes $\Omega(n \log n)$ steps.

Is there a better way? No:

Theorem

Sorting procedures that are based on comparison require in the worst case and on average at least $\Omega(n \log n)$ key comparisons.

Comparison based sorting

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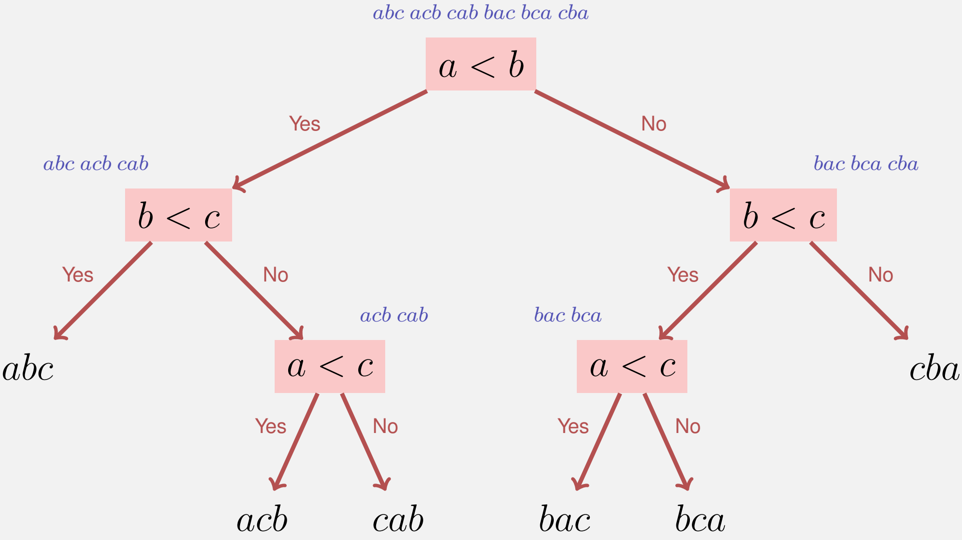
  - Nodes contain the remaining possibilities.
  - Edges contain the decisions.

Decision tree



Decision tree

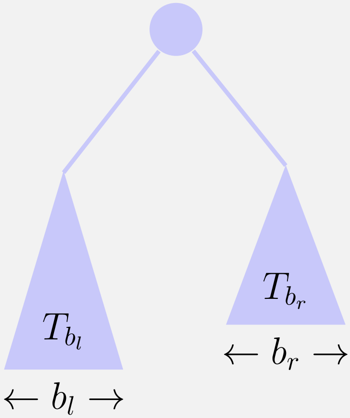
The height of a binary tree with L leaves is at least $\log_2 L$. $\Rightarrow$ The heigh of the decision tree $h \geq \log n! \in \Omega(n \log n)$.[12]

Thus the length of the longest path in the decision tree $\in \Omega(n \log n)$.

Remaining to show: mean length $M(n)$ of a path $M(n) \in \Omega(n \log n)$.

[12] $\log n! \in \Theta(n \log n)$:
$\log n! = \sum_{k=1}^{n} \log k \leq n \log n$.
$\log n! = \sum_{k=1}^{n} \log k \geq \sum_{k=n/2}^{n} \log k \geq \frac{n}{2} \cdot \log \frac{n}{2}$.

Average lower bound



- Decision tree T_n with n leaves, average height of a leaf $m(T_n)$
- Assumption $m(T_n) \geq \log n$ not for all n.
- Choose smalles b with $m(T_b) < \log n \Rightarrow b \geq 2$
- $b_l + b_r = b$, wlog $b_l > 0$ und $b_r > 0 \Rightarrow$ $b_l < b, b_r < b \Rightarrow m(T_{b_l}) \geq \log b_l$ und $m(T_{b_r}) \geq \log b_r$

Average lower bound

Average height of a leaf:

$$\begin{aligned} m(T_b) &= \frac{b_l}{b}(m(T_{b_l})+1) + \frac{b_r}{b}(m(T_{b_r})+1) \\ &\geq \frac{1}{b}(b_l(\log b_l + 1) + b_r(\log b_r + 1)) = \frac{1}{b}(b_l \log 2b_l + b_r \log 2b_r) \\ &\geq \frac{1}{b}(b \log b) = \log b. \end{aligned}$$

Contradiction. ■

The last inequality holds because $f(x) = x \log x$ is convex and for a convex function it holds that $f((x+y)/2) \leq 1/2 f(x) + 1/2 f(y)$ ($x = 2b_l$, $y = 2b_r$).[13]

Enter $x = 2b_l$, $y = 2b_r$, and $b_l + b_r = b$.

[13] generally $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$ for $0 \leq \lambda \leq 1$.

10.2 Radixsort and Bucketsort

Radixsort, Bucketsort [Ottman/Widmayer, Kap. 2.5, Cormen et al, Kap. 8.3]

Radix Sort

Sorting based on comparison: comparable keys ($<$ or $>$, often $=$).
No further assumptions.

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Different idea: use more information about the keys.

Annahmen

Assumption: keys representable as words from an alphabet containing m elements.

m is called the radix of the representation.

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Examples

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$m = 10$ decimal numbers $183 = 183_{10}$

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Examples

$m = 10$	decimal numbers	$183 = 183_{10}$
$m = 2$	dual numbers	101_2
$m = 16$	hexadecimal numbers	$A0_{16}$

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Assumption: keys representable as words from an alphabet containing m elements.

Examples

$m = 10$	decimal numbers	$183 = 183_{10}$
$m = 2$	dual numbers	101_2
$m = 16$	hexadecimal numbers	$A0_{16}$
$m = 26$	words	‘‘INFORMATIK’’

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Assumptions

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Example

$$z_{10}(0, 85) = 5$$
$$z_{10}(1, 85) = 8$$
$$z_{10}(2, 85) = 0$$

Radix-Exchange-Sort

Keys with radix 2.

Observation: if $k \geq 0$,

$$z_2(i, x) = z_2(i, y) \text{ for all } i > k$$

and

$$z_2(k, x) < z_2(k, y),$$

then $x < y$.

Radix-Exchange-Sort

Idea:

- Start with a maximal k.
- Binary partition the data sets with $z_2(k, \cdot) = 0$ vs. $z_2(k, \cdot) = 1$ like with quicksort.
- $k \leftarrow k - 1$.

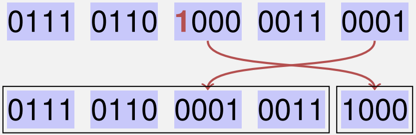
Radix-Exchange-Sort

0111 0110 1000 0011 0001

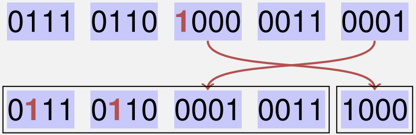
Radix-Exchange-Sort

0111 0110 **1**000 0011 0001

Radix-Exchange-Sort



Radix-Exchange-Sort



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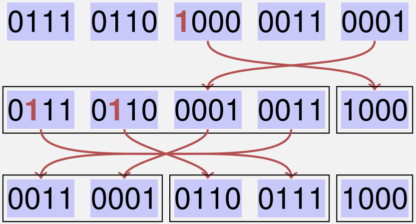

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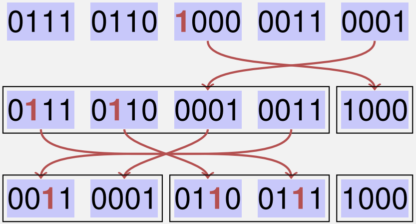
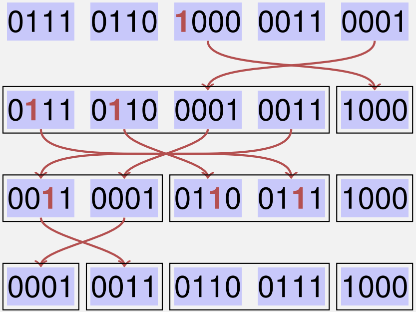


Radix-Exchange-Sort



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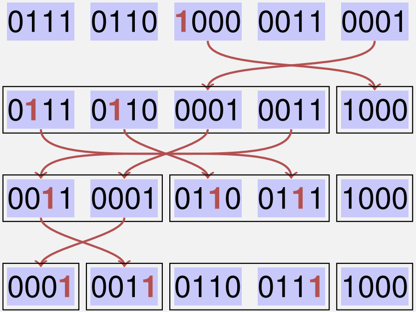

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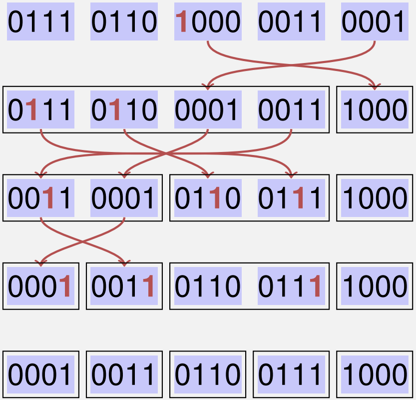


Algorithm RadixExchangeSort(A, l, r, b)

Input : Array A with length n, left and right bounds $1 \leq l \leq r \leq n$, bit position b
Output : Array A, sorted in the domain $[l, r]$ by bits $[0, \ldots, b]$.

```
if l > r and b ≥ 0 then
    i ← l − 1
    j ← r + 1
    repeat
        repeat i ← i + 1 until z_2(b, A[i]) = 1 and i ≥ j
        repeat j ← j + 1 until z_2(b, A[j]) = 0 and i ≥ j
        if i < j then swap(A[i], A[j])
    until i ≥ j
    RadixExchangeSort(A, l, i − 1, b − 1)
    RadixExchangeSort(A, i, r, b − 1)
```

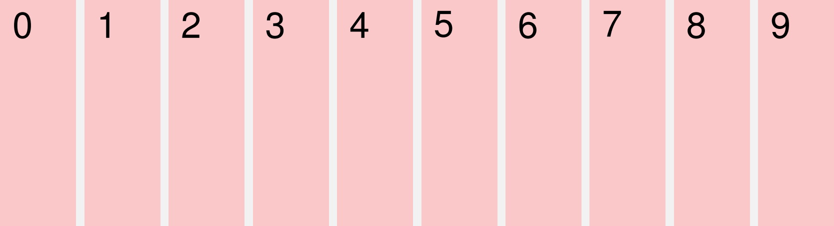
Analysis

RadixExchangeSort provide recursion with maximal recursion depth = maximal number of digits p.

Worst case run time $\mathcal{O}(p \cdot n)$.

Bucket Sort

3 8 18 122 121 131 23 21 19 29



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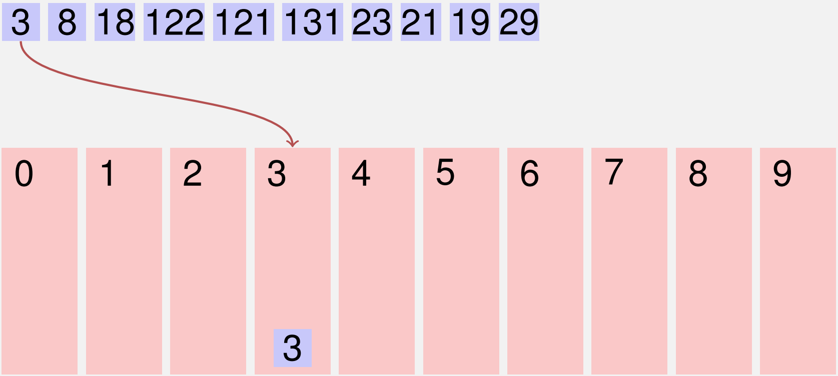

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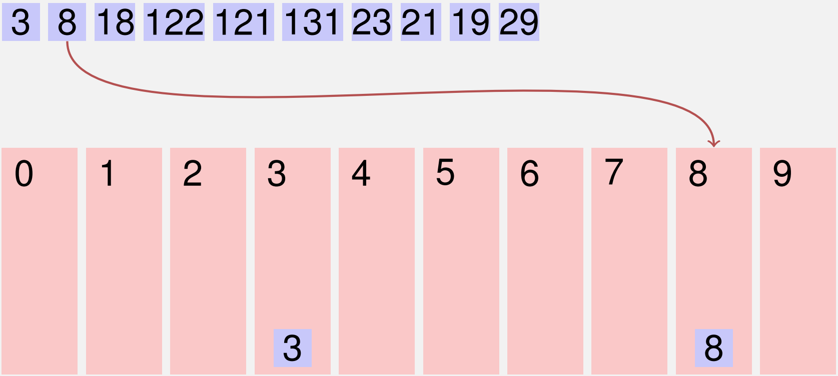

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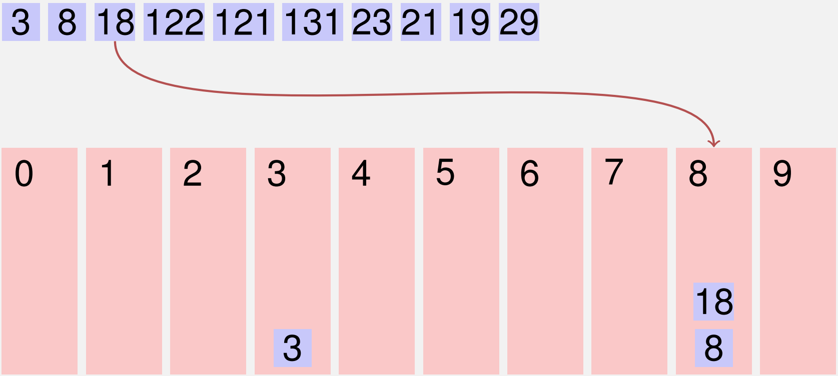

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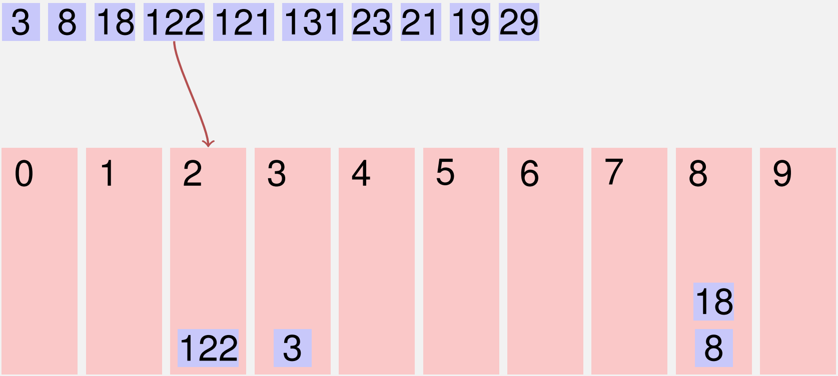

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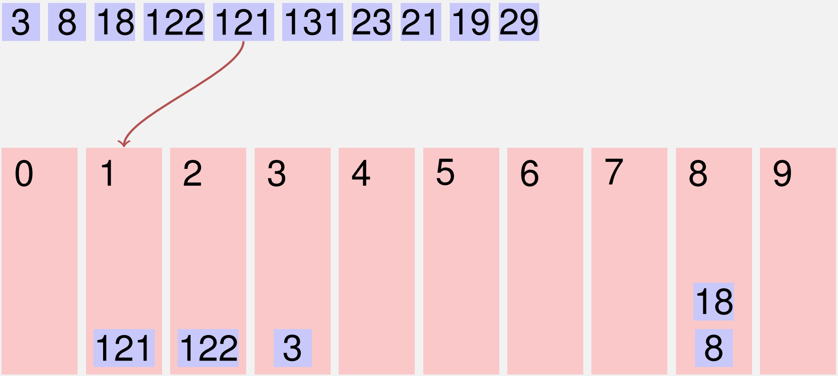

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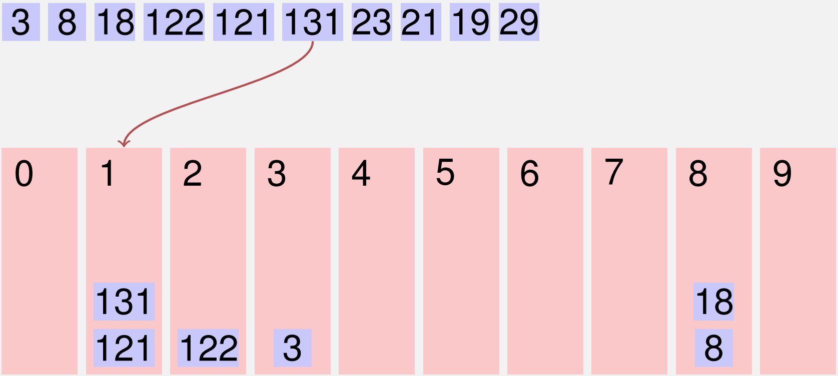

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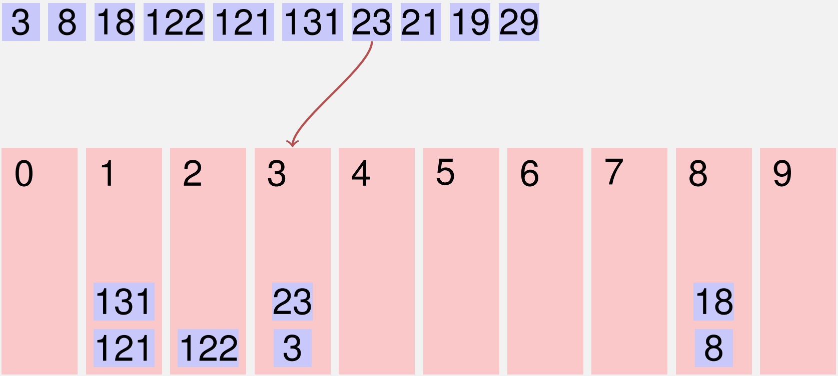

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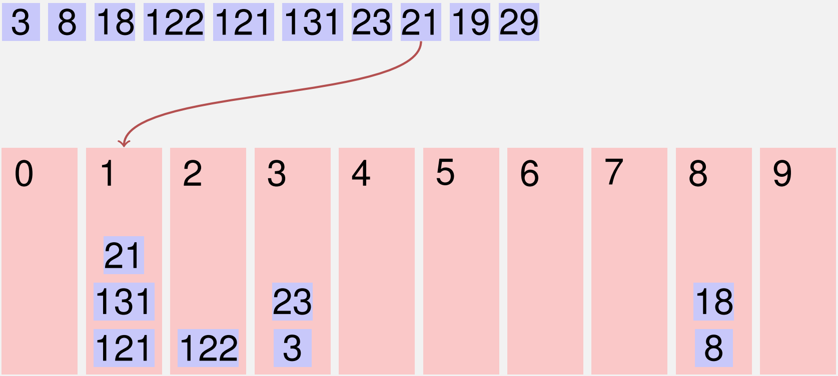

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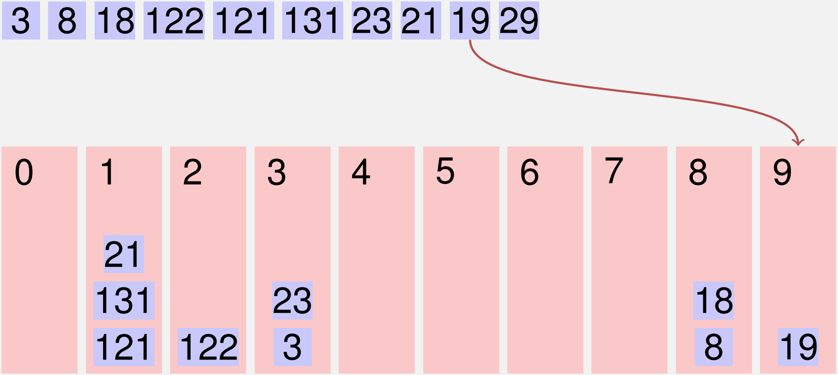

3 8 18 122 121 131 23 21 19 29
0
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18
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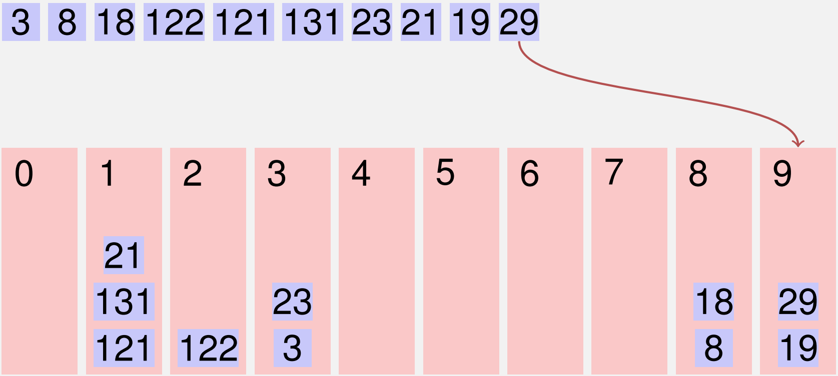
9
19

Bucket Sort



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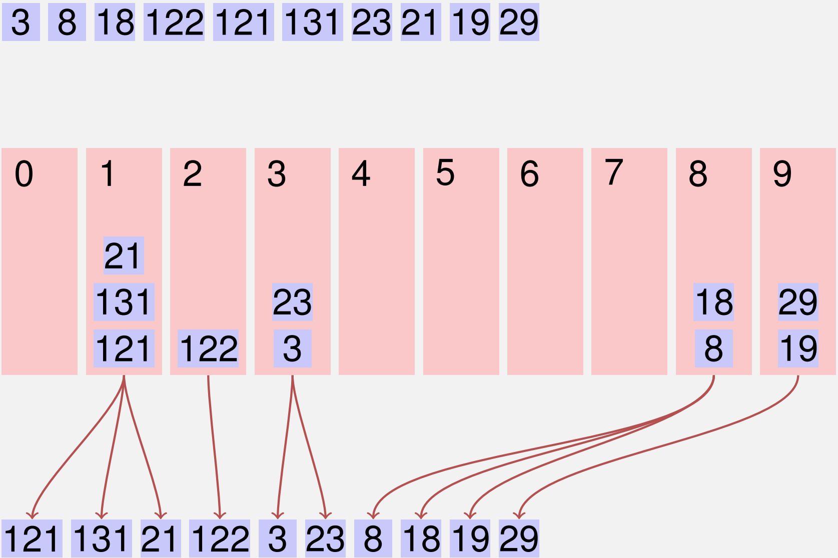

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121 131 21 122 3 23 8 18 19 29

Bucket Sort

121	131	21	122	3	23	8	18	19	29

Bucket Sort

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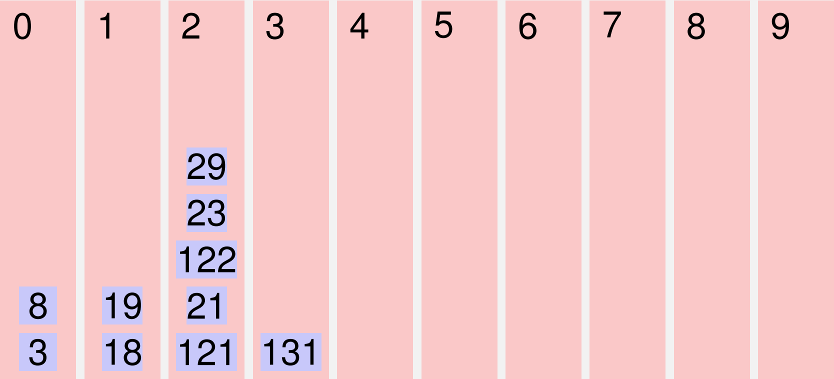

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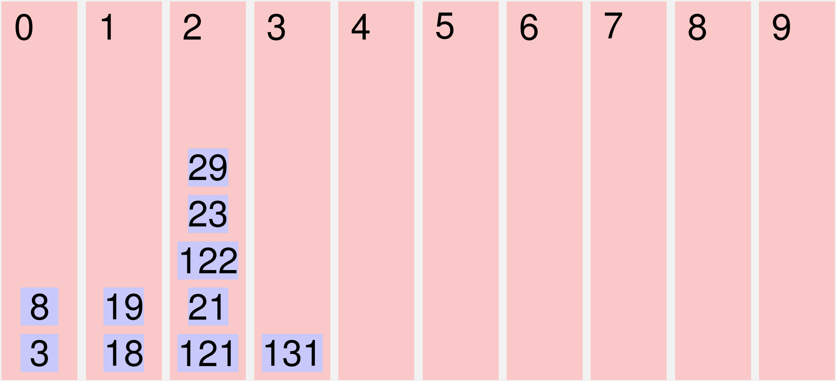

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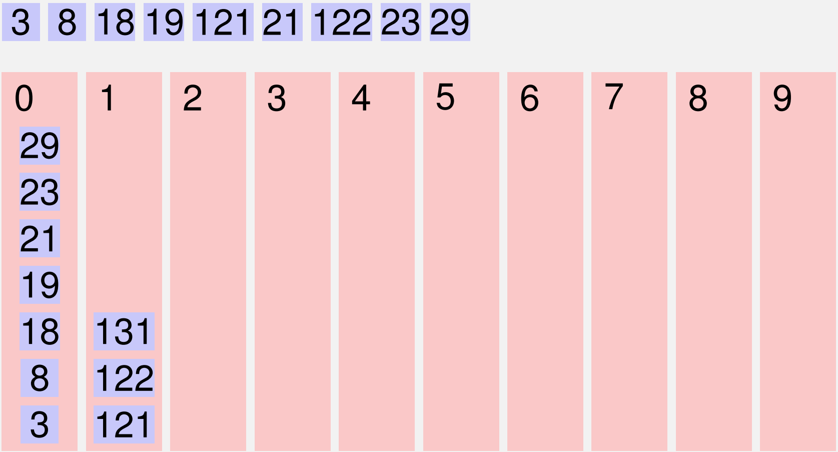

Bucket Sort

3 8 18 19 121 21 122 23 29

0	1	2	3	4	5	6	7	8	9
29									
23									
21									
19									
18	131								
8	122								
3	121								

3 8 18 19 21 23 29 121 122 131

implementation details

Bucket size varies greatly. Two possibilities

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- Linked list for each digit.

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- Linked list for each digit.
- One array of length n. compute offsets for each digit in the first iteration.