10. Sorting III

Lower bounds for the comparison based sorting, radix- and bucket-sort

10.1 Lower bounds for comparison based sorting

[Ottman/Widmayer, Kap. 2.8, Cormen et al, Kap. 8.1]

Lower bound for sorting

Up to here: worst case sorting takes $\Omega(n \log n)$ steps.

Is there a better way? No:

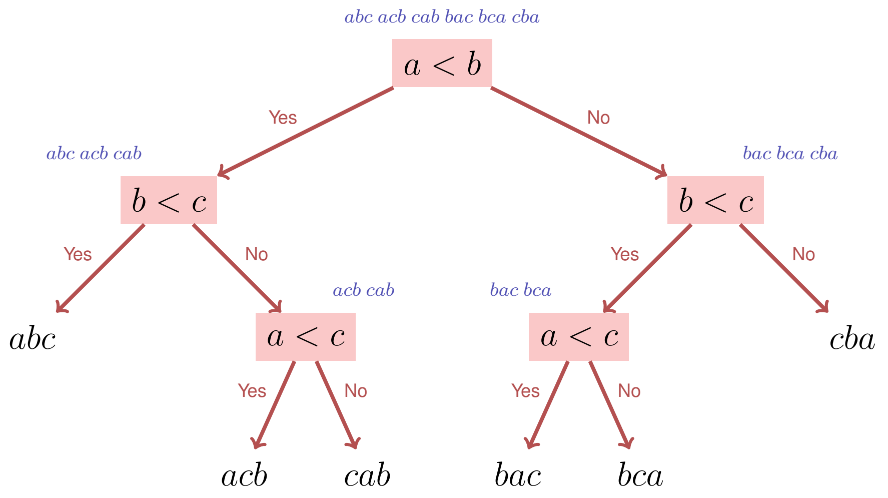
Theorem

Sorting procedures that are based on comparison require in the worst case and on average at least $\Omega(n \log n)$ key comparisons.

Comparison based sorting

- An algorithm must identify the correct one of $n!$ permutations of an array $(A_i)_{i=1,\dots,n}$.
- At the beginning the algorithm know nothing about the array structure.
- We consider the knowledge gain of the algorithm in the form of a decision tree:
  - Nodes contain the remaining possibilities.
  - Edges contain the decisions.

Decision tree



Decision tree

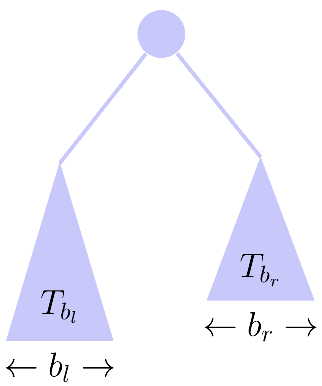
The height of a binary tree with L leaves is at least $\log_2 L$. $\Rightarrow$ The heigh of the decision tree $h \geq \log n! \in \Omega(n \log n)$.[12]

Thus the length of the longest path in the decision tree $\in \Omega(n \log n)$.

Remaining to show: mean length $M(n)$ of a path $M(n) \in \Omega(n \log n)$.

[12] $\log n! \in \Theta(n \log n)$:
$\log n! = \sum_{k=1}^{n} \log k \leq n \log n$.
$\log n! = \sum_{k=1}^{n} \log k \geq \sum_{k=n/2}^{n} \log k \geq \frac{n}{2} \cdot \log \frac{n}{2}$.

Average lower bound



- Decision tree T_n with n leaves, average height of a leaf $m(T_n)$
- Assumption $m(T_n) \geq \log n$ not for all n.
- Choose smalles b with $m(T_b) < \log n \Rightarrow b \geq 2$
- $b_l + b_r = b$, wlog $b_l > 0$ und $b_r > 0 \Rightarrow$ $b_l < b, b_r < b \Rightarrow m(T_{b_l}) \geq \log b_l$ und $m(T_{b_r}) \geq \log b_r$

Average lower bound

Average height of a leaf:

$$\begin{aligned} m(T_b) &= \frac{b_l}{b}(m(T_{b_l}) + 1) + \frac{b_r}{b}(m(T_{b_r}) + 1) \\ &\geq \frac{1}{b}(b_l(\log b_l + 1) + b_r(\log b_r + 1)) = \frac{1}{b}(b_l \log 2b_l + b_r \log 2b_r) \\ &\geq \frac{1}{b}(b \log b) = \log b. \end{aligned}$$

Contradiction. ■

The last inequality holds because $f(x) = x \log x$ is convex and for a convex function it holds that $f((x+y)/2) \leq 1/2 f(x) + 1/2 f(y)$ ($x = 2b_l$, $y = 2b_r$).[13] Enter $x = 2b_l$, $y = 2b_r$, and $b_l + b_r = b$.

[13] generally $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$ for $0 \leq \lambda \leq 1$.

10.2 Radixsort and Bucketsort

Radixsort, Bucketsort [Ottman/Widmayer, Kap. 2.5, Cormen et al, Kap. 8.3]

Radix Sort

Sorting based on comparison: comparable keys ($<$ or $>$, often $=$).
No further assumptions.

Different idea: use more information about the keys.

Annahmen

Assumption: keys representable as words from an alphabet containing m elements.

Examples

$m = 10$	decimal numbers	$183 = 183_{10}$
$m = 2$	dual numbers	101_2
$m = 16$	hexadecimal numbers	$A0_{16}$
$m = 26$	words	“INFORMATIK”

m is called the radix of the representation.

Assumptions

- keys = m-adic numbers with same length.
- Procedure z for the extraction of digit k in $\mathcal{O}(1)$ steps.

Example

$z_{10}(0, 85) = 5$
$z_{10}(1, 85) = 8$
$z_{10}(2, 85) = 0$

Radix-Exchange-Sort

Keys with radix 2.

Observation: if $k \geq 0$,

$$z_2(i, x) = z_2(i, y) \text{ for all } i > k$$

and

$$z_2(k, x) < z_2(k, y),$$

then $x < y$.

Radix-Exchange-Sort

Idea:

- Start with a maximal k.
- Binary partition the data sets with $z_2(k, \cdot) = 0$ vs. $z_2(k, \cdot) = 1$ like with quicksort.
- $k \leftarrow k - 1$.

Radix-Exchange-Sort

0111 0110 1000 0011 0001

0111 0110 0001 0011 1000

0011 0001 0110 0111 1000

0001 0011 0110 0111 1000

0001 0011 0110 0111 1000

Algorithm RadixExchangeSort(A, l, r, b)

Input : Array A with length n, left and right bounds $1 \leq l \leq r \leq n$, bit position b

Output : Array A, sorted in the domain $[l, r]$ by bits $[0, \ldots, b]$.

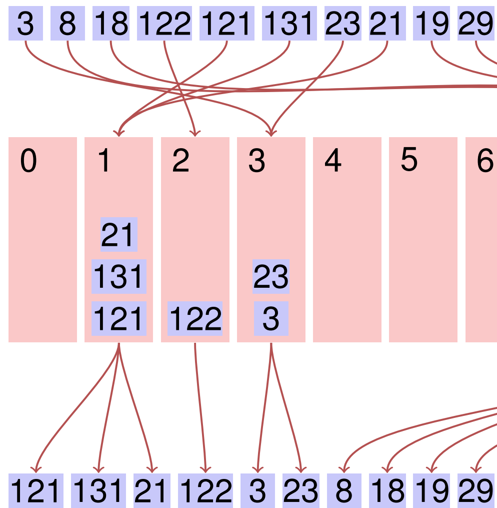
if $l > r$ **and** $b \geq 0$ **then**
    $i \leftarrow l - 1$
    $j \leftarrow r + 1$
    repeat
        repeat $i \leftarrow i + 1$ **until** $z_2(b, A[i]) = 1$ **and** $i \geq j$
        repeat $j \leftarrow j + 1$ **until** $z_2(b, A[j]) = 0$ **and** $i \geq j$
        if $i < j$ **then** swap$(A[i], A[j])$
    until $i \geq j$
    RadixExchangeSort$(A, l, i - 1, b - 1)$
    RadixExchangeSort$(A, i, r, b - 1)$

Analysis

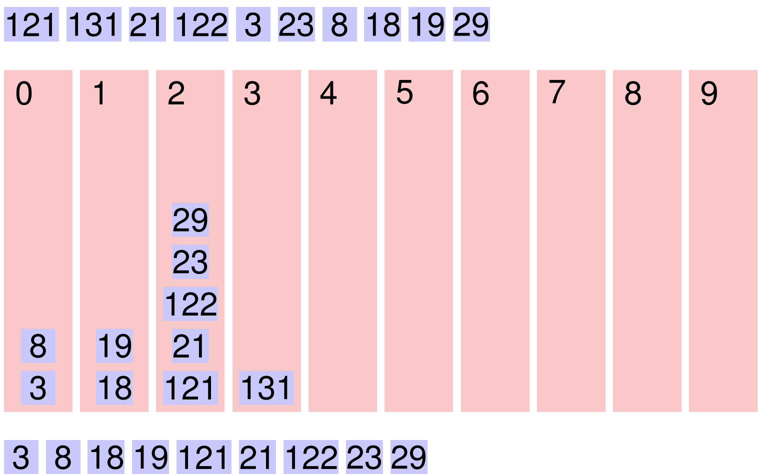
RadixExchangeSort provide recursion with maximal recursion depth = maximal number of digits p.

Worst case run time $\mathcal{O}(p \cdot n)$.

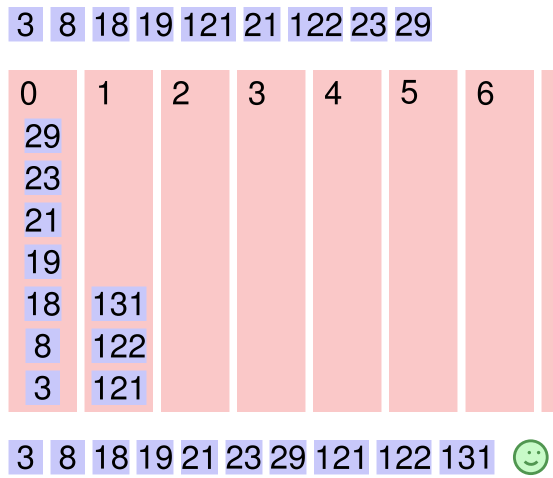
Bucket Sort



Bucket Sort



Bucket Sort



implementation details

Bucket size varies greatly. Two possibilities

- Linked list for each digit.
- One array of length n. compute offsets for each digit in the first iteration.