7. Sorting I

Simple Sorting

7.1 Simple Sorting

Selection Sort, Insertion Sort, Bubblesort [Ottman/Widmayer, Kap. 2.1, Cormen et al, Kap. 2.1, 2.2, Exercise 2.2-2, Problem 2-2

Problem

Input: An array $A = (A[1], ..., A[n])$ with length n.

Output: a permutation A' of A, that is sorted: $A'[i] \leq A'[j]$ for all $1 \leq i \leq j \leq n$.

Algorithm: IsSorted(A)

Input : Array $A = (A[1], ..., A[n])$ with length n.
Output : Boolean decision "sorted" or "not sorted"

for $i \leftarrow 1$ **to** $n - 1$ **do**
    if $A[i] > A[i+1]$ **then**
        return "not sorted";

return "sorted";

Observation

IsSorted(A):“not sorted”, if $A[i] > A[i+1]$ for an i.

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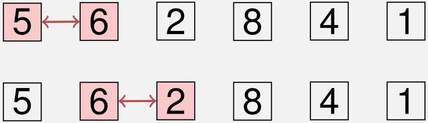
$\Rightarrow$ idea:

for $j \leftarrow 1$ **to** $n-1$ **do**
    if $A[j] > A[j+1]$ **then**
        swap($A[j], A[j+1]$);

Give it a try

5 ⟷	6	2	8	4	1	$(j = 1)$

Give it a try



$(j = 1)$

$(j = 2)$

Give it a try

5 ⟷ 6   2   8   4   1 $(j = 1)$

5   6 ⟷ 2   8   4   1 $(j = 2)$

5   2   6 ⟷ 8   4   1 $(j = 3)$

Give it a try

5 ↔ 6   2   8   4   1 $(j = 1)$

5   6 ↔ 2   8   4   1 $(j = 2)$

5   2   6 ↔ 8   4   1 $(j = 3)$

5   2   6   8 ↔ 4   1 $(j = 4)$

Give it a try

5 ⟷	6	2	8	4	1	$(j = 1)$
5	6 ⟷	2	8	4	1	$(j = 2)$
5	2	6 ⟷	8	4	1	$(j = 3)$
5	2	6	8 ⟷	4	1	$(j = 4)$
5	2	6	4	8 ⟷	1	$(j = 5)$

Give it a try

5 ⟷ 6 2 8 4 1 $(j = 1)$

5 6 ⟷ 2 8 4 1 $(j = 2)$

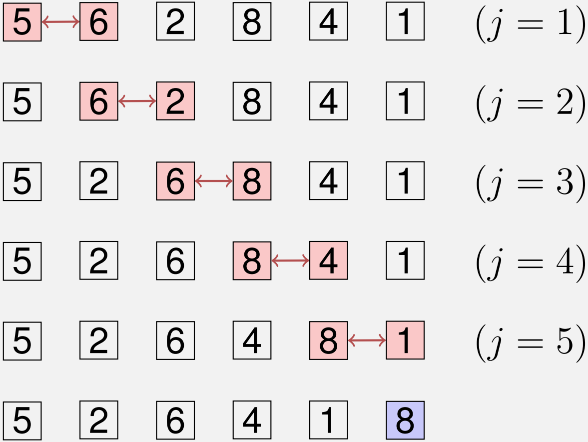
5 2 6 ⟷ 8 4 1 $(j = 3)$

5 2 6 8 ⟷ 4 1 $(j = 4)$

5 2 6 4 8 ⟷ 1 $(j = 5)$

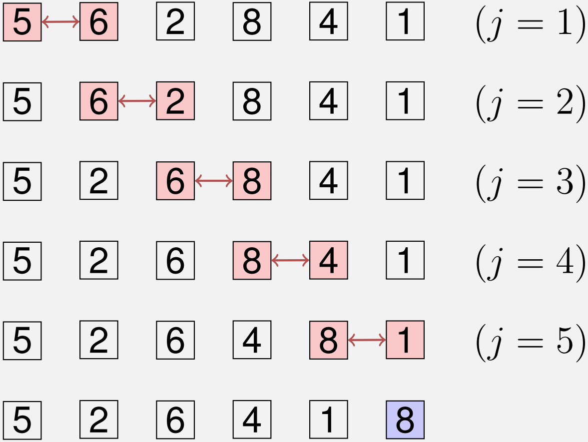
5 2 6 4 1 8

Give it a try



- Not sorted! ☹.

Give it a try



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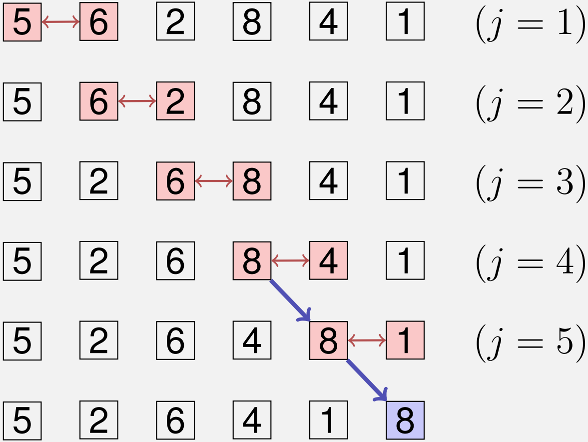

- Not sorted! ☹.
- But the greatest element moves to the right
  $\Rightarrow$ new idea! ☺

Try it out

5	6	2	8	4	1	$(j = 1, i = 1)$
5	6	2	8	4	1	$(j = 2)$
5	2	6	8	4	1	$(j = 3)$
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5	2	6	4	8	1	$(j = 5)$

- Apply the procedure iteratively.

Try it out

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5	6	2	8	4	1	$(j = 2)$
5	2	6	8	4	1	$(j = 3)$
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- Apply the procedure iteratively.
- For $A[1, \ldots, n]$,

Try it out

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5	2	6	8	4	1	$(j = 4)$
5	2	6	4	8	1	$(j = 5)$
5	2	6	4	1	8	$(j = 1, i = 2)$
2	5	6	4	1	8	$(j = 2)$

- Apply the procedure iteratively.
- For $A[1, \ldots, n]$, then $A[1, \ldots, n-1]$,

Try it out

5	6	2	8	4	1	$(j = 1, i = 1)$
5	6	2	8	4	1	$(j = 2)$
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5	2	6	8	4	1	$(j = 4)$
5	2	6	4	8	1	$(j = 5)$
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2	5	6	4	1	8	$(j = 3)$
2	5	4	6	1	8	$(j = 4)$
2	5	4	1	6	8	$(j = 1, i = 3)$

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2	5	4	1	6	8	$(j = 2)$
2	4	5	1	6	8	$(j = 3)$
2	4	1	5	6	8	$(j = 1, i = 4)$

- Apply the procedure iteratively.
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2	4	5	1	6	8	$(j = 3)$
2	4	1	5	6	8	$(j = 1, i = 4)$
2	4	1	5	6	8	$(j = 2)$
2	1	4	5	6	8	$(i = 1, j = 5)$
1	2	4	5	6	8	

- Apply the procedure iteratively.
- For $A[1, \ldots, n]$, then $A[1, \ldots, n-1]$, then $A[1, \ldots, n-2]$, etc.

Algorithm: Bubblesort

Input : Array $A = (A[1], \dots, A[n])$, $n \geq 0$.
Output : Sorted Array A

```
for i ← 1 to n − 1 do
    for j ← 1 to n − i do
        if A[j] > A[j + 1] then
            swap(A[j], A[j + 1]);
```

Analysis

Number key comparisons $\sum_{i=1}^{n-1}(n-i) = \frac{n(n-1)}{2} = \Theta(n^2)$.

Number swaps in the worst case: $\Theta(n^2)$

? What is the worst case?

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? Algorithm can be adapted such that it terminates when the array is sorted. Key comparisons and swaps of the modified algorithm in the best case?

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! Key comparisons = $n-1$. Swaps = 0.

Selection Sort

5	6	2	8	4	1	$(i = 1)$
↑						

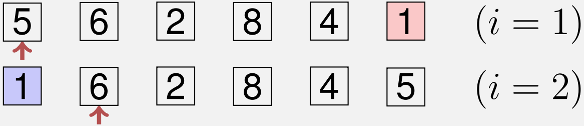
- Iterative procedure as for Bubblesort.

Selection Sort

5 6 2 8 4 1 $(i = 1)$

- Iterative procedure as for Bubblesort.
- Selection of the smallest (or largest) element by immediate search.

Selection Sort



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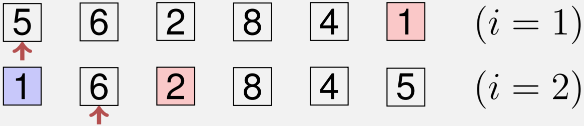

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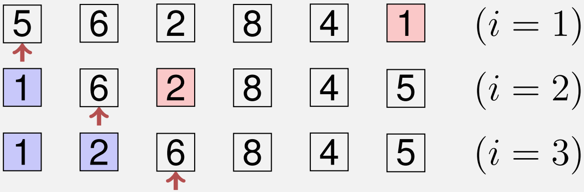

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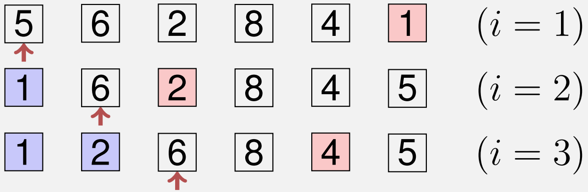

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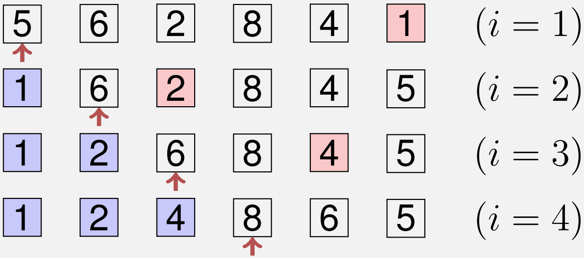

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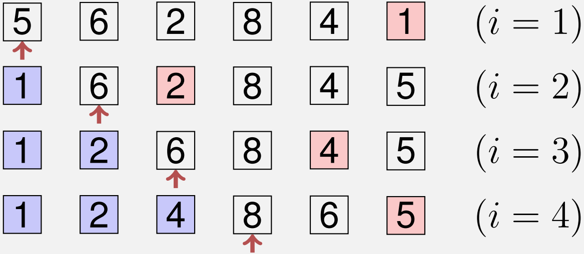

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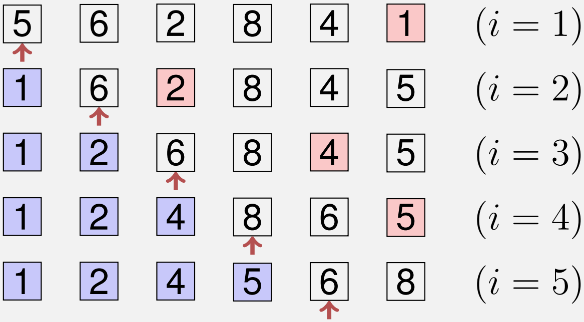

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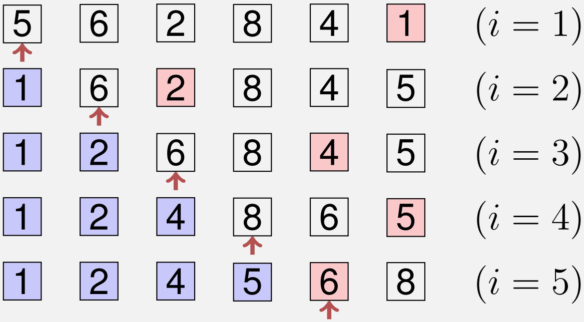

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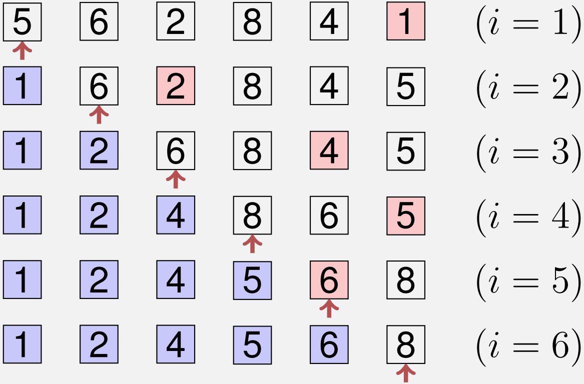

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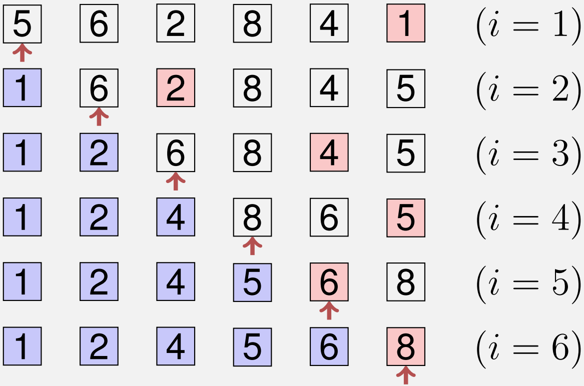

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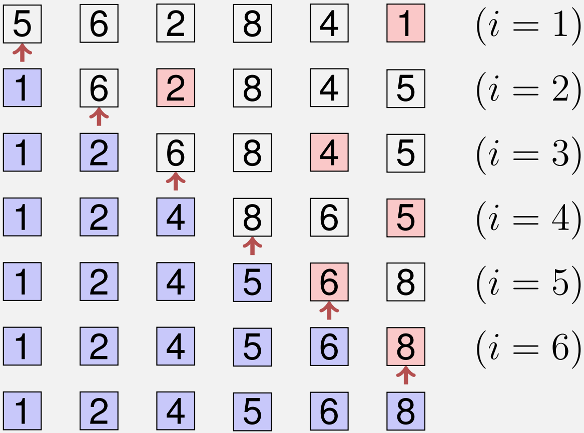

- Iterative procedure as for Bubblesort.
- Selection of the smallest (or largest) element by immediate search.

Algorithm: Selection Sort

Input : Array $A = (A[1], \ldots, A[n])$, $n \geq 0$.
Output : Sorted Array A

for $i \leftarrow 1$ **to** $n-1$ **do**
    $p \leftarrow i$
    for $j \leftarrow i+1$ **to** n **do**
        if $A[j] < A[p]$ **then**
            $p \leftarrow j$;
    swap($A[i], A[p]$)

Analysis

Number comparisons in worst case:

Analysis

Number comparisons in worst case: $\Theta(n^2)$.

Number swaps in the worst case:

Analysis

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Number swaps in the worst case: $n - 1 = \Theta(n)$

Best case number comparisons:

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Best case number comparisons: $\Theta(n^2)$.

Insertion Sort

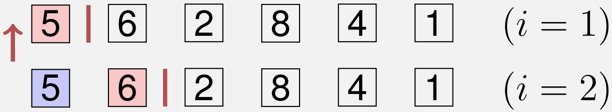
5 | 6 2 8 4 1 $(i = 1)$

Insertion Sort

$(i = 1)$

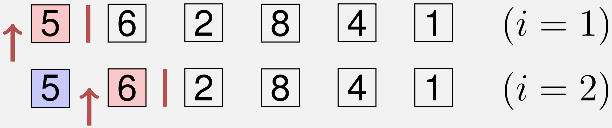
- Iterative procedure: $i = 1 ... n$

Insertion Sort



- Iterative procedure: $i = 1...n$
- Determine insertion position for element i.

Insertion Sort



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- Determine insertion position for element i.
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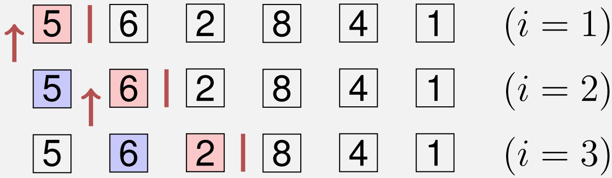
Insertion Sort



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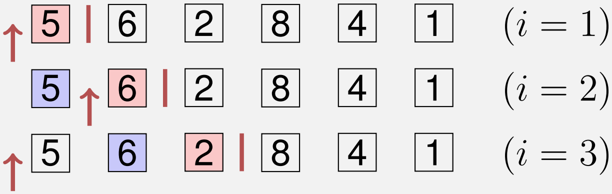

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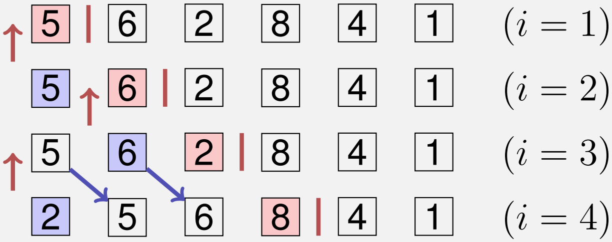

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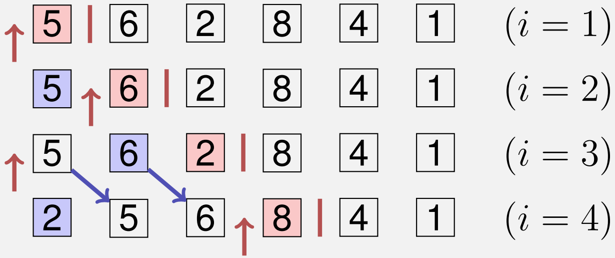
- Determine insertion position for element i.
- Insert element i array block movement potentially required

Insertion Sort



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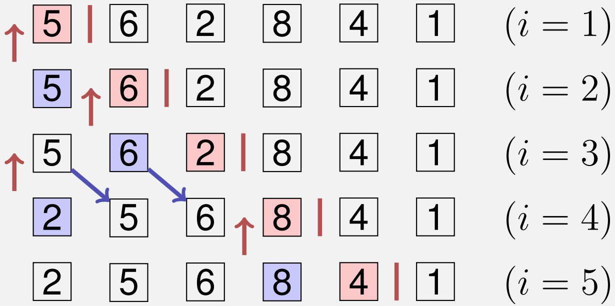

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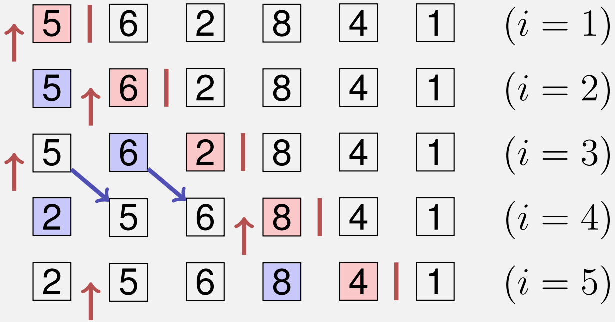

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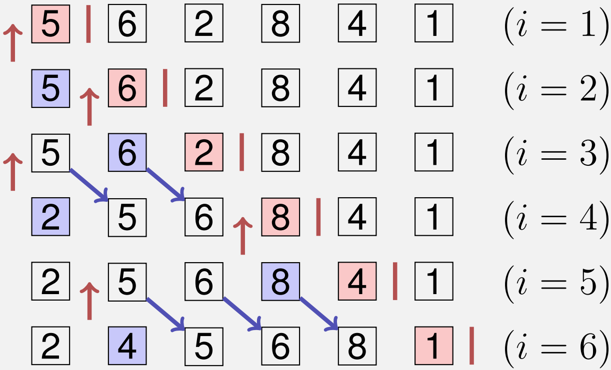

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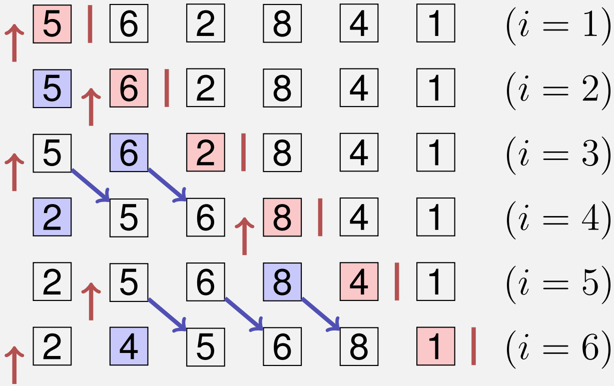

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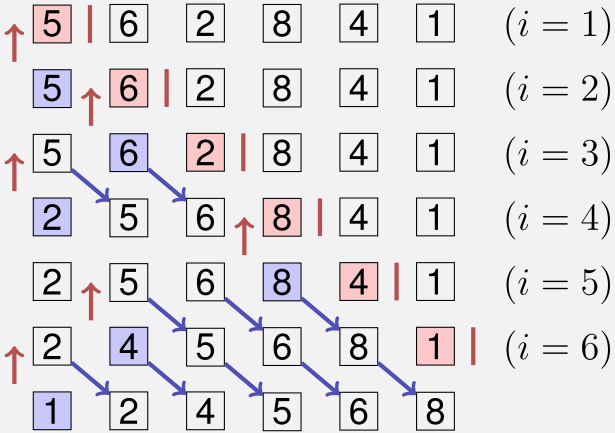

- Iterative procedure: $i = 1...n$
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Insertion Sort

? What is the disadvantage of this algorithm compared to sorting by selection?

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! Many element movements in the worst case.

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? What is the advantage of this algorithm compared to selection sort?

! The search domain (insertion interval) is already sorted. Consequently: binary search possible.

Algorithm: Insertion Sort

Input : Array $A = (A[1], \ldots, A[n])$, $n \geq 0$.
Output : Sorted Array A

for $i \leftarrow 2$ **to** n **do**
    $x \leftarrow A[i]$
    $p \leftarrow$ BinarySearch($A[1...i-1], x$); // Smallest $p \in [1, i]$ with $A[p] \geq x$
    for $j \leftarrow i - 1$ **downto** p **do**
        $A[j+1] \leftarrow A[j]$
    $A[p] \leftarrow x$

Analysis

Number comparisons in the worst case:

[4] With slight modification of the function BinarySearch for the minimum / maximum: $\Theta(n)$

Analysis

Number comparisons in the worst case:
$\sum_{k=1}^{n-1} a \cdot \log k = a \log((n-1)!) \in \mathcal{O}(n \log n)$.

Number comparisons in the best case

[4] With slight modification of the function BinarySearch for the minimum / maximum: $\Theta(n)$

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Number comparisons in the worst case:
$\sum_{k=1}^{n-1} a \cdot \log k = a \log((n-1)!) \in \mathcal{O}(n \log n)$.

Number comparisons in the best case $\Theta(n \log n)$.[4]

Number swaps in the worst case

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Number comparisons in the worst case:
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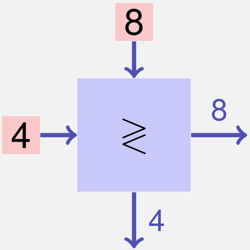
Number comparisons in the best case $\Theta(n \log n)$.[4]

Number swaps in the worst case $\sum_{k=2}^{n} (k-1) \in \Theta(n^2)$

[4] With slight modification of the function BinarySearch for the minimum / maximum: $\Theta(n)$

Different point of view

Sorting node:



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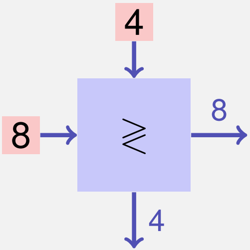
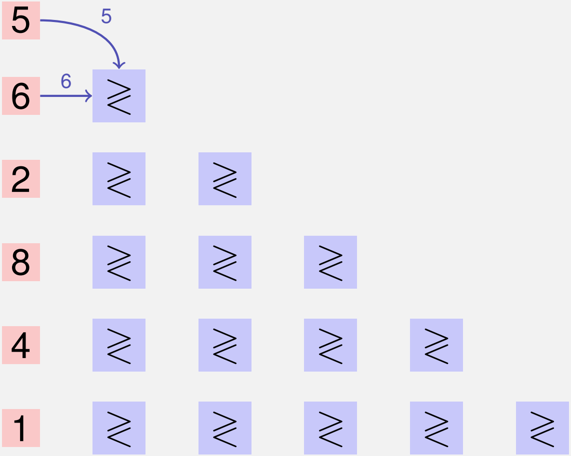


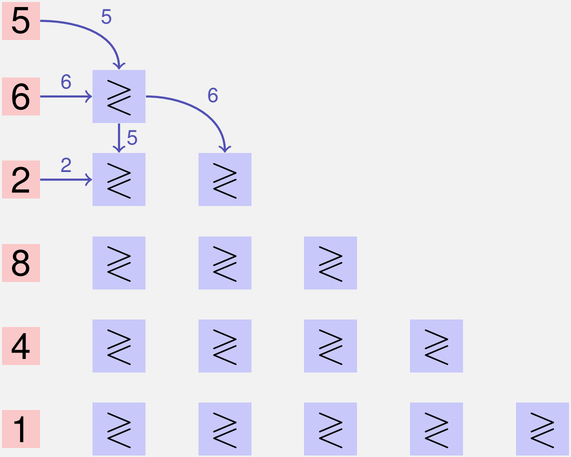
Different point of view

5					
6	$\gtrless$				
2	$\gtrless$	$\gtrless$			
8	$\gtrless$	$\gtrless$	$\gtrless$		
4	$\gtrless$	$\gtrless$	$\gtrless$	$\gtrless$	
1	$\gtrless$	$\gtrless$	$\gtrless$	$\gtrless$	$\gtrless$

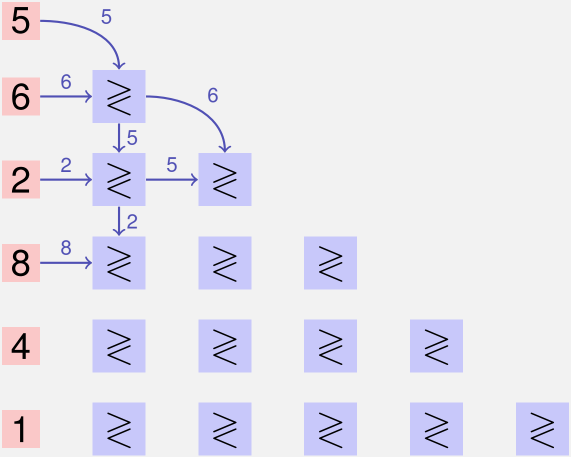
Different point of view



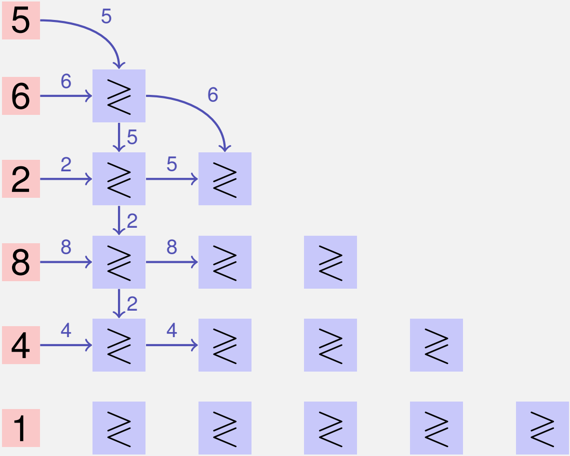
Different point of view



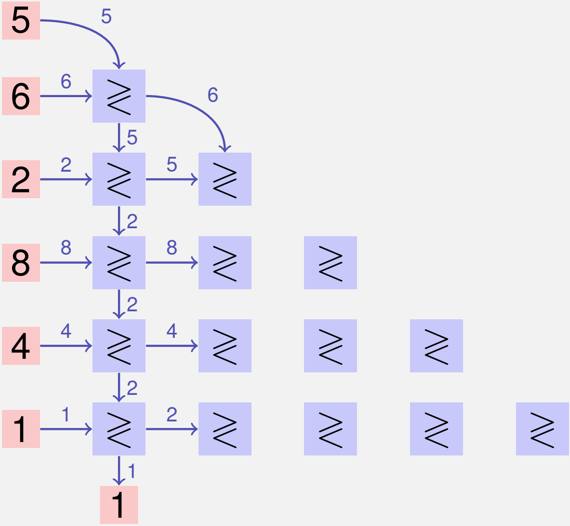
Different point of view



Different point of view



Different point of view



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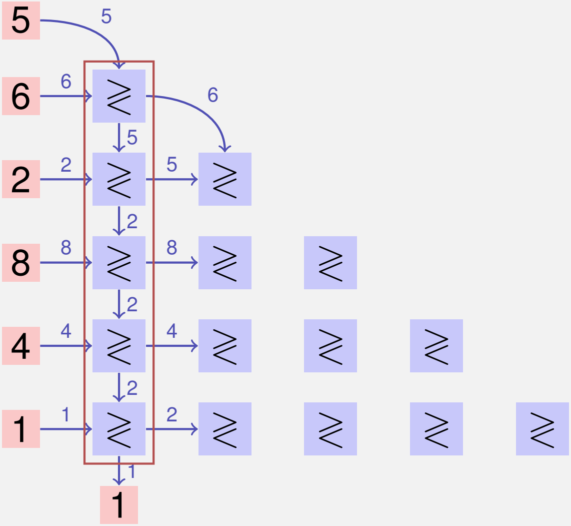
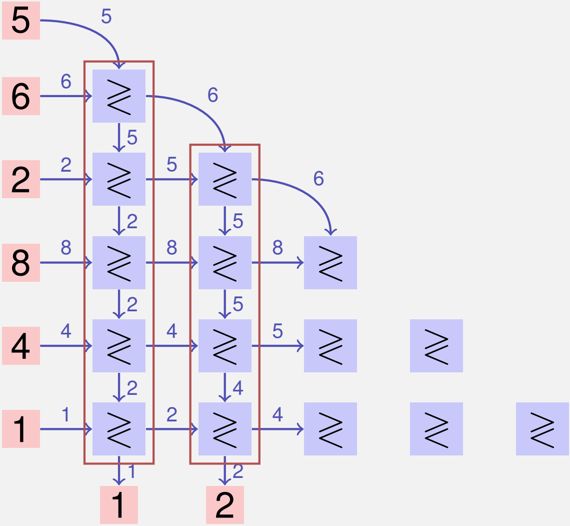
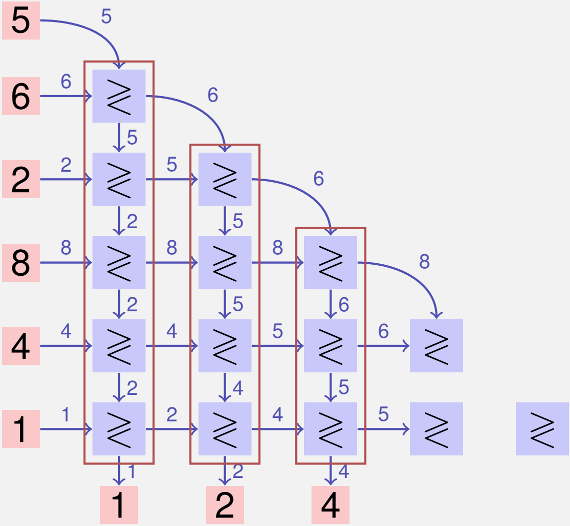


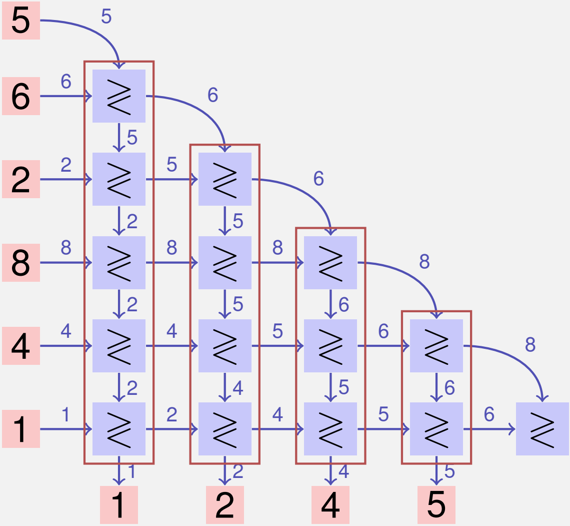
Different point of view



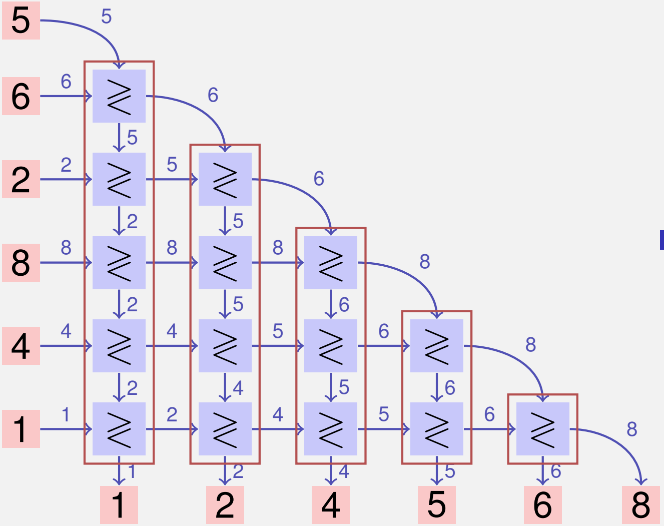
Different point of view



Different point of view

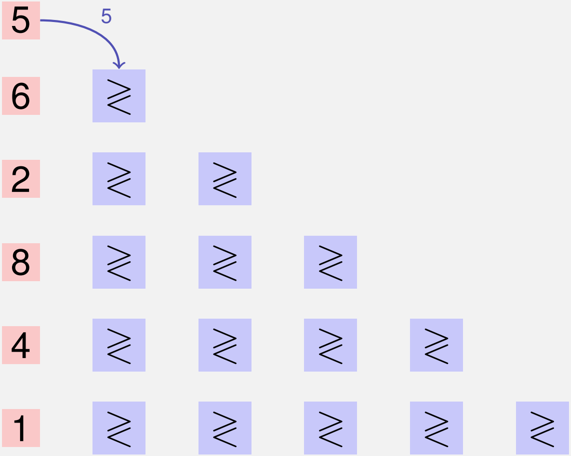


Different point of view

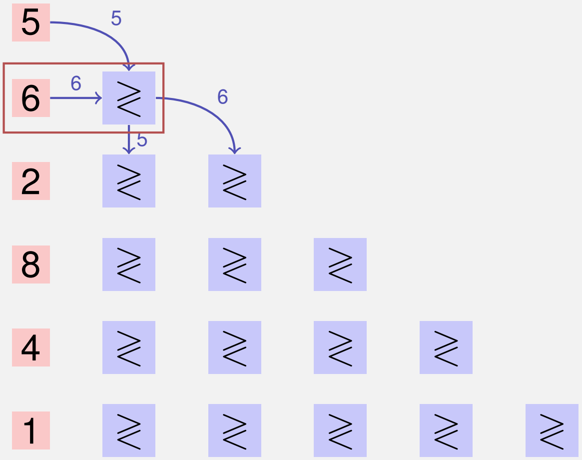


- Like selection sort
  [and like Bubblesort]

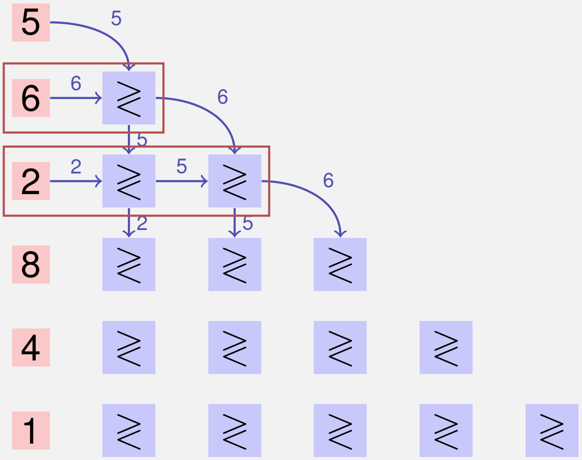
Different point of view



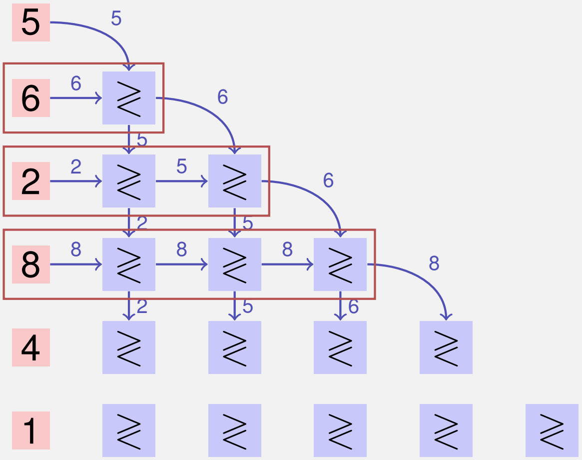
Different point of view



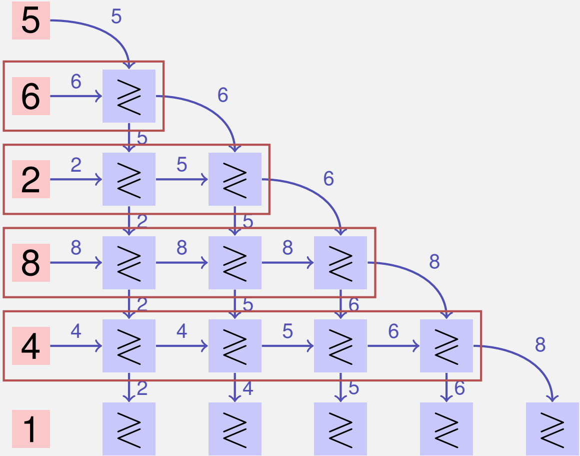
Different point of view



Different point of view



Different point of view



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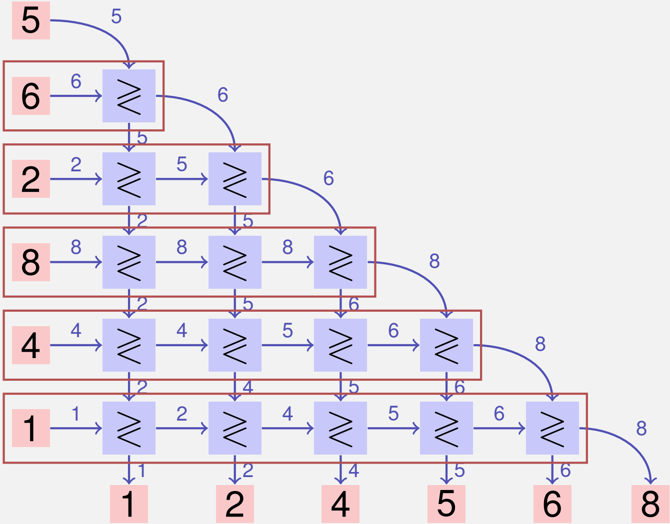
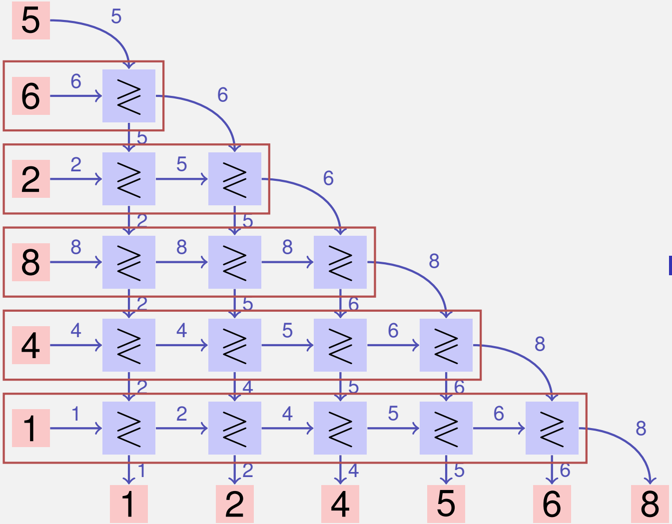


Different point of view



- Like insertion sort

Conclusion

In a certain sense, Selection Sort, Bubble Sort and Insertion Sort provide the same kind of sort strategy. Will be made more precise. [5]

[5] In the part about parallel sorting networks. For the sequential code of course the observations as described above still hold.

Shellsort

Insertion sort on subsequences of the form $(A_{k \cdot i})$ ($i \in \mathbb{N}$) with decreasing distances k. Last considered distance must be $k = 1$.

Good sequences: for example sequences with distances $k \in \{2^i 3^j | 0 \leq i, j\}$.

Shellsort

9 8 7 6 5 4 3 2 1 0

Shellsort

9	8	7	6	5	4	3	2	1	0	
1	8	7	6	5	4	3	2	9	0	insertion sort, $k = 4$

Shellsort

9	8	7	6	5	4	3	2	1	0	
1	8	7	6	5	4	3	2	9	0	insertion sort, $k = 4$
1	0	7	6	5	4	3	2	9	8	

Shellsort

9	8	7	6	5	4	3	2	1	0	
1	8	7	6	5	4	3	2	9	0	insertion sort, $k = 4$
1	0	7	6	5	4	3	2	9	8	
1	0	3	6	5	4	7	2	9	8	

Shellsort

9	8	7	6	5	4	3	2	1	0	
1	8	7	6	5	4	3	2	9	0	insertion sort, $k = 4$
1	0	7	6	5	4	3	2	9	8	
1	0	3	6	5	4	7	2	9	8	
1	0	3	2	5	4	7	6	9	8	

Shellsort

9	8	7	6	5	4	3	2	1	0	
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1	0	7	6	5	4	3	2	9	8	
1	0	3	6	5	4	7	2	9	8	
1	0	3	2	5	4	7	6	9	8	
1	0	3	2	5	4	7	6	9	8	insertion sort, $k = 2$

Shellsort

9	8	7	6	5	4	3	2	1	0	
1	8	7	6	5	4	3	2	9	0	insertion sort, $k = 4$
1	0	7	6	5	4	3	2	9	8	
1	0	3	6	5	4	7	2	9	8	
1	0	3	2	5	4	7	6	9	8	
1	0	3	2	5	4	7	6	9	8	insertion sort, $k = 2$
1	0	3	2	5	4	7	6	9	8	

Shellsort

9	8	7	6	5	4	3	2	1	0	
1	8	7	6	5	4	3	2	9	0	insertion sort, $k = 4$
1	0	7	6	5	4	3	2	9	8	
1	0	3	6	5	4	7	2	9	8	
1	0	3	2	5	4	7	6	9	8	
1	0	3	2	5	4	7	6	9	8	insertion sort, $k = 2$
1	0	3	2	5	4	7	6	9	8	
0	1	2	3	4	5	6	7	8	9	insertion sort, $k = 1$

8. Sorting II

Heapsort, Quicksort, Mergesort

8.1 Heapsort

[Ottman/Widmayer, Kap. 2.3, Cormen et al, Kap. 6]

Heapsort

Inspiration from selectsort: fast insertion

Inspiration from insertion sort: fast determination of position

? Can we have the best of both worlds?

Heapsort

Inspiration from selectsort: fast insertion

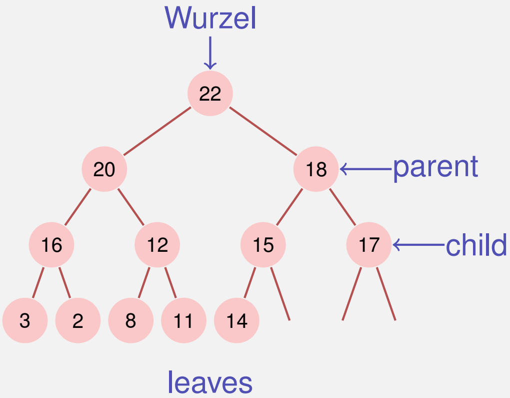
Inspiration from insertion sort: fast determination of position

? Can we have the best of both worlds?

! Yes, but it requires some more thinking...

[Max-]Heap[6]

Binary tree with the following properties

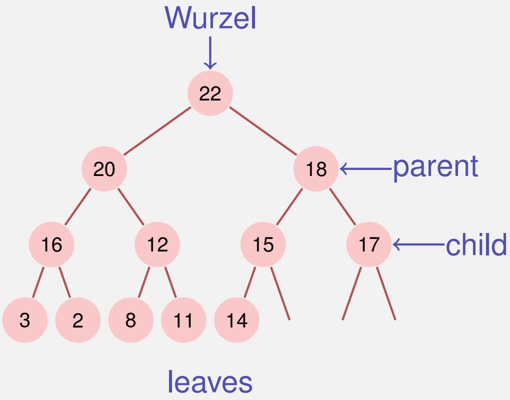


[6]Heap(data structure), not: as in "heap and stack" (memory allocation)

[Max-]Heap[6]

Binary tree with the following properties

1. complete up to the lowest level



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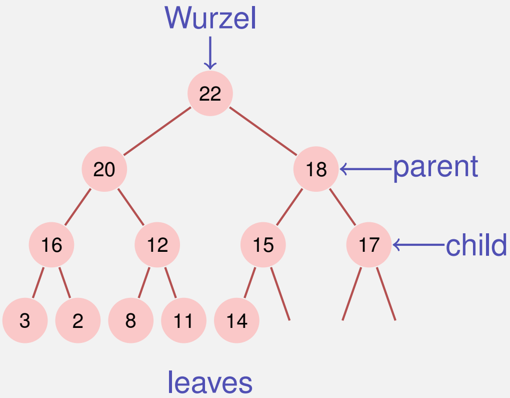
Binary tree with the following properties

1. complete up to the lowest level
2. Gaps (if any) of the tree in the last level to the right

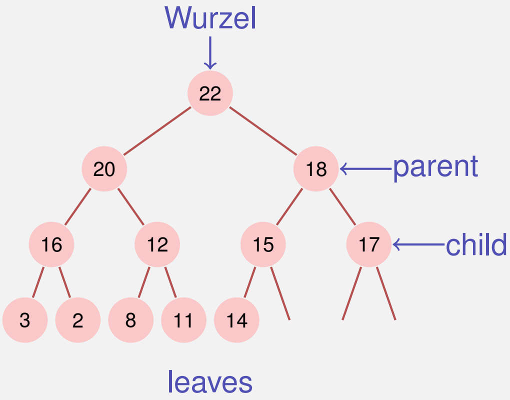


[6]Heap(data structure), not: as in "heap and stack" (memory allocation)

[Max-]Heap[6]

Binary tree with the following properties

1. complete up to the lowest level
2. Gaps (if any) of the tree in the last level to the right
3. *Heap-Condition:* Max-(Min-)Heap: key of a child smaller (greater) thant that of the parent node



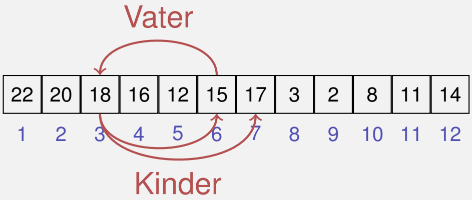
---

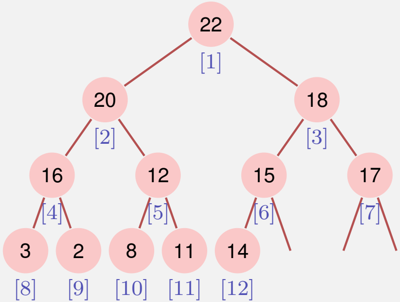
[6]Heap(data structure), not: as in “heap and stack” (memory allocation)

Heap and Array

Tree $\rightarrow$ Array:

- children$(i) = \{2i, 2i+1\}$
- parent$(i) = \lfloor i/2 \rfloor$



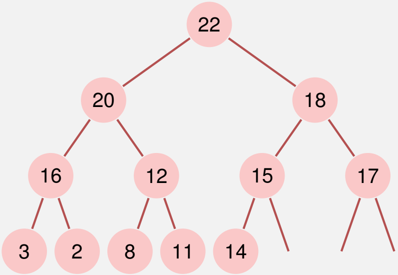


Depends on the starting index[7]

[7] For array that start at 0: $\{2i, 2i+1\} \rightarrow \{2i+1, 2i+2\}$, $\lfloor i/2 \rfloor \rightarrow \lfloor (i-1)/2 \rfloor$

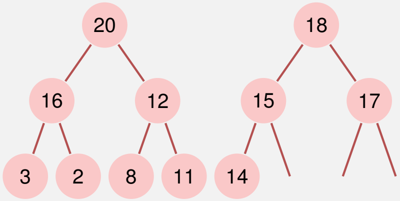
Recursive heap structure

A heap consists of two heaps:

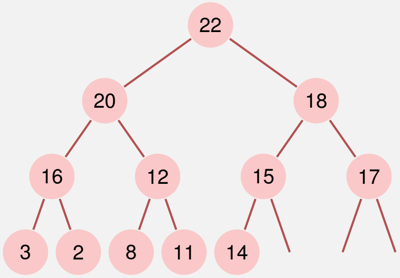


Recursive heap structure

A heap consists of two heaps:

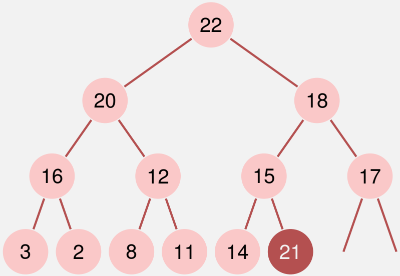


Insert



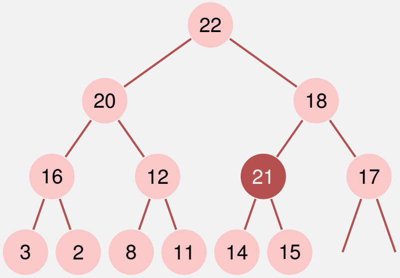
Insert

- Insert new element at the first free position. Potentially violates the heap property.



Insert

- Insert new element at the first free position. Potentially violates the heap property.
- Reestablish heap property: climb successively



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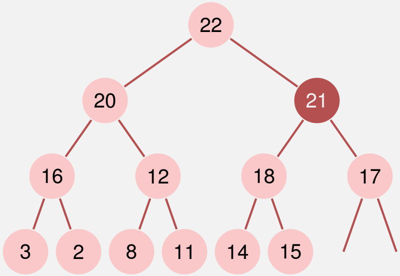
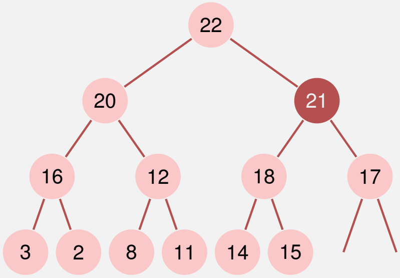
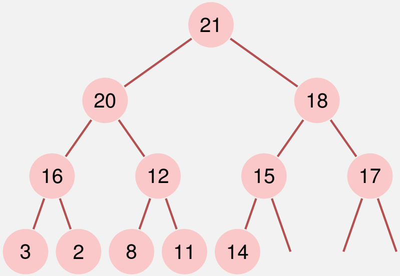


Insert

- Insert new element at the first free position. Potentially violates the heap property.
- Reestablish heap property: climb successively
- Worst case number of operations: $\mathcal{O}(\log n)$

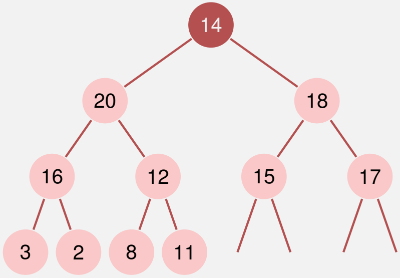


Remove the maximum



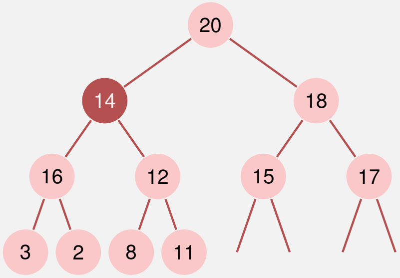
Remove the maximum

- Replace the maximum by the lower right element



Remove the maximum

- Replace the maximum by the lower right element
- Reestablish heap property: sink successively (in the direction of the greater child)



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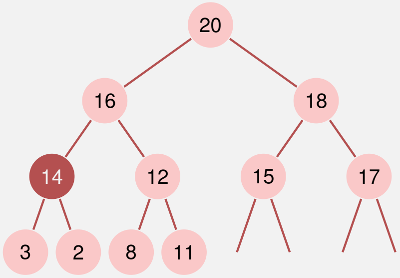
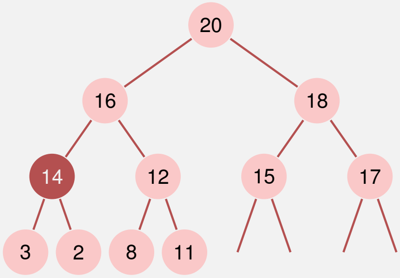


Remove the maximum

- Replace the maximum by the lower right element
- Reestablish heap property: sink successively (in the direction of the greater child)
- Worst case number of operations: $\mathcal{O}(\log n)$

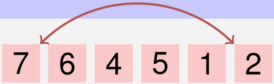


Algorithm Sink(A, i, m)

Input : Array A with heap structure for the children of i. Last element m.
Output : Array A with heap structure for i with last element m.

```
while 2i ≤ m do
    j ← 2i; // j left child
    if j < m and A[j] < A[j + 1] then
        j ← j + 1; // j right child with greater key
    if A[i] < A[j] then
        swap(A[i], A[j])
        i ← j; // keep sinking
    else
        i ← m; // sinking finished
```

Sort heap



$A[1, ..., n]$ is a Heap.

While $n > 1$

- swap($A[1]$, $A[n]$)
- Sink($A, 1, n - 1$);
- $n \leftarrow n - 1$

Sort heap

	7	6	4	5	1	2
swap $\Rightarrow$	2	6	4	5	1	7

$A[1, ..., n]$ is a Heap.
While $n > 1$

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Sort heap

		7	6	4	5	1	2
swap	$\Rightarrow$	2	6	4	5	1	7
sink	$\Rightarrow$	6	5	4	2	1	7

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While $n > 1$

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- $n \leftarrow n - 1$

		7	6	4	5	1	2
swap	$\Rightarrow$	2	6	4	5	1	**7**
sink	$\Rightarrow$	6	5	4	2	1	**7**
swap	$\Rightarrow$	1	5	4	2	**6**	**7**

Sort heap

$A[1, ..., n]$ is a Heap.
While $n > 1$

- swap($A[1]$, $A[n]$)
- Sink($A, 1, n-1$);
- $n \leftarrow n - 1$

		7	6	4	5	1	2
swap	$\Rightarrow$	2	6	4	5	1	7
sink	$\Rightarrow$	6	5	4	2	1	7
swap	$\Rightarrow$	1	5	4	2	6	7
sink	$\Rightarrow$	5	4	2	1	6	7
swap	$\Rightarrow$	1	4	2	5	6	7
sink	$\Rightarrow$	4	1	2	5	6	7
swap	$\Rightarrow$	2	1	4	5	6	7
sink	$\Rightarrow$	2	1	4	5	6	7
swap	$\Rightarrow$	1	2	4	5	6	7

Heap creation

Observation: Every leaf of a heap is trivially a correct heap.

Consequence:

Heap creation

Observation: Every leaf of a heap is trivially a correct heap.

Consequence: Induction from below!

Algorithm HeapSort(A, n)

Input : Array A with length n.
Output : A sorted.

```
// Build the heap.
for i ← n/2 downto 1 do
    Sink(A, i, n);
// Now A is a heap.
for i ← n downto 2 do
    swap(A[1], A[i])
    Sink(A, 1, i − 1)
// Now A is sorted.
```

Analysis: sorting a heap

Sink traverses at most $\log n$ nodes. For each node 2 key comparisons. $\Rightarrow$ sorting a heap costs in the worst case $2 \log n$ comparisons.

Number of memory movements of sorting a heap also $\mathcal{O}(n \log n)$.

Analysis: creating a heap

Calls to sink: $n/2$. Thus number of comparisons and movements: $v(n) \in \mathcal{O}(n \log n)$.

[8] $f(x) = \frac{1}{1-x} = 1 + x + x^2 ... \Rightarrow f'(x) = \frac{1}{(1-x)^2} = 1 + 2x + ...$

Analysis: creating a heap

Calls to sink: $n/2$. Thus number of comparisons and movements: $v(n) \in \mathcal{O}(n \log n)$.

But mean length of sinking paths is much smaller:

$$v(n) = \sum_{h=0}^{\lfloor \log n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil \cdot c \cdot h \in \mathcal{O}(n \sum_{h=0}^{\lfloor \log n \rfloor} \frac{h}{2^h})$$

with $s(x) := \sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2} \quad (0 < x < 1)$ [8] and $s(\frac{1}{2}) = 2$:

$$v(n) \in \mathcal{O}(n).$$

---

[8] $f(x) = \frac{1}{1-x} = 1 + x + x^2 ... \Rightarrow f'(x) = \frac{1}{(1-x)^2} = 1 + 2x + ...$

8.2 Mergesort

[Ottman/Widmayer, Kap. 2.4, Cormen et al, Kap. 2.3],

Intermediate result

Heapsort: $\mathcal{O}(n \log n)$ Comparisons and movements.

Disadvantages of heapsort?

Intermediate result

Heapsort: $\mathcal{O}(n \log n)$ Comparisons and movements.

? Disadvantages of heapsort?

! Missing locality: heapsort jumps around in the sorted array (negative cache effect).

Intermediate result

Heapsort: $\mathcal{O}(n \log n)$ Comparisons and movements.

Disadvantages of heapsort?

- Missing locality: heapsort jumps around in the sorted array (negative cache effect).
- Two comparisons required before each necessary memory movement.

Mergesort

Divide and Conquer!

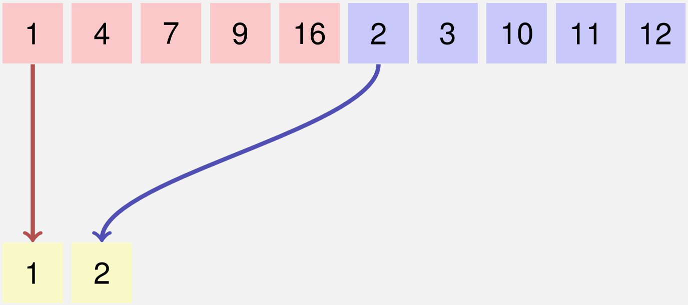
- Assumption: two halves of the array A are already sorted.
- Minimum of A can be evaluated with two comparisons.
- Iteratively: sort the pre-sorted array A in $\mathcal{O}(n)$.

Merge

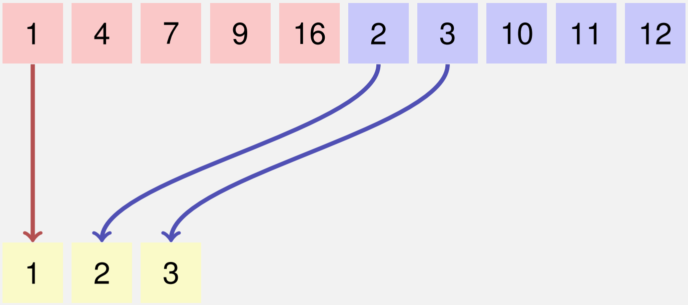
1	4	7	9	16	2	3	10	11	12

Merge

Merge



Merge



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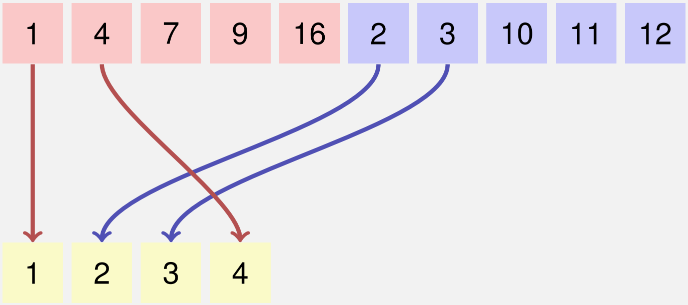

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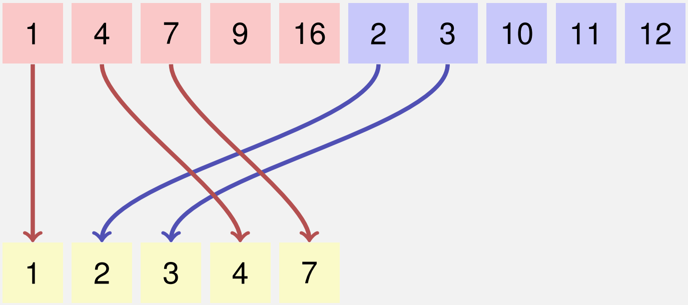

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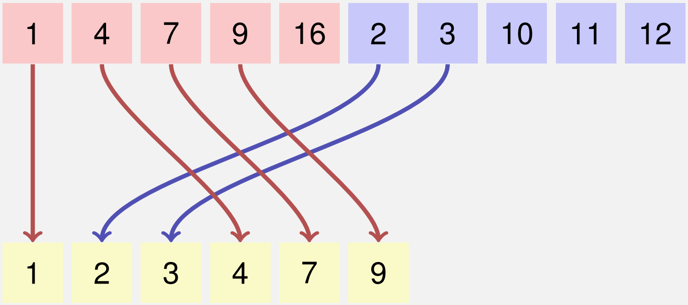

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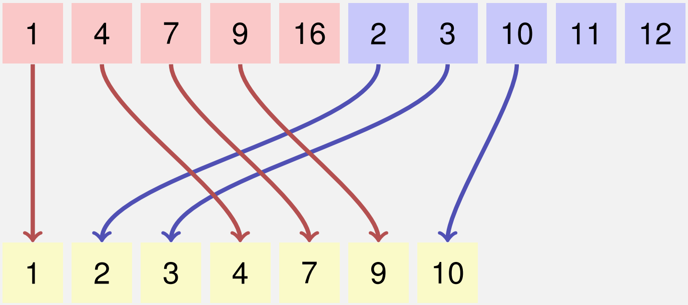
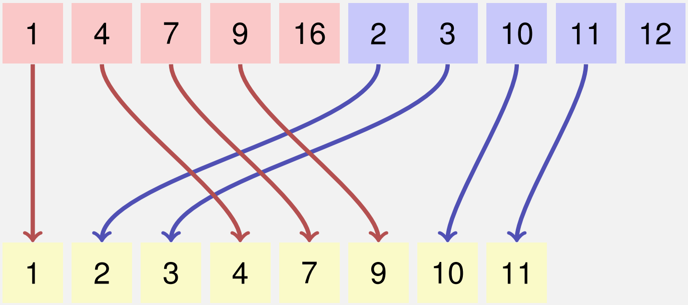
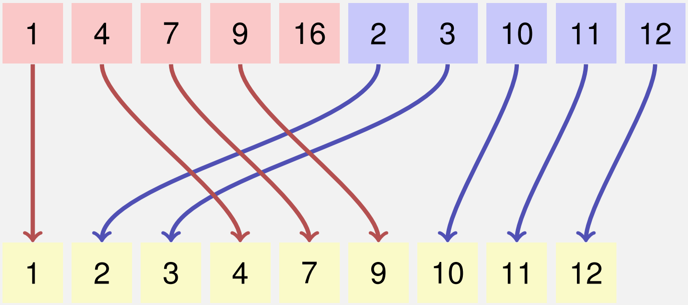


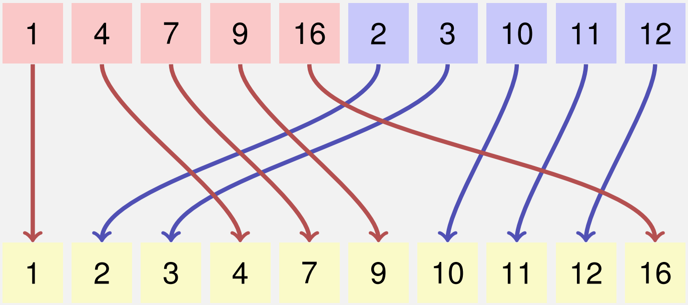
Merge



Merge



Merge



Algorithm Merge(A, l, m, r)

Input : Array A with length n, indexes $1 \leq l \leq m \leq r \leq n$. $A[l, \ldots, m]$, $A[m+1, \ldots, r]$ sorted

Output : $A[l, \ldots, r]$ sorted

$B \leftarrow$ new Array$(r - l + 1)$
$i \leftarrow l$; $j \leftarrow m + 1$; $k \leftarrow 1$
while $i \leq m$ and $j \leq r$ **do**
    if $A[i] \leq A[j]$ **then** $B[k] \leftarrow A[i]$; $i \leftarrow i + 1$
    else $B[k] \leftarrow A[j]$; $j \leftarrow j + 1$
    $k \leftarrow k + 1$;
while $i \leq m$ **do** $B[k] \leftarrow A[i]$; $i \leftarrow i + 1$; $k \leftarrow k + 1$
while $j \leq r$ **do** $B[k] \leftarrow A[j]$; $j \leftarrow j + 1$; $k \leftarrow k + 1$
for $k \leftarrow l$ **to** r **do** $A[k] \leftarrow B[k - l + 1]$

Correctness

Hypothesis: after k iterations of the loop in line 3 $B[1,\dots,k]$ is sorted and $B[k] \leq A[i]$, if $i \leq m$ and $B[k] \leq A[j]$ if $j \leq r$.

Proof by induction:
Base case: the empty array $B[1,\dots,0]$ is trivially sorted.
Induction step ($k \to k+1$):

- wlog $A[i] \leq A[j]$, $i \leq m, j \leq r$.
- $B[1,\dots,k]$ is sorted by hypothesis and $B[k] \leq A[i]$.
- After $B[k+1] \leftarrow A[i]$ $B[1,\dots,k+1]$ is sorted.
- $B[k+1] = A[i] \leq A[i+1]$ (if $i+1 \leq m$) and $B[k+1] \leq A[j]$ if $j \leq r$.
- $k \leftarrow k+1, i \leftarrow i+1$: Statement holds again.

Analysis (Merge)

Lemma

If: array A with length n, indexes $1 \leq l < r \leq n$. $m = \lfloor (l+r)/2 \rfloor$ and $A[l, \dots, m]$, $A[m+1, \dots, r]$ sorted.
Then: in the call of Merge(A, l, m, r) a number of $\Theta(r-l)$ key movements and comparisons are executed.

Proof: straightforward(Inspect the algorithm and count the operations.)

Mergesort

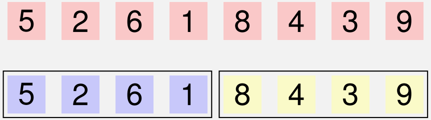
5	2	6	1	8	4	3	9

Mergesort

5	2	6	1	8	4	3	9

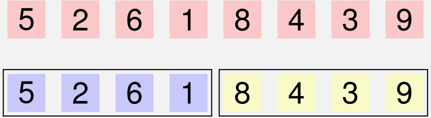
Split

Mergesort



Split

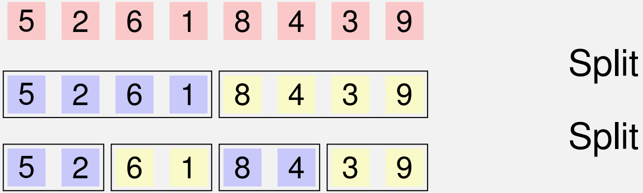
Mergesort



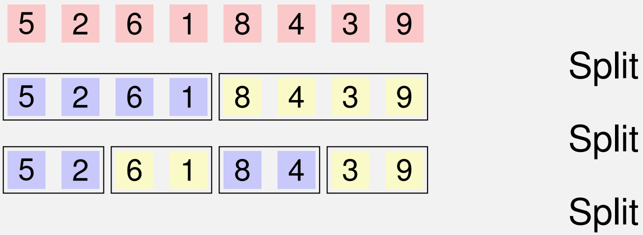
Split

Split

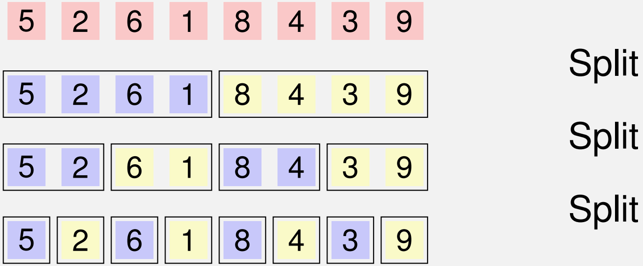
Mergesort



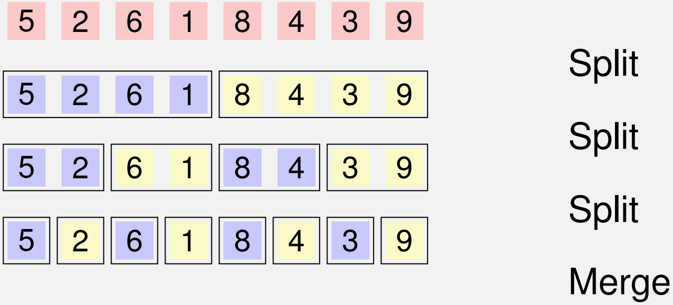
Mergesort



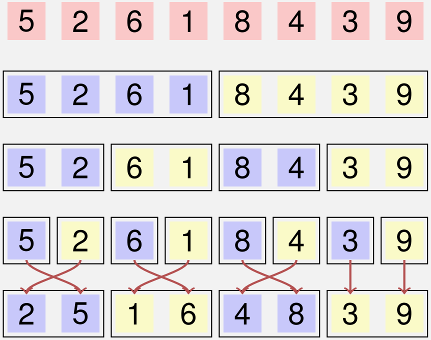
Mergesort



Mergesort


5 2 6 1 8 4 3 9
5 2 6 1 8 4 3 9
5 2 6 1 8 4 3 9
5 2 6 1 8 4 3 9
Split
Split
Split
Merge

Mergesort



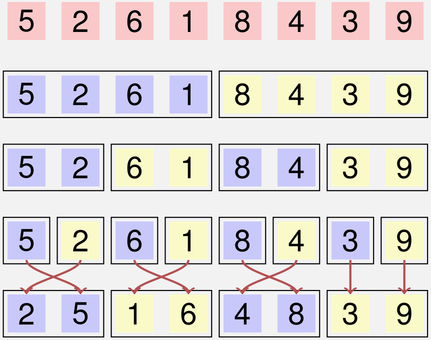
Split

Split

Split

Merge

Mergesort



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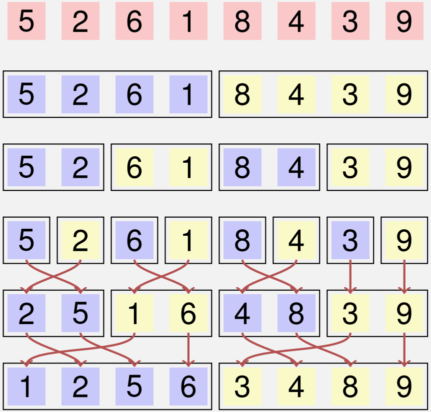

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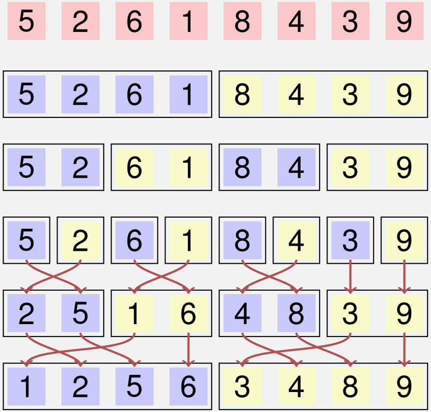

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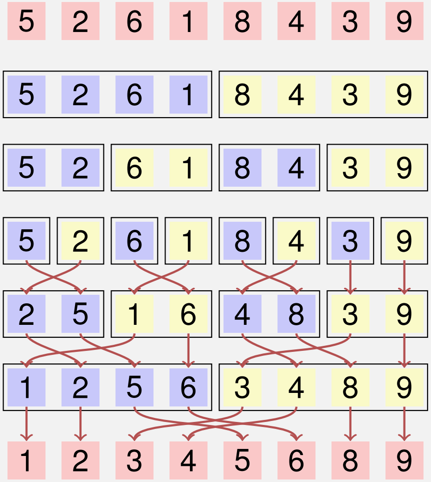

Split

Split

Split

Merge

Merge

Merge

Algorithm recursive 2-way Mergesort(A, l, r)

Input : Array A with length n. $1 \leq l \leq r \leq n$
Output : Array $A[l, \dots, r]$ sorted.

```
if l < r then
    m ← ⌊(l + r)/2⌋              // middle position
    Mergesort(A, l, m)           // sort lower half
    Mergesort(A, m + 1, r)       // sort higher half
    Merge(A, l, m, r)            // Merge subsequences
```

Analysis

Recursion equation for the number of comparisons and key movements:

$$C(n) = C(\left\lceil \frac{n}{2} \right\rceil) + C(\left\lfloor \frac{n}{2} \right\rfloor) + \Theta(n)$$

Analysis

Recursion equation for the number of comparisons and key movements:

$$C(n) = C(\left\lceil \frac{n}{2} \right\rceil) + C(\left\lfloor \frac{n}{2} \right\rfloor) + \Theta(n) \in \Theta(n \log n)$$

Algorithm StraightMergesort(A)

Avoid recursion: merge sequences of length $1, 2, 4, \ldots$ directly

```
Input :     Array A with length n
Output :    Array A sorted
length ← 1
while length < n do                      // Iterate over lengths n
    r ← 0
    while r + length < n do              // Iterate over subsequences
        l ← r + 1
        m ← l + length − 1
        r ← min(m + length, n)
        Merge(A, l, m, r)
    length ← length · 2
```

Analysis

Like the recursive variant, the straight 2-way mergesort always executes a number of $\Theta(n \log n)$ key comparisons and key movements.

Natural 2-way mergesort

Observation: the variants above do not make use of any presorting and always execute $\Theta(n \log n)$ memory movements.

? How can partially presorted arrays be sorted better?

Natural 2-way mergesort

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? How can partially presorted arrays be sorted better?

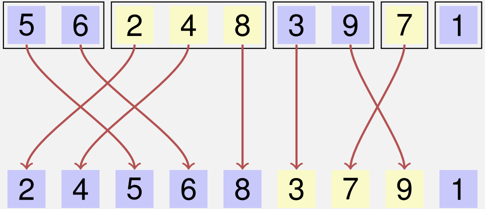
! Recursive merging of previously sorted parts (*runs*) *of* A.

Natural 2-way mergesort

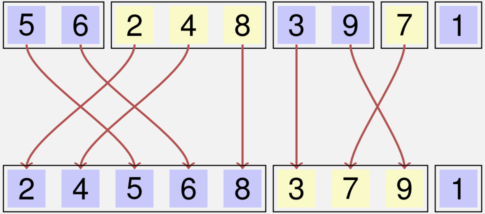
5	6	2	4	8	3	9	7	1

Natural 2-way mergesort

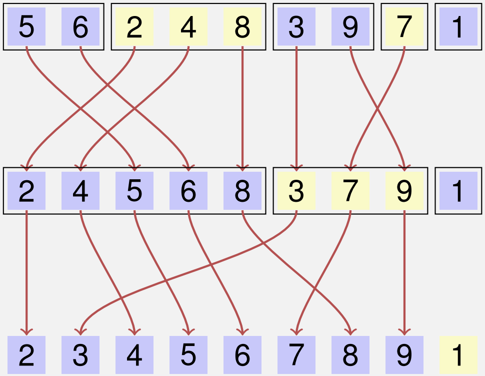
Natural 2-way mergesort



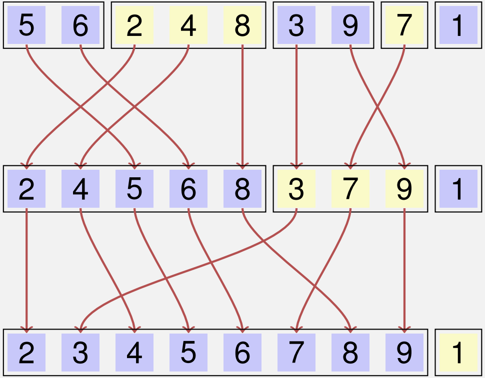
Natural 2-way mergesort



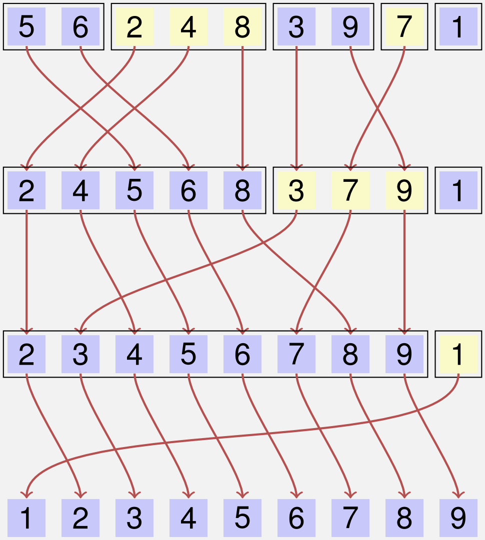
Natural 2-way mergesort



Natural 2-way mergesort



Natural 2-way mergesort



Algorithm NaturalMergesort(A)

```
Input :      Array A with length n > 0
Output :     Array A sorted
repeat
    r ← 0
    while r < n do
        l ← r + 1
        m ← l; while m < n and A[m + 1] ≥ A[m] do m ← m + 1
        if m < n then
            r ← m + 1; while r < n and A[r + 1] ≥ A[r] do r ← r + 1
            Merge(A, l, m, r);
        else
            r ← n
until l = 1
```

Analysis

In the best case, natural merge sort requires only $n-1$ comparisons.

? Is it also asymptotically better than StraightMergesort on average?

Analysis

In the best case, natural merge sort requires only $n-1$ comparisons.

? Is it also asymptotically better than StraightMergesort on average?

! No. Given the assumption of pairwise distinct keys, on average there are $n/2$ positions i with $k_i > k_{i+1}$, i.e. $n/2$ runs. Only one iteration is saved on average.

Natural mergesort executes in the worst case and on average a number of $\Theta(n \log n)$ comparisons and memory movements.

8.3 Quicksort

[Ottman/Widmayer, Kap. 2.2, Cormen et al, Kap. 7]

Quicksort

What is the disadvantage of Mergesort?

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? How could we reduce the merge costs?

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? How?

! Pivot and Partition!

Quicksort (arbitrary pivot)

2	4	5	6	8	3	7	9	1

Quicksort (arbitrary pivot)

2	4	5	6	8	3	7	9	1

Quicksort (arbitrary pivot)

2	4	5	6	8	3	7	9	1
2	1	3	6	8	5	7	9	4

Quicksort (arbitrary pivot)

2	4	5	6	8	3	7	9	1
2	1	3	6	8	5	7	9	4

Quicksort (arbitrary pivot)

2	4	5	6	8	3	7	9	1
2	1	3	6	8	5	7	9	4
1	2	3	4	5	8	7	9	6

Quicksort (arbitrary pivot)

2	4	5	6	8	3	7	9	1
2	1	3	6	8	5	7	9	4
1	2	3	4	5	8	7	9	6

Quicksort (arbitrary pivot)

2	4	5	6	8	3	7	9	1
2	1	3	6	8	5	7	9	4
1	2	3	4	5	8	7	9	6
1	2	3	4	5	6	7	9	8

Quicksort (arbitrary pivot)

Quicksort (arbitrary pivot)

2	4	5	6	8	3	7	9	1
2	1	3	6	8	5	7	9	4
1	2	3	4	5	8	7	9	6
1	2	3	4	5	6	7	9	8
1	2	3	4	5	6	7	8	9

Quicksort (arbitrary pivot)

2	4	5	6	8	3	7	9	1
2	1	3	6	8	5	7	9	4
1	2	3	4	5	8	7	9	6
1	2	3	4	5	6	7	9	8
1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9

Algorithm Quicksort($A[l, \dots, r]$

Input : Array A with length n. $1 \leq l \leq r \leq n$.
Output : Array A, sorted between l and r.

if $l < r$ **then**
  Choose pivot $p \in A[l, \dots, r]$
  $k \leftarrow$ Partition$(A[l, \dots, r], p)$
  Quicksort$(A[l, \dots, k-1])$
  Quicksort$(A[k+1, \dots, r])$

Reminder: algorithm Partition($A[l, \dots, r], p$)

Input : Array A, that contains the pivot p in $[l, r]$ at least once.
Output : Array A partitioned around p. Returns the position of p.

```
while l ≤ r do
    while A[l] < p do
        l ← l + 1
    while A[r] > p do
        r ← r − 1
    swap(A[l], A[r])
    if A[l] = A[r] then        // Only for keys that are not pairwise different
        l ← l + 1
return l-1
```

Analysis: number comparisons

Best case.

Analysis: number comparisons

Best case. Pivot = median; number comparisons:

$$T(n) = 2T(n/2) + c \cdot n,\ T(1) = 0 \quad \Rightarrow \quad T(n) \in \mathcal{O}(n \log n)$$

Worst case.

Analysis: number comparisons

Best case. Pivot = median; number comparisons:

$$T(n) = 2T(n/2) + c \cdot n, \; T(1) = 0 \quad \Rightarrow \quad T(n) \in \mathcal{O}(n \log n)$$

Worst case. Pivot = min or max; number comparisons:

$$T(n) = T(n-1) + c \cdot n, \; T(1) = 0 \quad \Rightarrow \quad T(n) \in \Theta(n^2)$$

Analysis: number swaps

Result of a call to partition (pivot 3):

2	1	3	6	8	5	7	9	4

? How many swaps have taken place?

Analysis: number swaps

Result of a call to partition (pivot 3):

2	1	3	6	8	5	7	9	4

? How many swaps have taken place?

! 2. The maximum number of swaps is given by the number of keys in the smaller part.

Analysis: number swaps

Intellectual game

Analysis: number swaps

Intellectual game

- Each key from the smaller part pay a coin when swapped.

Analysis: number swaps

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- Each key from the smaller part pay a coin when swapped.
- When a key has paid a coin then the domain containing the key is less than or equal to half the previous size.

Analysis: number swaps

Intellectual game

- Each key from the smaller part pay a coin when swapped.
- When a key has paid a coin then the domain containing the key is less than or equal to half the previous size.
- Every key needs to pay at most $\log n$ coins. But there are only n keys.

Analysis: number swaps

Intellectual game

- Each key from the smaller part pay a coin when swapped.
- When a key has paid a coin then the domain containing the key is less than or equal to half the previous size.
- Every key needs to pay at most $\log n$ coins. But there are only n keys.

Consequence: there are $\mathcal{O}(n \log n)$ key swaps in the worst case.

Randomized Quicksort

Despite the worst case running time of $\Theta(n^2)$, quicksort is used practically very often.

Reason: quadratic running time unlikely provided that the choice of the pivot and the pre-sorting are not very disadvantageous.

Avoidance: randomly choose pivot. Draw uniformly from $[l, r]$.

Analysis (randomized quicksort)

Expected number of compared keys with input length n:

$$T(n) = (n-1) + \frac{1}{n}\sum_{k=1}^{n}\left(T(k-1) + T(n-k)\right),\ T(0) = T(1) = 0$$

Claim $T(n) \leq 4n\log n$.

Proof by induction:

Base case straightforward for $n = 0$ (with $0\log 0 := 0$) and for $n = 1$.

Hypothesis: $T(n) \leq 4n\log n$ for some n.

Induction step: $(n-1 \to n)$

Analysis (randomized quicksort)

$$
\begin{aligned}
T(n) &= n-1+\frac{2}{n}\sum_{k=0}^{n-1} T(k) \overset{\text{H}}{\leq} n-1+\frac{2}{n}\sum_{k=0}^{n-1} 4k\log k \\
&= n-1+\sum_{k=1}^{n/2} 4k \underbrace{\log k}_{\leq \log n - 1} + \sum_{k=n/2+1}^{n-1} 4k \underbrace{\log k}_{\leq \log n} \\
&\leq n-1+\frac{8}{n}\left((\log n - 1)\sum_{k=1}^{n/2} k + \log n \sum_{k=n/2+1}^{n-1} k\right) \\
&= n-1+\frac{8}{n}\left((\log n)\cdot\frac{n(n-1)}{2} - \frac{n}{4}\left(\frac{n}{2}+1\right)\right) \\
&= 4n\log n - 4\log n - 3 \leq 4n\log n
\end{aligned}
$$

Analysis (randomized quicksort)

Theorem

On average randomized quicksort requires $\mathcal{O}(n \cdot \log n)$ *comparisons.*

Practical considerations

Worst case recursion depth $n - 1$[9]. Then also a memory consumption of $\mathcal{O}(n)$.

Can be avoided: recursion only on the smaller part. Then guaranteed $\mathcal{O}(\log n)$ worst case recursion depth and memory consumption.

[9] stack overflow possible!

Quicksort with logarithmic memory consumption

Input : Array A with length n. $1 \leq l \leq r \leq n$.
Output : Array A, sorted between l and r.

while $l < r$ **do**
  Choose pivot $p \in A[l, \dots, r]$
  $k \leftarrow$ Partition$(A[l, \dots, r], p)$
  if $k - l < r - k$ **then**
    Quicksort$(A[l, \dots, k-1])$
    $l \leftarrow k + 1$
  else
    Quicksort$(A[k+1, \dots, r])$
    $r \leftarrow k - 1$

The call of Quicksort$(A[l, \dots, r])$ in the original algorithm has moved to iteration (tail recursion!): the if-statement became a while-statement.

Practical considerations.

Practically the pivot is often the median of three elements. For example: Median3($A[l], A[r], A[\lfloor l + r/2 \rfloor]$).

There is a variant of quicksort that requires only constant storage. Idea: store the old pivot at the position of the new pivot.