# 7. Sorting I

Simple Sorting



## 7.1 Simple Sorting

Selection Sort, Insertion Sort, Bubblesort [Ottman/Widmayer, Kap. 2.1, Cormen et al, Kap. 2.1, 2.2, Exercise 2.2-2, Problem 2-2



## Problem

**Input:** An array $A = (A[1], ..., A[n])$ with length $n$.

**Output:** a permutation $A'$ of $A$, that is sorted: $A'[i] \leq A'[j]$ for all $1 \leq i \leq j \leq n$.



## Algorithm: IsSorted($A$)

**Input** : Array $A = (A[1], ..., A[n])$ with length $n$.
**Output** : Boolean decision "sorted" or "not sorted"

```
for i ← 1 to n − 1 do
    if A[i] > A[i + 1] then
        return "not sorted";
return "sorted";
```

## Observation

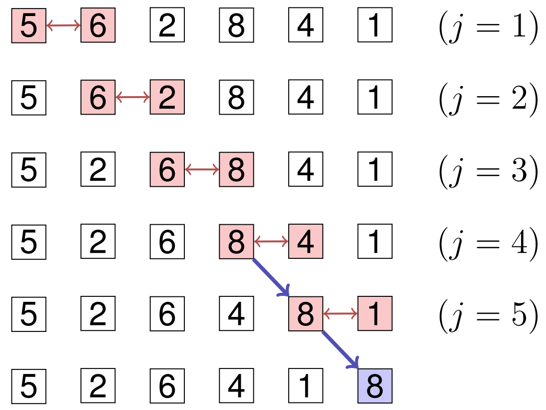
IsSorted($A$):"not sorted", if $A[i] > A[i+1]$ for an $i$.

$\Rightarrow$ idea:

**for** $j \leftarrow 1$ **to** $n-1$ **do**
    **if** $A[j] > A[j+1]$ **then**
        swap($A[j], A[j+1]$);



## Give it a try




- Not sorted! ☹.
- But the greatest element moves to the right
  $\Rightarrow$ new idea! ☺



## Try it out

| | | | | | | |
|---|---|---|---|---|---|---|
| 5 | 6 | 2 | 8 | 4 | 1 | $(j=1, i=1)$ |
| 5 | 6 | 2 | 8 | 4 | 1 | $(j=2)$ |
| 5 | 2 | 6 | 8 | 4 | 1 | $(j=3)$ |
| 5 | 2 | 6 | 8 | 4 | 1 | $(j=4)$ |
| 5 | 2 | 6 | 4 | 8 | 1 | $(j=5)$ |
| 5 | 2 | 6 | 4 | 1 | 8 | $(j=1, i=2)$ |
| 2 | 5 | 6 | 4 | 1 | 8 | $(j=2)$ |
| 2 | 5 | 6 | 4 | 1 | 8 | $(j=3)$ |
| 2 | 5 | 4 | 6 | 1 | 8 | $(j=4)$ |
| 2 | 5 | 4 | 1 | 6 | 8 | $(j=1, i=3)$ |
| 2 | 5 | 4 | 1 | 6 | 8 | $(j=2)$ |
| 2 | 4 | 5 | 1 | 6 | 8 | $(j=3)$ |
| 2 | 4 | 1 | 5 | 6 | 8 | $(j=1, i=4)$ |
| 2 | 4 | 1 | 5 | 6 | 8 | $(j=2)$ |
| 2 | 1 | 4 | 5 | 6 | 8 | $(i=1, j=5)$ |
| 1 | 2 | 4 | 5 | 6 | 8 | |

- Apply the procedure iteratively.
- For $A[1, \dots, n]$, then $A[1, \dots, n-1]$, then $A[1, \dots, n-2]$, etc.



## Algorithm: Bubblesort

**Input :** Array $A = (A[1], \dots, A[n])$, $n \geq 0$.
**Output :** Sorted Array $A$

**for** $i \leftarrow 1$ **to** $n-1$ **do**
    **for** $j \leftarrow 1$ **to** $n-i$ **do**
        **if** $A[j] > A[j+1]$ **then**
            swap($A[j], A[j+1]$);

## Analysis

Number key comparisons $\sum_{i=1}^{n-1}(n-i) = \frac{n(n-1)}{2} = \Theta(n^2)$.

Number swaps in the worst case: $\Theta(n^2)$

? What is the worst case?

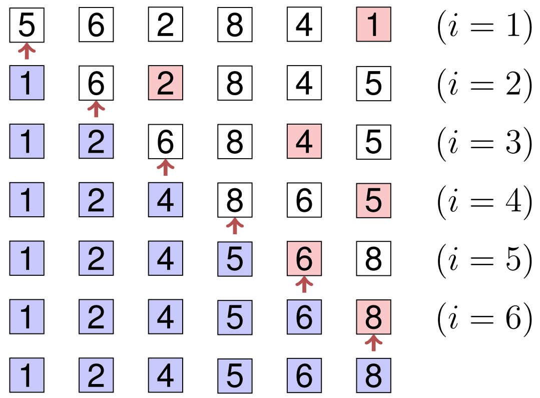
! If $A$ is sorted in decreasing order.

? Algorithm can be adapted such that it terminates when the array is sorted. Key comparisons and swaps of the modified algorithm in the best case?

! Key comparisons = $n-1$. Swaps = $0$.



## Selection Sort




- Iterative procedure as for Bubblesort.
- Selection of the smallest (or largest) element by immediate search.



## Algorithm: Selection Sort

**Input :** Array $A = (A[1], \dots, A[n])$, $n \geq 0$.
**Output :** Sorted Array $A$

```
for i ← 1 to n − 1 do
    p ← i
    for j ← i + 1 to n do
        if A[j] < A[p] then
            p ← j;
    swap(A[i], A[p])
```



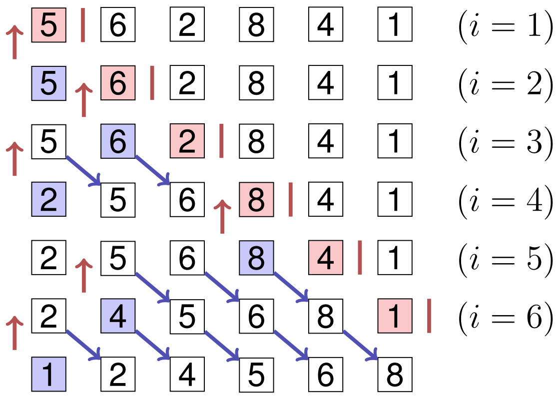
## Analysis

Number comparisons in worst case: $\Theta(n^2)$.

Number swaps in the worst case: $n-1 = \Theta(n)$

Best case number comparisons: $\Theta(n^2)$.

# Insertion Sort




- Iterative procedure: $i = 1...n$
- Determine insertion position for element $i$.
- Insert element $i$ array block movement potentially required



# Insertion Sort

? What is the disadvantage of this algorithm compared to sorting by selection?

! Many element movements in the worst case.

? What is the advantage of this algorithm compared to selection sort?

! The search domain (insertion interval) is already sorted. Consequently: binary search possible.



# Algorithm: Insertion Sort

**Input :** Array $A = (A[1], \dots, A[n])$, $n \geq 0$.
**Output :** Sorted Array $A$

**for** $i \leftarrow 2$ **to** $n$ **do**
  $x \leftarrow A[i]$
  $p \leftarrow$ BinarySearch($A[1...i-1], x$); // Smallest $p \in [1, i]$ with $A[p] \geq x$
  **for** $j \leftarrow i - 1$ **downto** $p$ **do**
    $A[j+1] \leftarrow A[j]$
  $A[p] \leftarrow x$



# Analysis

Number comparisons in the worst case:
$\sum_{k=1}^{n-1} a \cdot \log k = a \log((n-1)!) \in \mathcal{O}(n \log n)$.

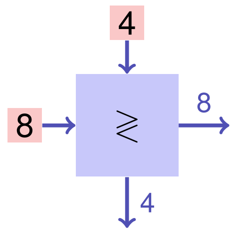
Number comparisons in the best case $\Theta(n \log n)$.[4]

Number swaps in the worst case $\sum_{k=2}^{n}(k-1) \in \Theta(n^2)$

[4] With slight modification of the function BinarySearch for the minimum / maximum: $\Theta(n)$

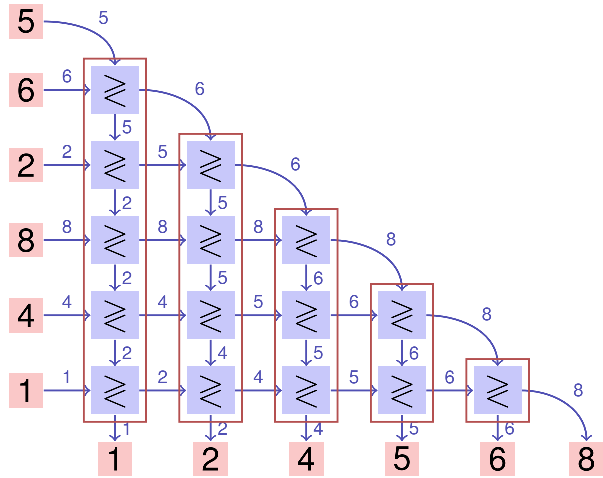
# Different point of view

Sorting node:





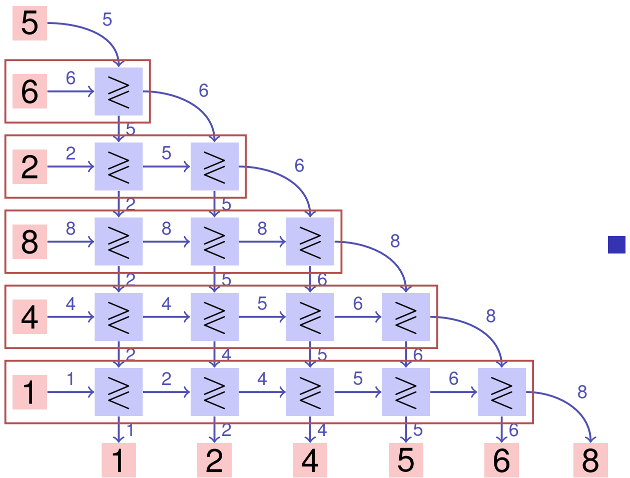

# Different point of view




- Like selection sort
  [and like Bubblesort]



# Different point of view




- Like insertion sort



# Conclusion

In a certain sense, Selection Sort, Bubble Sort and Insertion Sort provide the same kind of sort strategy. Will be made more precise. [5]

[5] In the part about parallel sorting networks. For the sequential code of course the observations as described above still hold.

## Shellsort

Insertion sort on subsequences of the form $(A_{k\cdot i})$ $(i \in \mathbb{N})$ with decreasing distances $k$. Last considered distance must be $k = 1$.

Good sequences: for example sequences with distances $k \in \{2^i 3^j | 0 \le i, j\}$.



## Shellsort

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| 9 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 1 | 0 | |
| 1 | 8 | 7 | 6 | 5 | 4 | 3 | 2 | 9 | 0 | insertion sort, $k = 4$ |
| 1 | 0 | 7 | 6 | 5 | 4 | 3 | 2 | 9 | 8 | |
| 1 | 0 | 3 | 6 | 5 | 4 | 7 | 2 | 9 | 8 | |
| 1 | 0 | 3 | 2 | 5 | 4 | 7 | 6 | 9 | 8 | |
| 1 | 0 | 3 | 2 | 5 | 4 | 7 | 6 | 9 | 8 | insertion sort, $k = 2$ |
| 1 | 0 | 3 | 2 | 5 | 4 | 7 | 6 | 9 | 8 | |
| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | insertion sort, $k = 1$ |



# 8. Sorting II

Heapsort, Quicksort, Mergesort



## 8.1 Heapsort

[Ottman/Widmayer, Kap. 2.3, Cormen et al, Kap. 6]

# Heapsort

Inspiration from selectsort: fast insertion

Inspiration from insertion sort: fast determination of position

? Can we have the best of both worlds?

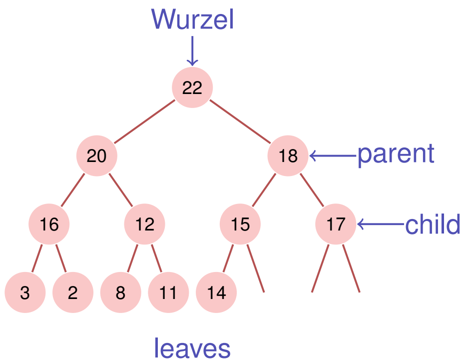
! Yes, but it requires some more thinking...



# [Max-]Heap[6]

Binary tree with the following properties

1. complete up to the lowest level
2. Gaps (if any) of the tree in the last level to the right
3. *Heap-Condition:* Max-(Min-)Heap: key of a child smaller (greater) thant that of the parent node



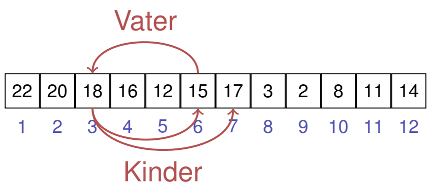

[6] Heap(data structure), not: as in "heap and stack" (memory allocation)

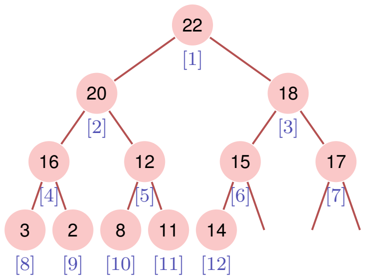


# Heap and Array

Tree → Array:

- children$(i) = \{2i, 2i+1\}$
- parent$(i) = \lfloor i/2 \rfloor$



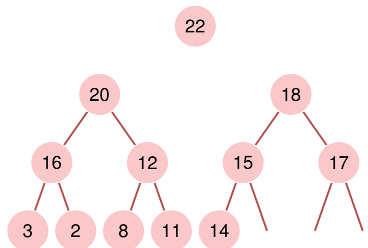

Depends on the starting index[7]

[7] For array that start at 0: $\{2i, 2i+1\} \to \{2i+1, 2i+2\}$, $\lfloor i/2 \rfloor \to \lfloor (i-1)/2 \rfloor$



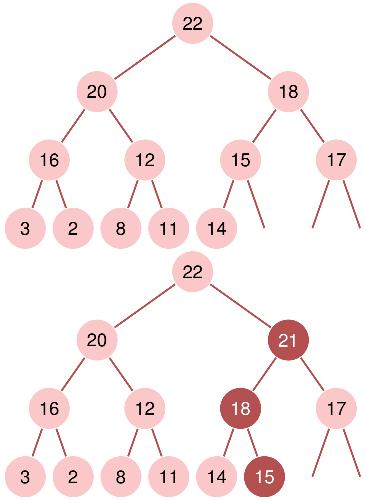
# Recursive heap structure

A heap consists of two heaps:



## Insert

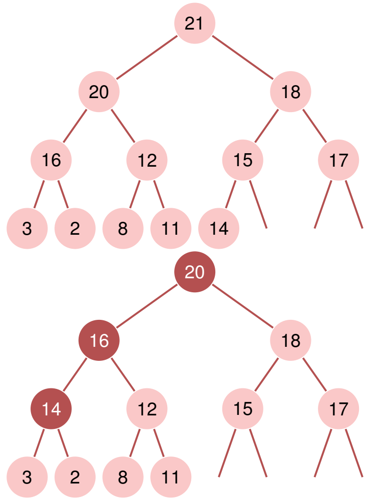
- Insert new element at the first free position. Potentially violates the heap property.
- Reestablish heap property: climb successively
- Worst case number of operations: $\mathcal{O}(\log n)$






## Remove the maximum

- Replace the maximum by the lower right element
- Reestablish heap property: sink successively (in the direction of the greater child)
- Worst case number of operations: $\mathcal{O}(\log n)$






## Algorithm Sink($A, i, m$)

**Input :** Array $A$ with heap structure for the children of $i$. Last element $m$.
**Output :** Array $A$ with heap structure for $i$ with last element $m$.

**while** $2i \leq m$ **do**
  $j \leftarrow 2i$; // $j$ left child
  **if** $j < m$ and $A[j] < A[j+1]$ **then**
    $j \leftarrow j + 1$; // $j$ right child with greater key
  **if** $A[i] < A[j]$ **then**
    swap($A[i], A[j]$)
    $i \leftarrow j$; // keep sinking
  **else**
    $i \leftarrow m$; // sinking finished



## Sort heap

$A[1, ..., n]$ is a Heap.
While $n > 1$

- swap($A[1]$, $A[n]$)
- Sink($A, 1, n-1$);
- $n \leftarrow n - 1$

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | 7 | 6 | 4 | 5 | 1 | 2 |
| swap | $\Rightarrow$ | 2 | 6 | 4 | 5 | 1 | 7 |
| sink | $\Rightarrow$ | 6 | 5 | 4 | 2 | 1 | 7 |
| swap | $\Rightarrow$ | 1 | 5 | 4 | 2 | 6 | 7 |
| sink | $\Rightarrow$ | 5 | 4 | 2 | 1 | 6 | 7 |
| swap | $\Rightarrow$ | 1 | 4 | 2 | 5 | 6 | 7 |
| sink | $\Rightarrow$ | 4 | 1 | 2 | 5 | 6 | 7 |
| swap | $\Rightarrow$ | 2 | 1 | 4 | 5 | 6 | 7 |
| sink | $\Rightarrow$ | 2 | 1 | 4 | 5 | 6 | 7 |
| swap | $\Rightarrow$ | 1 | 2 | 4 | 5 | 6 | 7 |

## Heap creation

Observation: Every leaf of a heap is trivially a correct heap.

Consequence: Induction from below!



## Algorithm HeapSort($A, n$)

**Input :** Array $A$ with length $n$.
**Output :** $A$ sorted.
// Build the heap.
**for** $i \leftarrow n/2$ **downto** $1$ **do**
  Sink($A, i, n$);
// Now $A$ is a heap.
**for** $i \leftarrow n$ **downto** $2$ **do**
  swap($A[1], A[i]$)
  Sink($A, 1, i-1$)
// Now $A$ is sorted.



## Analysis: sorting a heap

Sink traverses at most $\log n$ nodes. For each node $2$ key comparisons. $\Rightarrow$ sorting a heap costs in the worst case $2 \log n$ comparisons.

Number of memory movements of sorting a heap also $\mathcal{O}(n \log n)$.



## Analysis: creating a heap

Calls to sink: $n/2$. Thus number of comparisons and movements: $v(n) \in \mathcal{O}(n \log n)$.

But mean length of sinking paths is much smaller:

$$v(n) = \sum_{h=0}^{\lfloor \log n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil \cdot c \cdot h \in \mathcal{O}(n \sum_{h=0}^{\lfloor \log n \rfloor} \frac{h}{2^h})$$

with $s(x) := \sum_{k=0}^{\infty} k x^k = \frac{x}{(1-x)^2} \quad (0 < x < 1)$ [8] and $s(\frac{1}{2}) = 2$:

$$v(n) \in \mathcal{O}(n).$$

[8] $f(x) = \frac{1}{1-x} = 1 + x + x^2 ... \Rightarrow f'(x) = \frac{1}{(1-x)^2} = 1 + 2x + ...$

## 8.2 Mergesort

[Ottman/Widmayer, Kap. 2.4, Cormen et al, Kap. 2.3],



## Intermediate result

Heapsort: $\mathcal{O}(n \log n)$ Comparisons and movements.

**?** Disadvantages of heapsort?

- **!** Missing locality: heapsort jumps around in the sorted array (negative cache effect).
- **!** Two comparisons required before each necessary memory movement.



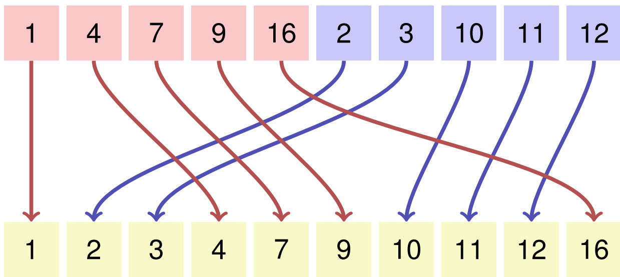
## Mergesort

Divide and Conquer!

- Assumption: two halves of the array $A$ are already sorted.
- Minimum of $A$ can be evaluated with two comparisons.
- Iteratively: sort the pre-sorted array $A$ in $\mathcal{O}(n)$.



## Merge



## Algorithm Merge($A, l, m, r$)

**Input :** Array $A$ with length $n$, indexes $1 \leq l \leq m \leq r \leq n$. $A[l, \dots, m]$, $A[m+1, \dots, r]$ sorted

**Output :** $A[l, \dots, r]$ sorted

```
1 B ← new Array(r − l + 1)
2 i ← l; j ← m + 1; k ← 1
3 while i ≤ m and j ≤ r do
4     if A[i] ≤ A[j] then B[k] ← A[i]; i ← i + 1
5     else B[k] ← A[j]; j ← j + 1
6     k ← k + 1;
7 while i ≤ m do B[k] ← A[i]; i ← i + 1; k ← k + 1
8 while j ≤ r do B[k] ← A[j]; j ← j + 1; k ← k + 1
9 for k ← l to r do A[k] ← B[k − l + 1]
```



## Correctness

Hypothesis: after $k$ iterations of the loop in line 3 $B[1, \dots, k]$ is sorted and $B[k] \leq A[i]$, if $i \leq m$ and $B[k] \leq A[j]$ if $j \leq r$.

Proof by induction:
*Base case:* the empty array $B[1, \dots, 0]$ is trivially sorted.
*Induction step* ($k \to k+1$):

- wlog $A[i] \leq A[j]$, $i \leq m, j \leq r$.
- $B[1, \dots, k]$ is sorted by hypothesis and $B[k] \leq A[i]$.
- After $B[k+1] \leftarrow A[i]$ $B[1, \dots, k+1]$ is sorted.
- $B[k+1] = A[i] \leq A[i+1]$ (if $i+1 \leq m$) and $B[k+1] \leq A[j]$ if $j \leq r$.
- $k \leftarrow k+1, i \leftarrow i+1$: Statement holds again.



## Analysis (Merge)

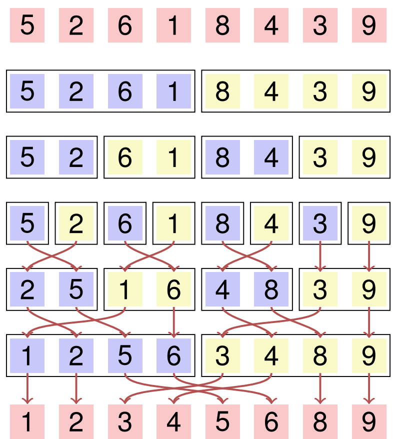
**Lemma**

*If: array $A$ with length $n$, indexes $1 \leq l < r \leq n$. $m = \lfloor (l+r)/2 \rfloor$ and $A[l, \dots, m]$, $A[m+1, \dots, r]$ sorted.*
*Then: in the call of Merge($A, l, m, r$) a number of $\Theta(r-l)$ key movements and comparisons are executed.*

Proof: straightforward(Inspect the algorithm and count the operations.)



## Mergesort




Split
Split
Split
Merge
Merge
Merge

## Algorithm recursive 2-way Mergesort($A, l, r$)

**Input :** Array $A$ with length $n$. $1 \leq l \leq r \leq n$
**Output :** Array $A[l, \ldots, r]$ sorted.

```
if l < r then
    m ← ⌊(l + r)/2⌋              // middle position
    Mergesort(A, l, m)           // sort lower half
    Mergesort(A, m + 1, r)       // sort higher half
    Merge(A, l, m, r)            // Merge subsequences
```



## Analysis

Recursion equation for the number of comparisons and key movements:

$$C(n) = C(\left\lceil \frac{n}{2} \right\rceil) + C(\left\lfloor \frac{n}{2} \right\rfloor) + \Theta(n) \in \Theta(n \log n)$$



## Algorithm StraightMergesort($A$)

*Avoid recursion:* merge sequences of length $1, 2, 4, \ldots$ directly

**Input :** Array $A$ with length $n$
**Output :** Array $A$ sorted

```
length ← 1
while length < n do                  // Iterate over lengths n
    r ← 0
    while r + length < n do          // Iterate over subsequences
        l ← r + 1
        m ← l + length − 1
        r ← min(m + length, n)
        Merge(A, l, m, r)
    length ← length · 2
```



## Analysis

Like the recursive variant, the straight 2-way mergesort always executes a number of $\Theta(n \log n)$ key comparisons and key movements.

## Natural 2-way mergesort

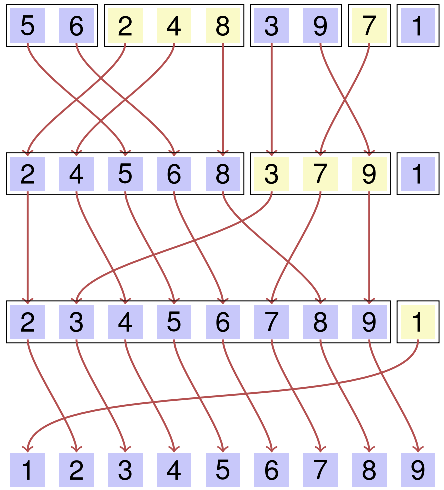
Observation: the variants above do not make use of any presorting and always execute $\Theta(n \log n)$ memory movements.

**?** How can partially presorted arrays be sorted better?

**!** Recursive merging of previously sorted parts (*runs*) of $A$.



## Natural 2-way mergesort






## Algorithm NaturalMergesort($A$)

**Input :** Array $A$ with length $n > 0$
**Output :** Array $A$ sorted

```
repeat
    r ← 0
    while r < n do
        l ← r + 1
        m ← l; while m < n and A[m + 1] ≥ A[m] do m ← m + 1
        if m < n then
            r ← m + 1; while r < n and A[r + 1] ≥ A[r] do r ← r + 1
            Merge(A, l, m, r);
        else
            r ← n
until l = 1
```



## Analysis

In the best case, natural merge sort requires only $n-1$ comparisons.

**?** Is it also asymptotically better than StraightMergesort on average?

**!** No. Given the assumption of pairwise distinct keys, on average there are $n/2$ positions $i$ with $k_i > k_{i+1}$, i.e. $n/2$ runs. Only one iteration is saved on average.

Natural mergesort executes in the worst case and on average a number of $\Theta(n \log n)$ comparisons and memory movements.

## 8.3 Quicksort

[Ottman/Widmayer, Kap. 2.2, Cormen et al, Kap. 7]



# Quicksort

? What is the disadvantage of Mergesort?

! Requires $\Theta(n)$ storage for merging.

? How could we reduce the merge costs?

! Make sure that the left part contains only smaller elements than the right part.

? How?

! Pivot and Partition!



# Quicksort (arbitrary pivot)

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 2 | 4 | 5 | 6 | 8 | 3 | 7 | 9 | 1 |
| 2 | 1 | 3 | 6 | 8 | 5 | 7 | 9 | 4 |
| 1 | 2 | 3 | 4 | 5 | 8 | 7 | 9 | 6 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 9 | 8 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |



# Algorithm Quicksort($A[l, \dots, r]$)

**Input :** Array $A$ with length $n$. $1 \le l \le r \le n$.
**Output :** Array $A$, sorted between $l$ and $r$.
**if** $l < r$ **then**
  Choose pivot $p \in A[l, \dots, r]$
  $k \leftarrow$ Partition($A[l, \dots, r], p$)
  Quicksort($A[l, \dots, k-1]$)
  Quicksort($A[k+1, \dots, r]$)

## Reminder: algorithm Partition($A[l, \dots, r], p$)

**Input :** Array $A$, that contains the pivot $p$ in $[l, r]$ at least once.
**Output :** Array $A$ partitioned around $p$. Returns the position of $p$.

```
while l ≤ r do
    while A[l] < p do
        l ← l + 1
    while A[r] > p do
        r ← r − 1
    swap(A[l], A[r])
    if A[l] = A[r] then          // Only for keys that are not pairwise different
        l ← l + 1
return l-1
```



## Analysis: number comparisons

*Best case.* Pivot = median; number comparisons:

$$T(n) = 2T(n/2) + c \cdot n, \; T(1) = 0 \quad \Rightarrow \quad T(n) \in \mathcal{O}(n \log n)$$

*Worst case.* Pivot = min or max; number comparisons:

$$T(n) = T(n-1) + c \cdot n, \; T(1) = 0 \quad \Rightarrow \quad T(n) \in \Theta(n^2)$$



## Analysis: number swaps

Result of a call to partition (pivot 3):

2 1 3 6 8 5 7 9 4

? How many swaps have taken place?

! 2. The maximum number of swaps is given by the number of keys in the smaller part.



## Analysis: number swaps

*Intellectual game*

- Each key from the smaller part pay a coin when swapped.
- When a key has paid a coin then the domain containing the key is less than or equal to half the previous size.
- Every key needs to pay at most $\log n$ coins. But there are only $n$ keys.

*Consequence:* there are $\mathcal{O}(n \log n)$ key swaps in the worst case.

## Randomized Quicksort

Despite the worst case running time of $\Theta(n^2)$, quicksort is used practically very often.

Reason: quadratic running time unlikely provided that the choice of the pivot and the pre-sorting are not very disadvantageous.

Avoidance: randomly choose pivot. Draw uniformly from $[l, r]$.



## Analysis (randomized quicksort)

Expected number of compared keys with input length $n$:

$$T(n) = (n-1) + \frac{1}{n}\sum_{k=1}^{n}\left(T(k-1) + T(n-k)\right),\ T(0) = T(1) = 0$$

Claim $T(n) \leq 4n\log n$.

Proof by induction:
*Base case* straightforward for $n = 0$ (with $0 \log 0 := 0$) and for $n = 1$.
*Hypothesis:* $T(n) \leq 4n\log n$ for some $n$.
*Induction step:* $(n-1 \to n)$



## Analysis (randomized quicksort)

$$\begin{aligned}
T(n) &= n - 1 + \frac{2}{n}\sum_{k=0}^{n-1} T(k) \overset{\mathsf{H}}{\leq} n - 1 + \frac{2}{n}\sum_{k=0}^{n-1} 4k\log k \\
&= n - 1 + \sum_{k=1}^{n/2} 4k \underbrace{\log k}_{\leq \log n - 1} + \sum_{k=n/2+1}^{n-1} 4k \underbrace{\log k}_{\leq \log n} \\
&\leq n - 1 + \frac{8}{n}\left((\log n - 1)\sum_{k=1}^{n/2} k + \log n \sum_{k=n/2+1}^{n-1} k\right) \\
&= n - 1 + \frac{8}{n}\left((\log n)\cdot\frac{n(n-1)}{2} - \frac{n}{4}\left(\frac{n}{2} + 1\right)\right) \\
&= 4n\log n - 4\log n - 3 \leq 4n\log n
\end{aligned}$$



## Analysis (randomized quicksort)

**Theorem**

*On average randomized quicksort requires $\mathcal{O}(n \cdot \log n)$ comparisons.*

## Practical considerations

Worst case recursion depth $n - 1$[9]. Then also a memory consumption of $\mathcal{O}(n)$.

Can be avoided: recursion only on the smaller part. Then guaranteed $\mathcal{O}(\log n)$ worst case recursion depth and memory consumption.

[9] stack overflow possible!



## Quicksort with logarithmic memory consumption

**Input** : Array $A$ with length $n$. $1 \le l \le r \le n$.
**Output** : Array $A$, sorted between $l$ and $r$.

**while** $l < r$ **do**
  Choose pivot $p \in A[l, \dots, r]$
  $k \leftarrow$ Partition$(A[l, \dots, r], p)$
  **if** $k - l < r - k$ **then**
    Quicksort$(A[l, \dots, k-1])$
    $l \leftarrow k + 1$
  **else**
    Quicksort$(A[k+1, \dots, r])$
    $r \leftarrow k - 1$

The call of Quicksort($A[l, \dots, r]$) in the original algorithm has moved to iteration (tail recursion!): the if-statement became a while-statement.



## Practical considerations.

Practically the pivot is often the median of three elements. For example: Median3($A[l], A[r], A[\lfloor l + r/2 \rfloor]$).

There is a variant of quicksort that requires only constant storage. Idea: store the old pivot at the position of the new pivot.