4. Searching

Linear Search, Binary Search, Interpolation Search, Lower Bounds [Ottman/Widmayer, Kap. 3.2, Cormen et al, Kap. 2: Problems 2.1-3,2.2-3,2.3-5]

The Search Problem

Provided

- A set of data sets

examples

telephone book, dictionary, symbol table

- Each dataset has a key k.
- Keys are comparable: unique answer to the question $k_1 \leq k_2$ for keys k_1, k_2.

Task: find data set by key k.

The Selection Problem

Provided

- Set of data sets with comparable keys k.

Wanted: data set with smallest, largest, middle key value. Generally: find a data set with i-smallest key.

Search in Array

Provided

- Array A with n elements $(A[1], \ldots, A[n])$.
- Key b

Wanted: index k, $1 \leq k \leq n$ with $A[k] = b$ or "not found".

22	20	32	10	35	24	42	38	28	41
1	2	3	4	5	6	7	8	9	10

Linear Search

Traverse the array from $A[1]$ to $A[n]$.

- *Best case:* 1 comparison.
- *Worst case:* n comparisons.
- Assumption: each permutation of the n keys with same probability. *Expected* number of comparisons:

$$\frac{1}{n}\sum_{i=1}^{n} i = \frac{n+1}{2}.$$

Search in a Sorted Array

Provided

- Sorted array A with n elements $(A[1], \ldots, A[n])$ with $A[1] \leq A[2] \leq \cdots \leq A[n]$.
- Key b

Wanted: index k, $1 \leq k \leq n$ with $A[k] = b$ or "not found".

10	20	22	24	28	32	35	38	41	42
1	2	3	4	5	6	7	8	9	10

Divide and Conquer!

Search $b = 23$.

10	20	22	24	28	32	35	38	41	42	$b < 28$
1	2	3	4	5	6	7	8	9	10	
10	20	22	24	28	32	35	38	41	42	$b > 20$
1	2	3	4	5	6	7	8	9	10	
10	20	22	24	28	32	35	38	41	42	$b > 22$
1	2	3	4	5	6	7	8	9	10	
10	20	22	24	28	32	35	38	41	42	$b < 24$
1	2	3	4	5	6	7	8	9	10	
10	20	22	24	28	32	35	38	41	42	erfolglos
1	2	3	4	5	6	7	8	9	10	

Binary Search Algorithm BSearch$(A[l..r], b)$

Input : Sorted array A of n keys. Key b. Bounds $1 \leq l \leq r \leq n$ or $l > r$ beliebig.
Output : Index of the found element. 0, if not found.

```
m ← ⌊(l + r)/2⌋
if l > r then // Unsuccessful search
    return NotFound
else if b = A[m] then// found
    return m
else if b < A[m] then// element to the left
    return BSearch(A[l..m − 1], b)
else // b > A[m]: element to the right
    return BSearch(A[m + 1..r], b)
```

Analysis (worst case)

Recurrence ($n = 2^k$)

$$T(n) = \begin{cases} d & \text{falls } n = 1, \\ T(n/2) + c & \text{falls } n > 1. \end{cases}$$

Compute:

$$\begin{aligned} T(n) &= T\left(\frac{n}{2}\right) + c = T\left(\frac{n}{4}\right) + 2c \\ &= T\left(\frac{n}{2^i}\right) + i \cdot c \\ &= T\left(\frac{n}{n}\right) + c \cdot \log_2 n = d + c \cdot \log_2 n \end{aligned}$$

$\Rightarrow$ Assumption: $T(n) = d + c \log_2 n$

Analysis (worst case)

$$T(n) = \begin{cases} d & \text{if } n = 1, \\ T(n/2) + c & \text{if } n > 1. \end{cases}$$

Guess : $T(n) = d + c \cdot \log_2 n$

Proof by induction:

- Base clause: $T(1) = d$.
- Hypothesis: $T(n/2) = d + c \cdot \log_2 n/2$
- Step: $(n/2 \to n)$

$$T(n) = T(n/2) + c = d + c \cdot (\log_2 n - 1) + c = d + c \log_2 n.$$

Result

Theorem

The binary sorted search algorithm requires $\Theta(\log n)$ fundamental operations.

Iterative Binary Search Algorithm

Input : Sorted array A of n keys. Key b.
Output : Index of the found element. 0, if unsuccessful.

```
l ← 1; r ← n
while l ≤ r do
    m ← ⌊(l + r)/2⌋
    if A[m] = b then
        return m
    else if A[m] < b then
        l ← m + 1
    else
        r ← m − 1
return NotFound;
```

Correctness

Algorithm terminates only if A is empty or b is found.

Invariant: If b is in A then b is in domain $A[l..r]$

Proof by induction

- Base clause $b \in A[1..n]$ (oder nicht)
- Hypothesis: invariant holds after i steps.
- Step:
  $b < A[m] \Rightarrow b \in A[l..m-1]$
  $b > A[m] \Rightarrow b \in A[m+1..r]$

Can this be improved?

Assumption: *values* of the array are uniformly distributed.

Example

Search for "Becker" at the very beginning of a telephone book while search for "Wawrinka" rather close to the end.
Binary search always starts in the middle.

Binary search always takes $m = \left\lfloor l + \frac{r-l}{2} \right\rfloor$.

Interpolation search

Expected relative position of b in the search interval $[l, r]$

$$\rho = \frac{b - A[l]}{A[r] - A[l]} \in [0, 1].$$

New 'middle': $l + \rho \cdot (r - l)$

Expected number of comparisons $\mathcal{O}(\log \log n)$ (without proof).

? Would you always prefer interpolation search?

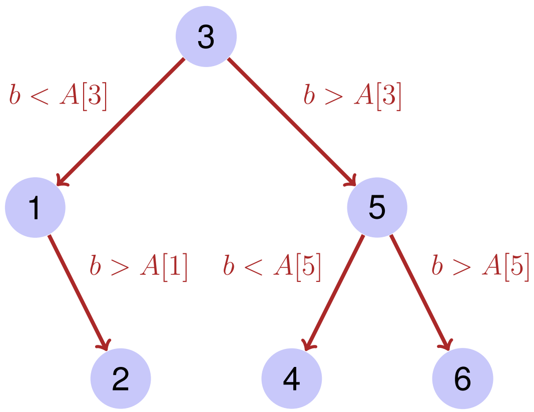
! No: worst case number of comparisons $\Omega(n)$.

Lower Bounds

Binary Search (worst case): $\Theta(\log n)$ comparisons.

Does for *any* search algorithm in a sorted array (worst case) hold that number comparisons = $\Omega(\log n)$?

Decision tree



- For any input $b = A[i]$ the algorithm must succeed $\Rightarrow$ decision tree comprises at least n nodes.
- Number comparisons in worst case = height of the tree = maximum number nodes from root to leaf.

Decision Tree

Binary tree with height h has at most $2^0 + 2^1 + \cdots + 2^{h-1} = 2^h - 1 < 2^h$ nodes.

$$2^h > n \Rightarrow h > \log_2 n$$

Decision tree with n node has at least height $\log_2 n$.

Number decisions = $\Omega(\log n)$.

Theorem

Any search algorithm on sorted data with length n requires in the worst case $\Omega(\log n)$ comparisons.

Lower bound for Search in Unsorted Array

Theorem

Any search algorithm with unsorted data of length n requires in the worst case $\Omega(n)$ comparisons.

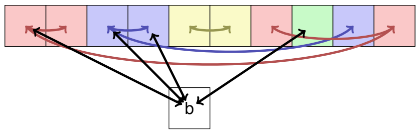
Attempt

? Correct?

"Proof": to find b in A, b must be compared with each of the n elements $A[i]$ $(1 \leq i \leq n)$.

! Wrong argument! It is still possible to compare elements within A.

Better Argument



- Different comparisons: Number comparisons with b: e Number comparisons without b: i
- Comparisons induce g groups. Initially $g = n$.
- To connect two groups at least one comparison is needed: $n - g \leq i$.
- At least one element per group must be compared with b.
- Number comparisons $i + e \geq n - g + g = n$. ■

5. Selection

The Selection Problem, Randomised Selection, Linear Worst-Case Selection [Ottman/Widmayer, Kap. 3.1, Cormen et al, Kap. 9]

Min and Max

? To separately find minimum an maximum in $(A[1], \ldots, A[n])$, $2n$ comparisons are required. (How) can an algorithm with less than $2n$ comparisons for both values at a time can be found?

! Possible with $\frac{3}{2}n$ comparisons: compare 2 elemetns each and then the smaller one with min and the greater one with max.

The Problem of Selection

Input

- unsorted array $A = (A_1, \ldots, A_n)$ with pairwise different values
- Number $1 \leq k \leq n$.

Output $A[i]$ with $|\{j : A[j] < A[i]\}| = k - 1$

Special cases

$k = 1$: Minimum: Algorithm with n comparison operations trivial.
$k = n$: Maximum: Algorithm with n comparison operations trivial.
$k = \lfloor n/2 \rfloor$: Median.

Approaches

- Repeatedly find and remove the minimum $\mathcal{O}(k \cdot n)$.
  Median: $\mathcal{O}(n^2)$
- Sorting (covered soon): $\mathcal{O}(n \log n)$
- Use a pivot $\mathcal{O}(n)$!

Use a pivot

1. Choose a *pivot* p
2. Partition A in two parts, thereby determining the rank of p.
3. Recursion on the relevant part. If $k = r$ then found.

$\leq$	$\leq$	$\leq$	$\leq$	$\leq$	p	$>$	$>$	$>$	$>$
1					r				n

Algorithmus Partition($A[l..r], p$)

Input : Array A, that contains the pivot p in the interval $[l, r]$ at least once.
Output : Array A partitioned in $[l..r]$ around p. Returns position of p.

while $l \leq r$ **do**
- **while** $A[l] < p$ **do**
  - $l \leftarrow l + 1$
- **while** $A[r] > p$ **do**
  - $r \leftarrow r - 1$
- swap($A[l]$, $A[r]$)
- **if** $A[l] = A[r]$ **then**
  - $l \leftarrow l + 1$

return l-1

Correctness: Invariant

Invariant I: $A_i \leq p \; \forall i \in [0, l)$, $A_i \geq p \; \forall i \in (r, n]$, $\exists k \in [l, r] : A_k = p$.

while $l \leq r$ **do**
- — I
- **while** $A[l] < p$ **do**
  - $l \leftarrow l + 1$
- — I und $A[l] \geq p$
- **while** $A[r] > p$ **do**
  - $r \leftarrow r - 1$
- — I und $A[r] \leq p$
- swap($A[l]$, $A[r]$)
- — I und $A[l] \leq p \leq A[r]$
- **if** $A[l] = A[r]$ **then**
  - $l \leftarrow l + 1$
- — I

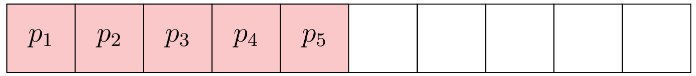
return l-1

Correctness: progress

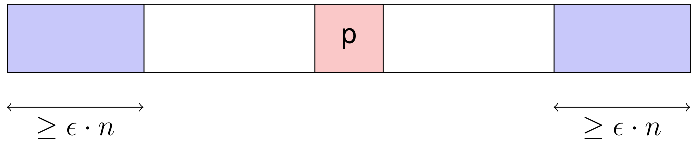
while $l \leq r$ **do**
    while $A[l] < p$ **do**   progress if $A[l] < p$
        $l \leftarrow l + 1$
    while $A[r] > p$ **do**   progress if $A[r] > p$
        $r \leftarrow r - 1$
    swap($A[l]$, $A[r]$)   progress if $A[l] > p$ oder $A[r] < p$
    if $A[l] = A[r]$ **then**   progress if $A[l] = A[r] = p$
        $l \leftarrow l + 1$
return l-1

Choice of the pivot.

The minimum is a bad pivot: worst case $\Theta(n^2)$



A good pivot has a linear number of elements on both sides.



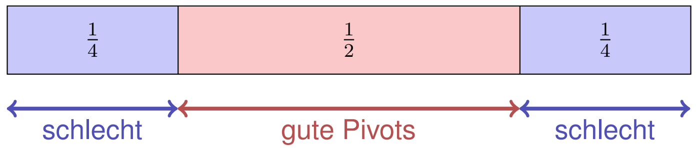
Analysis

Partitioning with factor q $(0 < q < 1)$: two groups with $q \cdot n$ and $(1 - q) \cdot n$ elements (without loss of generality $g \geq 1 - q$).

$$\begin{aligned} T(n) &\leq T(q \cdot n) + c \cdot n \\ &= c \cdot n + q \cdot c \cdot n + T(q^2 \cdot n) = \ldots = c \cdot n \sum_{i=0}^{\log_q(n)-1} q^i + T(1) \\ &\leq c \cdot n \underbrace{\sum_{i=0}^{\infty} q^i}_{\text{geom. Reihe}} + d = c \cdot n \cdot \frac{1}{1-q} + d = \mathcal{O}(n) \end{aligned}$$

How can we achieve this?

Randomness to our rescue (Tony Hoare, 1961). In each step choose a random pivot.



Probability for a good pivot in one trial: $\frac{1}{2} =: \rho$.

Probability for a good pivot after k trials: $(1 - \rho)^{k-1} \cdot \rho$.

Expected value of the geometric distribution: $1/\rho = 2$

[Expected value of the Geometric Distribution]

Random variable $X \in \mathbb{N}^+$ with $\mathbb{P}(X = k) = (1-p)^{k-1} \cdot p$.

Expected value

$$\begin{aligned}\mathbb{E}(X) &= \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} \cdot p = \sum_{k=1}^{\infty} k \cdot q^{k-1} \cdot (1-q) \\ &= \sum_{k=1}^{\infty} k \cdot q^{k-1} - k \cdot q^k = \sum_{k=0}^{\infty} (k+1) \cdot q^k - k \cdot q^k \\ &= \sum_{k=0}^{\infty} q^k = \frac{1}{1-q} = \frac{1}{p}.\end{aligned}$$

Algorithm Quickselect ($A[l..r], k$)

Input : Array A with length n. Indices $1 \le l \le k \le r \le n$, such that for all $x \in A[l..r] : |\{j | A[j] \le x\}| \ge l$ and $|\{j | A[j] \le x\}| \le r$.
Output : Value $x \in A[l..r]$ with $|\{j | A[j] \le x\}| \ge k$ and $|\{j | x \le A[j]\}| \ge n - k + 1$

if l=r **then**
  return $A[l]$;
$x \leftarrow$ RandomPivot$(A[l..r])$
$m \leftarrow$ Partition$(A[l..r], x)$
if $k < m$ **then**
  return QuickSelect$(A[l..m-1], k)$
else if $k > m$ **then**
  return QuickSelect$(A[m+1..r], k)$
else
  return $A[k]$

Algorithm RandomPivot ($A[l..r]$)

Input : Array A with length n. Indices $1 \le l \le i \le r \le n$
Output : Random "good" pivot $x \in A[l..r]$

repeat
  choose a random pivot $x \in A[l..r]$
  $p \leftarrow l$
  for $j = l$ **to** r **do**
    if $A[j] \le x$ **then** $p \leftarrow p + 1$
until $\left\lfloor \frac{3l+r}{4} \right\rfloor \le p \le \left\lceil \frac{l+3r}{4} \right\rceil$
return x

This algorithm is only of theoretical interest and delivers a good pivot in 2 expected iterations. Practically, in algorithm QuickSelect a uniformly chosen random pivot can be chosen or a deterministic one such as the median of three elements.

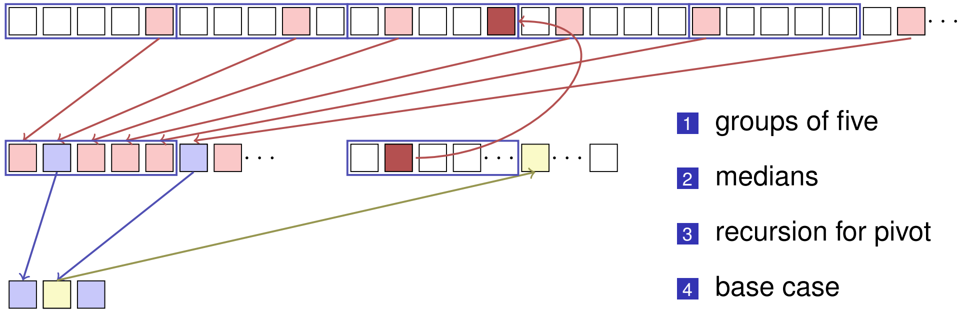
Median of medians

Goal: find an algorithm that even in worst case requires only linearly many steps.

Algorithm Select (k-smallest)

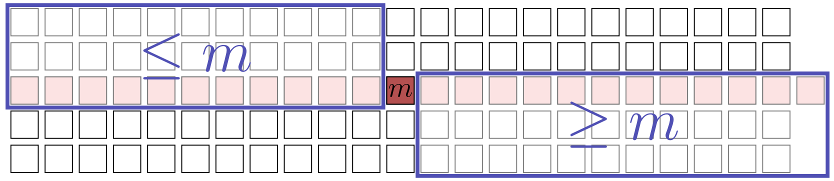
- Consider groups of five elements.
- Compute the median of each group (straighforward)
- Apply Select recursively on the group medians.
- Partition the array around the found median of medians. Result: i
- If $i = k$ then result. Otherwise: select recursively on the proper side.

Median of medians



1. groups of five
2. medians
3. recursion for pivot
4. base case
5. pivot (level 1)
6. partition (level 1)
7. median = pivot level 0
8. 2. recursion starts

How good is this?



Number points left / right of the median of medians (without median group and the rest group) $\geq 3 \cdot (\lceil \frac{1}{2} \lceil \frac{n}{5} \rceil \rceil - 2) \geq \frac{3n}{10} - 6$

Second call with maximally $\lceil \frac{7n}{10} + 6 \rceil$ elements.

Analysis

Recursion inequality:

$$T(n) \leq T\left(\left\lceil \frac{n}{5} \right\rceil\right) + T\left(\left\lceil \frac{7n}{10} + 6 \right\rceil\right) + d \cdot n.$$

with some constant d.

Claim:

$$T(n) = \mathcal{O}(n).$$

Proof

Base clause: choose c large enough such that

$$T(n) \leq c \cdot n \text{ für alle } n \leq n_0.$$

Induction hypothesis:

$$T(i) \leq c \cdot i \text{ für alle } i < n.$$

Induction step:

$$\begin{aligned} T(n) &\leq T\left(\left\lceil \frac{n}{5} \right\rceil\right) + T\left(\left\lceil \frac{7n}{10} + 6 \right\rceil\right) + d \cdot n \\ &= c \cdot \left\lceil \frac{n}{5} \right\rceil + c \cdot \left\lceil \frac{7n}{10} + 6 \right\rceil + d \cdot n. \end{aligned}$$

Proof

Induction step:

$$T(n) \le c \cdot \left\lceil \frac{n}{5} \right\rceil + c \cdot \left\lceil \frac{7n}{10} + 6 \right\rceil + d \cdot n$$
$$\le c \cdot \frac{n}{5} + c + c \cdot \frac{7n}{10} + 6c + c + d \cdot n = \frac{9}{10} \cdot c \cdot n + 8c + d \cdot n.$$

Choose $c \ge 80 \cdot d$ and $n_0 = 91$.

$$T(n) \le \frac{72}{80} \cdot c \cdot n + 8c + \frac{1}{80} \cdot c \cdot n = c \cdot \underbrace{\left(\frac{73}{80} n + 8\right)}_{\le n \text{ für } n > n_0} \le c \cdot n.$$

Result

Theorem

The k-the element of a sequence of n elements can be found in at most $\mathcal{O}(n)$ steps.

Overview

1.	Repeatedly find minimum	$\mathcal{O}(n^2)$
2.	Sorting and choosing $A[i]$	$\mathcal{O}(n \log n)$
3.	Quickselect with random pivot	$\mathcal{O}(n)$ expected
4.	Median of Medians (Blum)	$\mathcal{O}(n)$ worst case