19. Dynamic Programming I

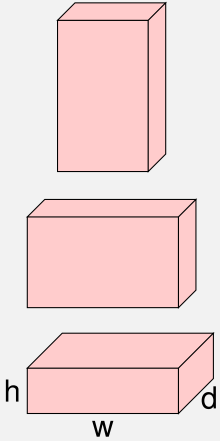
Fibonacci, Längste aufsteigende Teilfolge, längste gemeinsame Teilfolge, Editierdistanz, Matrixkettenmultiplikation, Matrixmultiplikation nach Strassen [Ottman/Widmayer, Kap. 1.2.3, 7.1, 7.4, Cormen et al, Kap. 15]

Quiz: Stacking Boxes

- Given: n boxes with sizes $w_i \times d_i \times h_i$
- Wanted: maximal height of a permitted stack
- Permitted stack: the base area of stacked boxes must become strictly smaller in both directions (width and depth)

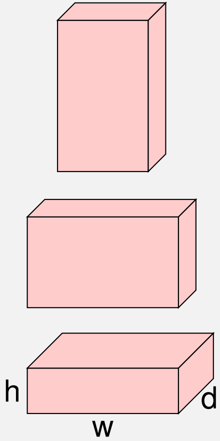
Boxen Stapeln

We assume that there are enough boxes of a kind such that each box is available in all orientations (right hand side of the figure below).



Box	1	2	3	4	5	6
$[w \times d \times h]$	$[1 \times 2 \times 3]$	$[1 \times 3 \times 2]$	$[2 \times 3 \times 1]$	$[3 \times 4 \times 5]$	$[3 \times 5 \times 4]$	$[4 \times 5 \times 3]$

Boxen Stapeln



We assume that there are enough boxes of a kind such that each box is available in all orientations (right hand side of the figure below).

Solution: later

Box	1	2	3	4	5	6
$[w \times d \times h]$	$[1 \times 2 \times 3]$	$[1 \times 3 \times 2]$	$[2 \times 3 \times 1]$	$[3 \times 4 \times 5]$	$[3 \times 5 \times 4]$	$[4 \times 5 \times 3]$

Simpler: Fibonacci Numbers

 (again)

$$F_n := \begin{cases} n & \text{if } n < 2 \\ F_{n-1} + F_{n-2} & \text{if } n \geq 2. \end{cases}$$

Analysis: why ist the recursive algorithm so slow?

Algorithm FibonacciRecursive(n)

Input : $n \geq 0$
Output : n-th Fibonacci number

if $n < 2$ **then**
  $f \leftarrow n$
else
  $f \leftarrow$ FibonacciRecursive$(n-1)$ + FibonacciRecursive$(n-2)$
return f

Analysis

$T(n)$: Number executed operations.

- $n = 0, 1$: $T(n) = \Theta(1)$

Analysis

$T(n)$: Number executed operations.

- $n = 0, 1$: $T(n) = \Theta(1)$
- $n \geq 2$: $T(n) = T(n-2) + T(n-1) + c$.

Analysis

$T(n)$: Number executed operations.

- $n = 0, 1$: $T(n) = \Theta(1)$
- $n \geq 2$: $T(n) = T(n-2) + T(n-1) + c$.

$$T(n) = T(n-2) + T(n-1) + c \geq 2T(n-2) + c \geq 2^{n/2}c' = (\sqrt{2})^n c'$$

Analysis

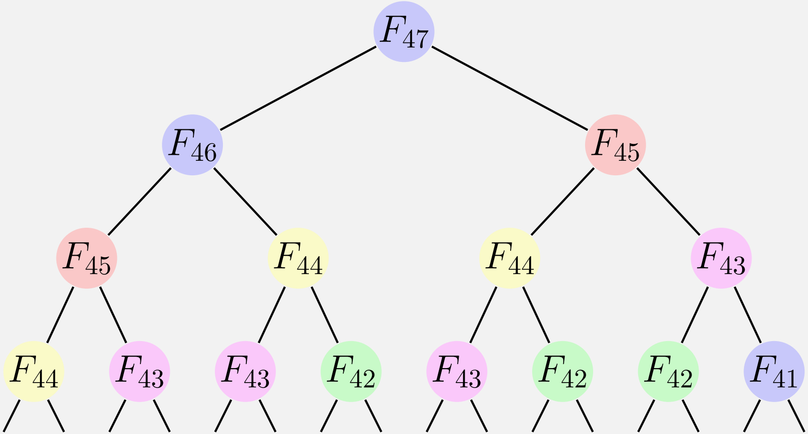
$T(n)$: Number executed operations.

- $n = 0, 1$: $T(n) = \Theta(1)$
- $n \geq 2$: $T(n) = T(n-2) + T(n-1) + c$.

$$T(n) = T(n-2) + T(n-1) + c \geq 2T(n-2) + c \geq 2^{n/2}c' = (\sqrt{2})^n c'$$

Algorithm is *exponential* in n.

Reason (visual)



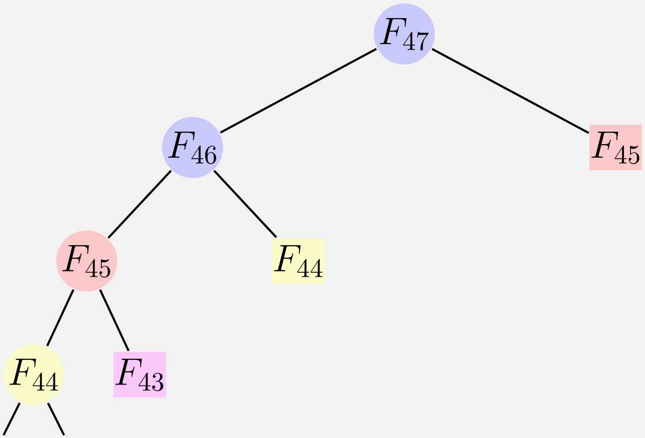
Nodes with same values are evaluated (too) often.

Memoization

Memoization (sic) saving intermediate results.

- Before a subproblem is solved, the existence of the corresponding intermediate result is checked.
- If an intermediate result exists then it is used.
- Otherwise the algorithm is executed and the result is saved accordingly.

Memoization with Fibonacci



Rechteckige Knoten wurden bereits ausgewertet.

Algorithm FibonacciMemoization(n)

Input : $n \geq 0$
Output : n-th Fibonacci number

if $n \leq 2$ **then**
    $f \leftarrow 1$
else if $\exists$memo$[n]$ **then**
    $f \leftarrow$ memo$[n]$
else
    $f \leftarrow$ FibonacciMemoization$(n-1)$ + FibonacciMemoization$(n-2)$
    memo$[n] \leftarrow f$

return f

Analysis

Computational complexity:

$$T(n) = T(n-1) + c = ... = \mathcal{O}(n).$$

Algorithm requires $\Theta(n)$ memory.[28]

[28] But the naive recursive algorithm also requires $\Theta(n)$ memory implicitly.

Looking closer ...

... the algorithm computes the values of F_1, F_2, F_3,... in the *top-down* approach of the recursion.

Can write the algorithm *bottom-up*. Then it is called *dynamic programming*.

Algorithm FibonacciDynamicProgram(n)

Input : $n \geq 0$
Output : n-th Fibonacci number

$F[1] \leftarrow 1$
$F[2] \leftarrow 1$
for $i \leftarrow 3, \ldots, n$ **do**
    $F[i] \leftarrow F[i-1] + F[i-2]$

return $F[n]$

Dynamic Programming: Idea

- Divide a complex problem into a reasonable number of sub-problems
- The solution of the sub-problems will be used to solve the more complex problem
- Identical problems will be computed only once

Dynamic Programming Consequence

Identical problems will be computed only once

$\Rightarrow$ Results are saved

We trade spee against memory consumption

Dynamic Programming = Divide-And-Conquer ?

- In both cases the original problem can be solved (more easily) by utilizing the solutions of sub-problems. The problem provides *optimal substructure*.
- Divide-And-Conquer algorithms (such as Mergesort): sub-problems are independent; their solutions are required only once in the algorithm.
- DP: sub-problems are dependent. The problem is said to have *overlapping sub-problems* that are required multiple-times in the algorithm. In order to avoid redundant computations, results have to be tabulated.

Dynamic Programming: Procedure

1. Use a *DP-table* with information to the subproblems.
   Dimension of the entries? Semantics of the entries?

Dynamic Programming: Procedure

1. Use a *DP-table* with information to the subproblems.
   Dimension of the entries? Semantics of the entries?
2. Computation of the *base cases*
   Which entries do not depend on others?

Dynamic Programming: Procedure

1. Use a *DP-table* with information to the subproblems.
   Dimension of the entries? Semantics of the entries?
2. Computation of the *base cases*
   Which entries do not depend on others?
3. Determine *computation order*.
   In which order can the entries be computed such that dependencies are fulfilled?

Dynamic Programming: Procedure

1. Use a *DP-table* with information to the subproblems.
   Dimension of the entries? Semantics of the entries?
2. Computation of the *base cases*
   Which entries do not depend on others?
3. Determine *computation order*.
   In which order can the entries be computed such that dependencies are fulfilled?
4. Read-out the *solution*
   How can the solution be read out from the table?

Dynamic Programming: Procedure

1. Use a *DP-table* with information to the subproblems.
   Dimension of the entries? Semantics of the entries?
2. Computation of the *base cases*
   Which entries do not depend on others?
3. Determine *computation order*.
   In which order can the entries be computed such that dependencies are fulfilled?
4. Read-out the *solution*
   How can the solution be read out from the table?

Runtime (typical) = number entries of the table times required operations per entry.

Dynamic Programing: Procedure with the example

1. Dimension of the table? Semantics of the entries?

Dynamic Programing: Procedure with the example

1. **Dimension of the table? Semantics of the entries?**

   $n \times 1$ table. nth entry contains nth Fibonacci number.

Dynamic Programing: Procedure with the example

1. **Dimension of the table? Semantics of the entries?**

   $n \times 1$ table. nth entry contains nth Fibonacci number.

2. **Which entries do not depend on other entries?**

Dynamic Programing: Procedure with the example

1. **Dimension of the table? Semantics of the entries?**

   $n \times 1$ table. nth entry contains nth Fibonacci number.

2. **Which entries do not depend on other entries?**

   Values F_1 and F_2 can be computed easily and independently.

Dynamic Programing: Procedure with the example

1. **Dimension of the table? Semantics of the entries?**

   $n \times 1$ table. nth entry contains nth Fibonacci number.

2. **Which entries do not depend on other entries?**

   Values F_1 and F_2 can be computed easily and independently.

3. **What is the execution order such that required entries are always available?**

Dynamic Programing: Procedure with the example

1. **Dimension of the table? Semantics of the entries?**

   $n \times 1$ table. nth entry contains nth Fibonacci number.

2. **Which entries do not depend on other entries?**

   Values F_1 and F_2 can be computed easily and independently.

3. **What is the execution order such that required entries are always available?**

   F_i with increasing i.

Dynamic Programing: Procedure with the example

1. **Dimension of the table? Semantics of the entries?**

   $n \times 1$ table. nth entry contains nth Fibonacci number.

2. **Which entries do not depend on other entries?**

   Values F_1 and F_2 can be computed easily and independently.

3. **What is the execution order such that required entries are always available?**

   F_i with increasing i.

4. **Wie kann sich Lösung aus der Tabelle konstruieren lassen?**

Dynamic Programing: Procedure with the example

1. **Dimension of the table? Semantics of the entries?**

   $n \times 1$ table. nth entry contains nth Fibonacci number.

2. **Which entries do not depend on other entries?**

   Values F_1 and F_2 can be computed easily and independently.

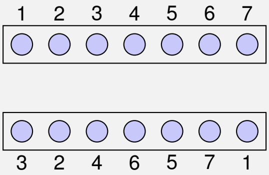
3. **What is the execution order such that required entries are always available?**

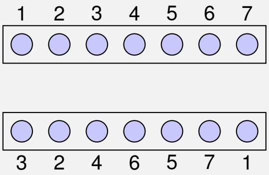
   F_i with increasing i.

4. **Wie kann sich Lösung aus der Tabelle konstruieren lassen?**

   F_n ist die n-te Fibonacci-Zahl.

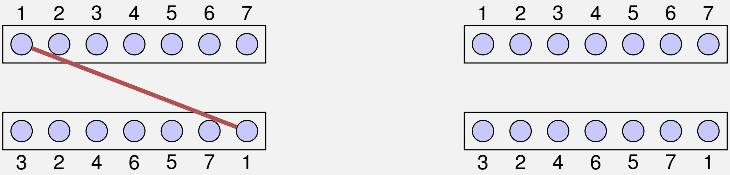
Longest Ascending Sequence (LAS)





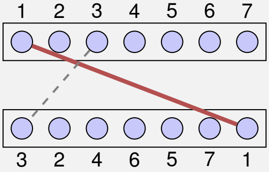
Connect as many as possible fitting ports without lines crossing.

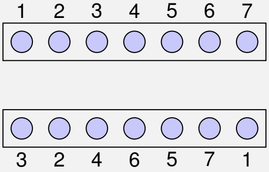
Longest Ascending Sequence (LAS)



Connect as many as possible fitting ports without lines crossing.

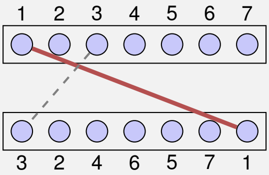
Longest Ascending Sequence (LAS)

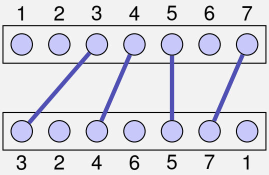




Connect as many as possible fitting ports without lines crossing.

Longest Ascending Sequence (LAS)

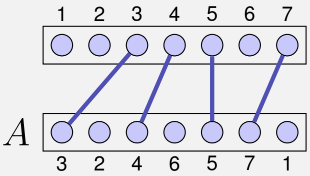




Connect as many as possible fitting ports without lines crossing.

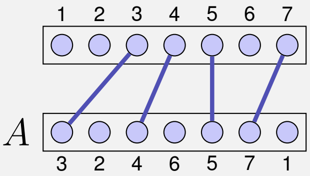
Formally

- Consider Sequence $A = (a_1, \dots, a_n)$.
- Search for a longest increasing subsequence of A.
- Examples of increasing subsequences: $(3,4,5)$, $(2,4,5,7)$, $(3,4,5,7)$, $(3,7)$.



Formally

- Consider Sequence $A = (a_1, \ldots, a_n)$.
- Search for a longest increasing subsequence of A.
- Examples of increasing subsequences: $(3, 4, 5)$, $(2, 4, 5, 7)$, $(3, 4, 5, 7)$, $(3, 7)$.



Generalization: allow any numbers, even with duplicates. But only strictly increasing subsequences are permitted. Example: $(2, 3, 3, 3, 5, 1)$ with increasing subsequence $(2, 3, 5)$.

First idea

Assumption: LAS L_k known for k Now want to compute L_{k+1} for $k+1$.

First idea

Assumption: LAS L_k known for k Now want to compute L_{k+1} for $k+1$.

If a_{k+1} fits to L_k, then $L_{k+1} = L_k \oplus a_{k+1}$

First idea

Assumption: LAS L_k known for k Now want to compute L_{k+1} for $k+1$.

If a_{k+1} fits to L_k, then $L_{k+1} = L_k \oplus a_{k+1}$

Counterexample $A_5 = (1, 2, 5, 3, 4)$. Let $A_3 = (1, 2, 5)$ with $L_3 = A$. Determine L_4 from L_3?

First idea

Assumption: LAS L_k known for k Now want to compute L_{k+1} for $k+1$.

If a_{k+1} fits to L_k, then $L_{k+1} = L_k \oplus a_{k+1}$

Counterexample $A_5 = (1, 2, 5, 3, 4)$. Let $A_3 = (1, 2, 5)$ with $L_3 = A$. Determine L_4 from L_3?

It does not work this way, we cannot infer L_{k+1} from L_k.

Second idea.

Assumption: a LAS L_j is known for each $j \leq k$. Now compute LAS L_{k+1} for $k + 1$.

Second idea.

Assumption: a LAS L_j is known for each $j \leq k$. Now compute LAS L_{k+1} for $k + 1$.

Look at all fitting $L_{k+1} = L_j \oplus a_{k+1}$ ($j \leq k$) and choose a longest sequence.

Second idea.

Assumption: a LAS L_j is known for each $j \leq k$. Now compute LAS L_{k+1} for $k+1$.

Look at all fitting $L_{k+1} = L_j \oplus a_{k+1}$ ($j \leq k$) and choose a longest sequence.

Counterexample: $A_5 = (1, 2, 5, 3, 4)$. Let $A_4 = (1, 2, 5, 3)$ with $L_1 = (1)$, $L_2 = (1, 2)$, $L_3 = (1, 2, 5)$, $L_4 = (1, 2, 5)$. Determine L_5 from $L_1, \ldots, L_4$?

Second idea.

Assumption: a LAS L_j is known for each $j \leq k$. Now compute LAS L_{k+1} for $k+1$.

Look at all fitting $L_{k+1} = L_j \oplus a_{k+1}$ ($j \leq k$) and choose a longest sequence.

Counterexample: $A_5 = (1, 2, 5, 3, 4)$. Let $A_4 = (1, 2, 5, 3)$ with $L_1 = (1)$, $L_2 = (1, 2)$, $L_3 = (1, 2, 5)$, $L_4 = (1, 2, 5)$. Determine L_5 from $L_1, \ldots, L_4$?

That does not work either: cannot infer L_{k+1} from only *an arbitrary solution* L_j. We need to consider all LAS. Too many.

Third approach

Assumption: the LAS L_j, *that ends with smallest element* is known for each of the lengths $1 \leq j \leq k$.

Example: $A = (1, 1000, 1001, 2, 3, 4, ..., 999)$

A	LAT

Third approach

Assumption: the LAS L_j, *that ends with smallest element* is known for each of the lengths $1 \leq j \leq k$.

Consider all fitting $L_j \oplus a_{k+1}$ ($j \leq k$) and update the table of the LAS,that end with smallest possible element.

Example: $A = (1, 1000, 1001, 2, 3, 4, ..., 999)$

A	LAT

Third approach

Assumption: the LAS L_j, *that ends with smallest element* is known for each of the lengths $1 \leq j \leq k$.

Consider all fitting $L_j \oplus a_{k+1}$ ($j \leq k$) and update the table of the LAS,that end with smallest possible element.

Example: $A = (1, 1000, 1001, 2, 3, 4, ..., 999)$

A	LAT
(1)	(1)

Third approach

Assumption: the LAS L_j, *that ends with smallest element* is known for each of the lengths $1 \leq j \leq k$.

Consider all fitting $L_j \oplus a_{k+1}$ ($j \leq k$) and update the table of the LAS,that end with smallest possible element.

Example: $A = (1, 1000, 1001, 2, 3, 4, ..., 999)$

A	LAT
(1)	(1)
$(1, 1000)$	$(1), (1, 1000)$

Third approach

Assumption: the LAS L_j, *that ends with smallest element* is known for each of the lengths $1 \leq j \leq k$.

Consider all fitting $L_j \oplus a_{k+1}$ ($j \leq k$) and update the table of the LAS,that end with smallest possible element.

Example: $A = (1, 1000, 1001, 2, 3, 4, ..., 999)$

A	LAT
(1)	(1)
$(1, 1000)$	$(1), (1, 1000)$
$(1, 1000, 1001)$	$(1), (1, 1000), (1, 1000, 1001)$

Third approach

Assumption: the LAS L_j, *that ends with smallest element* is known for each of the lengths $1 \leq j \leq k$.

Consider all fitting $L_j \oplus a_{k+1}$ ($j \leq k$) and update the table of the LAS,that end with smallest possible element.

Example: $A = (1, 1000, 1001, 2, 3, 4, ..., 999)$

A	LAT
(1)	(1)
$(1, 1000)$	$(1), (1, 1000)$
$(1, 1000, 1001)$	$(1), (1, 1000), (1, 1000, 1001)$
$(1, 1000, 1001, 2)$	$(1), (1, 2), (1, 1000, 1001)$

Third approach

Assumption: the LAS L_j, *that ends with smallest element* is known for each of the lengths $1 \leq j \leq k$.

Consider all fitting $L_j \oplus a_{k+1}$ ($j \leq k$) and update the table of the LAS,that end with smallest possible element.

Example: $A = (1, 1000, 1001, 2, 3, 4, ..., 999)$

A	LAT
(1)	(1)
$(1, 1000)$	$(1), (1, 1000)$
$(1, 1000, 1001)$	$(1), (1, 1000), (1, 1000, 1001)$
$(1, 1000, 1001, 2)$	$(1), (1, 2), (1, 1000, 1001)$
$(1, 1000, 1001, 2, 3)$	$(1), (1, 2), (1, 2, 3)$

DP Table

- Idea: save the last element of the increasing sequence L_j at slot j.

DP Table

- Idea: save the last element of the increasing sequence L_j at slot j.
- Example: 3 2 5 1 6 4

Index	1	2	3	4	5	6
Wert	3	2	5	1	6	4

DP Table

- Idea: save the last element of the increasing sequence L_j at slot j.
- Example: 3 2 5 1 6 4

Index	1	2	3	4	5	6
Wert	3	2	5	1	6	4

Index	0	1	2	3	4	...
$(L_j)_j$	$-\infty$	∞	∞	∞	∞	

DP Table

- Idea: save the last element of the increasing sequence L_j at slot j.
- Example: 3 2 5 1 6 4

Index	1	2	3	4	5	6
Wert	3	2	5	1	6	4

Index	0	1	2	3	4	...
$(L_j)_j$	-∞	3	∞	∞	∞	

DP Table

- Idea: save the last element of the increasing sequence L_j at slot j.
- Example: 3 2 5 1 6 4

Index	1	2	3	4	5	6
Wert	3	2	5	1	6	4

Index	0	1	2	3	4	...
$(L_j)_j$	$-\infty$	2	∞	∞	∞	

DP Table

- Idea: save the last element of the increasing sequence L_j at slot j.
- Example: 3 2 5 1 6 4

Index	1	2	3	4	5	6
Wert	3	2	5	1	6	4

Index	0	1	2	3	4	...
$(L_j)_j$	$-\infty$	2	5	∞	∞	

DP Table

- Idea: save the last element of the increasing sequence L_j at slot j.
- Example: 3 2 5 1 6 4

Index	1	2	3	4	5	6
Wert	3	2	5	1	6	4

Index	0	1	2	3	4	...
$(L_j)_j$	-∞	1	5	∞	∞	

DP Table

- Idea: save the last element of the increasing sequence L_j at slot j.
- Example: 3 2 5 1 6 4

Index	1	2	3	4	5	6
Wert	3	2	5	1	6	4

Index	0	1	2	3	4	...
$(L_j)_j$	-∞	1	5	6	∞	

DP Table

- Idea: save the last element of the increasing sequence L_j at slot j.
- Example: 3 2 5 1 6 4
- Problem: Table does not contain the subsequence, only the last value.

Index	1	2	3	4	5	6
Wert	3	2	5	1	6	4

Index	0	1	2	3	4	...
$(L_j)_j$	$-\infty$	1	4	6	∞	

DP Table

- Idea: save the last element of the increasing sequence L_j at slot j.
- Example: 3 2 5 1 6 4
- Problem: Table does not contain the subsequence, only the last value.
- Solution: second table with the predecessors.

Index	1	2	3	4	5	6
Wert	3	2	5	1	6	4

Index	0	1	2	3	4	...
$(L_j)_j$	-∞	1	4	6	∞	

DP Table

- Idea: save the last element of the increasing sequence L_j at slot j.
- Example: 3 2 5 1 6 4
- Problem: Table does not contain the subsequence, only the last value.
- Solution: second table with the predecessors.

Index	1	2	3	4	5	6
Wert	3	2	5	1	6	4
Predecessor	$-\infty$	$-\infty$	2	$-\infty$	5	1

Index	0	1	2	3	4	...
$(L_j)_j$	-∞	1	4	6	∞	

Dynamic Programming Algorithm LAS

Table dimension? Semantics?

1

Dynamic Programming Algorithm LAS

Table dimension? Semantics?

1. Two tables $T[0,\ldots,n]$ and $V[1,\ldots,n]$. Start with $T[0] \leftarrow -\infty$, $T[i] \leftarrow \infty \; \forall i > 1$

Dynamic Programming Algorithm LAS

1 **Table dimension? Semantics?**

Two tables $T[0,\ldots,n]$ and $V[1,\ldots,n]$. Start with $T[0] \leftarrow -\infty$, $T[i] \leftarrow \infty \; \forall i > 1$

2 **Computation of an entry**

Dynamic Programming Algorithm LAS

Table dimension? Semantics?

1. Two tables $T[0, \ldots, n]$ and $V[1, \ldots, n]$. Start with $T[0] \leftarrow -\infty$, $T[i] \leftarrow \infty \; \forall i > 1$

Computation of an entry

2. Entries in T sorted in ascending order. For each new entry a_{k+1} binary search for l, such that $T[l] < a_k < T[l+1]$. Set $T[l+1] \leftarrow a_{k+1}$. Set $V[k] = T[l]$.

Dynamic Programming algorithm LAS

3 **Computation order**

Dynamic Programming algorithm LAS

3

Computation order

Traverse the list anc compute $T[k]$ and $V[k]$ with ascending k

Dynamic Programming algorithm LAS

3 **Computation order**

Traverse the list anc compute $T[k]$ and $V[k]$ with ascending k

4 **How can the solution be determined from the table?**

Dynamic Programming algorithm LAS

3

Computation order

Traverse the list anc compute $T[k]$ and $V[k]$ with ascending k

4

How can the solution be determined from the table?

Search the largest l with $T[l] < \infty$. l is the last index of the LAS. Starting at l search for the index $i < l$ such that $V[l] = A[i]$, i is the predecessor of l. Repeat with $l \leftarrow i$ until $T[l] = -\infty$

Analysis

- Computation of the table:
  - Initialization: $\Theta(n)$ Operations
  - Computation of the kth entry: binary search on positions $\{1, \ldots, k\}$ plus constant number of assignments.

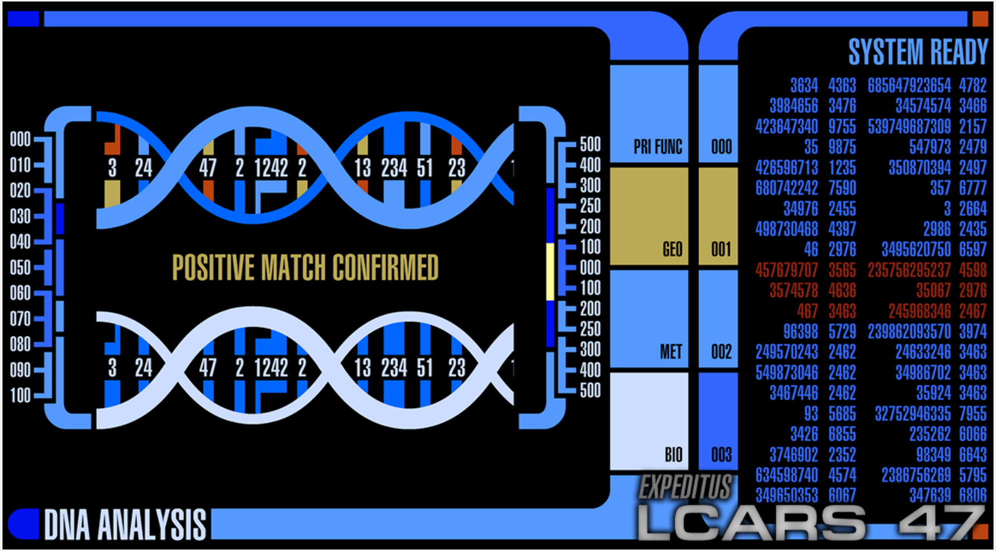
$$\sum_{k=1}^{n} (\log k + \mathcal{O}(1)) = \mathcal{O}(n) + \sum_{k=1}^{n} \log(k) = \Theta(n \log n).$$

- Reconstruction: traverse A from right to left: $\mathcal{O}(n)$.

Overal runtime:

$$\Theta(n \log n).$$

DNA - Comparison (Star Trek)



DNA - Comparison

- DNA consists of sequences of four different nucleotides **A**denine **G**uanine **T**hymine **C**ytosine
- DNA sequences (genes) thus can be described with strings of A, G, T and C.
- Possible comparison of two genes: determine the **longest common subsequence**

Longest common subsequence

Subsequences of a string:

> *Subsequences(KUH): (), (K), (U), (H), (KU), (KH), (UH), (KUH)*

Problem:

- Input: two strings $A = (a_1, \ldots, a_m)$, $B = (b_1, \ldots, b_n)$ with lengths $m > 0$ and $n > 0$.
- Wanted: Longest common subsequecnes (LCS) of A and B.

Longest Common Subsequence

Examples:

LGT(IGEL,KATZE)=E, LGT(TIGER,ZIEGE)=IGE

Ideas to solve?

T	I		G	E	R
Z	I	E	G	E	

Recursive Procedure

Assumption: solutions $L(i, j)$ known for $A[1, \ldots, i]$ and $B[1, \ldots, j]$ for all $1 \le i \le m$ and $1 \le j \le n$, but not for $i = m$ and $j = n$.

T	I		G	E	R
Z	I	E	G	E	

Consider characters a_m, b_n. Three possibilities:

1. A is enlarged by one whitespace. $L(m, n) = L(m, n-1)$
2. B is enlarged by one whitespace. $L(m, n) = L(m-1, n)$
3. $L(m, n) = L(m-1, n-1) + \delta_{mn}$ with $\delta_{mn} = 1$ if $a_m = b_n$ and $\delta_{mn} = 0$ otherwise

Recursion

$$L(m,n) \leftarrow \max\{L(m-1,n-1)+\delta_{mn}, L(m,n-1), L(m-1,n)\}$$

for $m,n > 0$ and base cases $L(\cdot,0)=0$, $L(0,\cdot)=0$.

	∅	Z	I	E	G	E
∅	0	0	0	0	0	0
T	0	0	0	0	0	0
I	0	0	1	1	1	1
G	0	0	1	1	2	2
E	0	0	1	2	2	3
R	0	0	1	2	2	3

Dynamic Programming algorithm LCS

Dimension of the table? Semantics?

1

Dynamic Programming algorithm LCS

Dimension of the table? Semantics?

1. Table $L[0, \ldots, m][0, \ldots, n]$. $L[i, j]$: length of a LCS of the strings $(a_1, \ldots, a_i)$ and $(b_1, \ldots, b_j)$

Dynamic Programming algorithm LCS

Dimension of the table? Semantics?

1 Table $L[0,\ldots,m][0,\ldots,n]$. $L[i,j]$: length of a LCS of the strings $(a_1,\ldots,a_i)$ and $(b_1,\ldots,b_j)$

Computation of an entry

2

Dynamic Programming algorithm LCS

Dimension of the table? Semantics?

1. Table $L[0,\dots,m][0,\dots,n]$. $L[i,j]$: length of a LCS of the strings $(a_1,\dots,a_i)$ and $(b_1,\dots,b_j)$

Computation of an entry

2. $L[0,i] \leftarrow 0 \;\forall 0 \le i \le m$, $L[j,0] \leftarrow 0 \;\forall 0 \le j \le n$. Computation of $L[i,j]$ otherwise via $L[i,j] = \max(L[i-1,j-1]+\delta_{ij}, L[i,j-1], L[i-1,j])$.

Dynamic Programming algorithm LCS

3 **Computation order**

Dynamic Programming algorithm LCS

3 **Computation order**

Rows increasing and within columns increasing (or the other way round).

Dynamic Programming algorithm LCS

3 **Computation order**

Rows increasing and within columns increasing (or the other way round).

4 **Reconstruct solution?**

Dynamic Programming algorithm LCS

3 **Computation order**

Rows increasing and within columns increasing (or the other way round).

4 **Reconstruct solution?**

Start with $j = m$, $i = n$. If $a_i = b_j$ then output a_i and continue with $(j, i) \leftarrow (j-1, i-1)$; otherwise, if $L[i, j] = L[i, j-1]$ continue with $j \leftarrow j - 1$ otherwise, if $L[i, j] = L[i-1, j]$ continue with $i \leftarrow i - 1$. Terminate for $i = 0$ or $j = 0$.

Analysis LCS

- Number table entries: $(m+1) \cdot (n+1)$.
- Constant number of assignments and comparisons each. Number steps: $\mathcal{O}(mn)$
- Determination of solition: decrease i or j. Maximally $\mathcal{O}(n+m)$ steps.

Runtime overal:

$$\mathcal{O}(mn).$$

Editing Distance

Editing distance of two sequences $A = (a_1, \ldots, a_m)$, $B = (b_1, \ldots, b_m)$.

Editing operations:

- Insertion of a character
- Deletion of a character
- Replacement of a character

Question: how many editing operations at least required in order to transform string A into string B.

TIGER ZIGER ZIEGER ZIEGE

Editing Distance = Levenshtein Distance

Procedure?

[29] or append character to B_j

[30] or delete last character of B_j

Procedure?

- Two dimensional table $E[0, \ldots, m][0, \ldots, n]$ with editing distances $E[i, j]$ of strings $A_i = (a_1, \ldots, a_i)$ and $B_j = (b_1, \ldots, b_j)$.

---

[29] or append character to B_j

[30] or delete last character of B_j

Procedure?

- Two dimensional table $E[0,\ldots,m][0,\ldots,n]$ with editing distances $E[i,j]$ of strings $A_i = (a_1,\ldots,a_i)$ and $B_j = (b_1,\ldots,b_j)$.
- Consider the last characters of A_i and B_j. Three possible cases:

  1. Delete last character of A_i: [29] $E[i-1,j]+1$.
  2. Append character to A_i:[30] $E[i,j-1]+1$.
  3. Replace A_i by B_j: $E[i-1,j-1]+1-\delta_{ij}$.

$$E[i,j] \leftarrow \min\left\{E[i-1,j]+1, E[i,j-1]+1, E[i-1,j-1]+1-\delta_{ij}\right\}$$

---

[29] or append character to B_j

[30] or delete last character of B_j

DP Table

$$E[i,j] \leftarrow \min\left\{E[i-1,j]+1, E[i,j-1]+1, E[i-1,j-1]+1-\delta_{ij}\right\}$$

	∅	Z	I	E	G	E
∅	0	1	2	3	4	5
T	1	1	2	3	4	5
I	2	2	1	2	3	4
G	3	3	2	2	2	3
E	4	4	3	2	3	2
R	5	5	4	3	3	3

Algorithm: exercise

Matrix-Chain-Multiplication

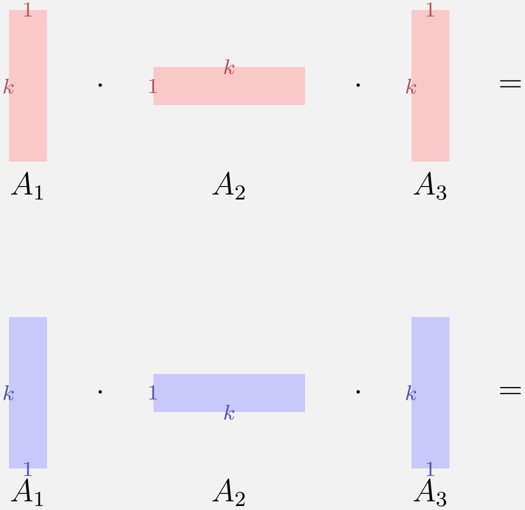
Task: Computation of the product $A_1 \cdot A_2 \cdot ... \cdot A_n$ of matrices $A_1, \ldots,$ A_n.

Matrix multiplication is associative, i.e. the order of evalution can be chosen arbitrarily

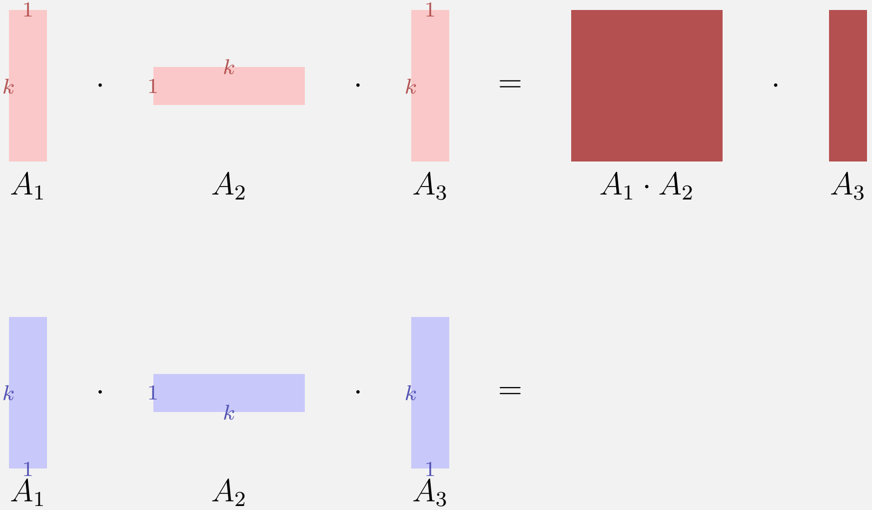
Goal: efficient computation of the product.

Assumption: multiplicaiton of an $(r \times s)$-matrix with an $(s \times u)$-matrix provides costs $r \cdot s \cdot u$.

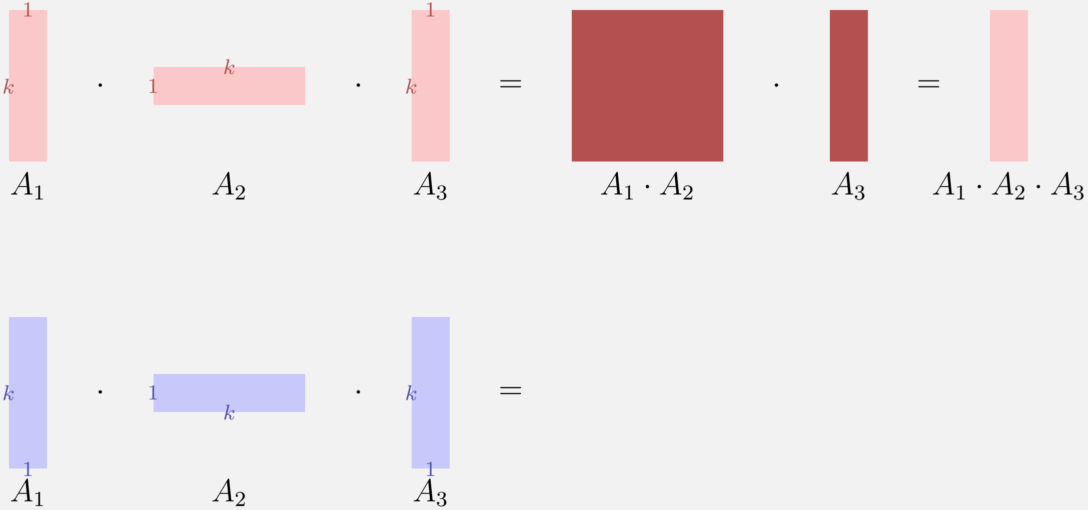
Does it matter?



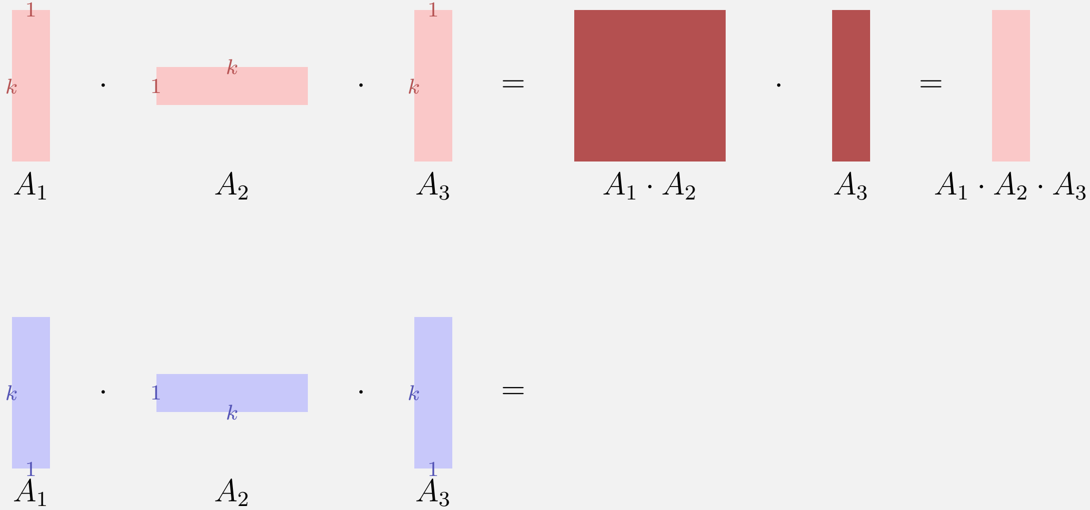
Does it matter?



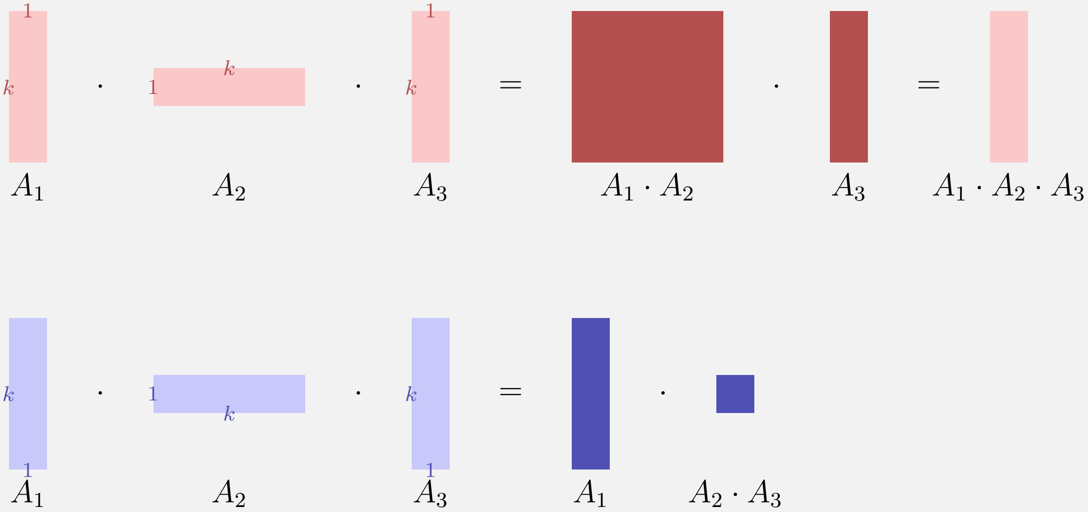
Does it matter?



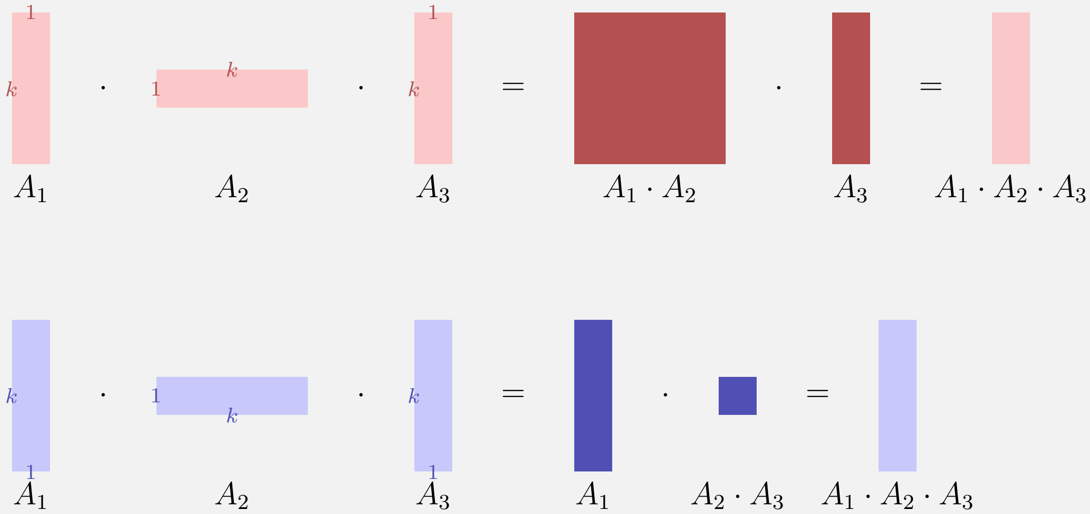
Does it matter?



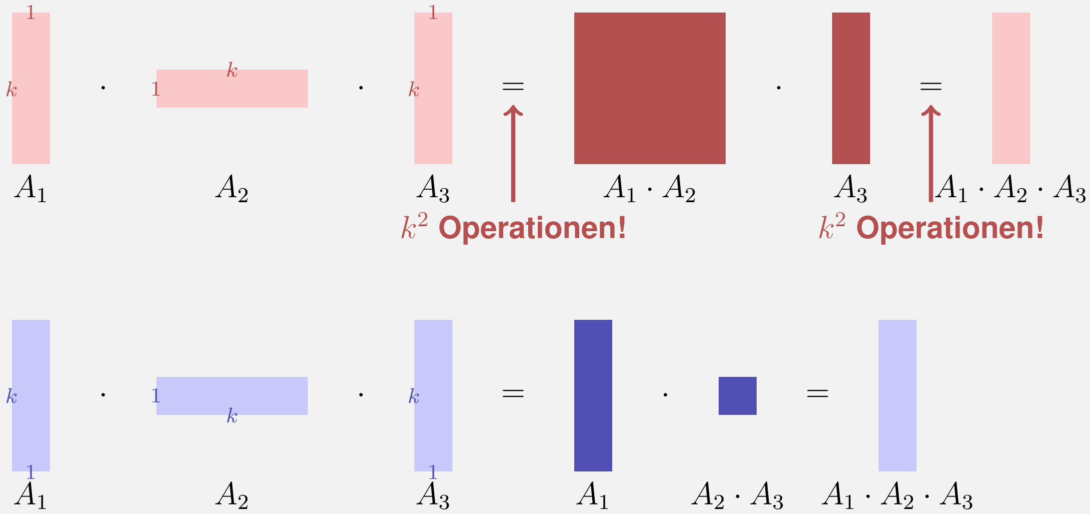
Does it matter?



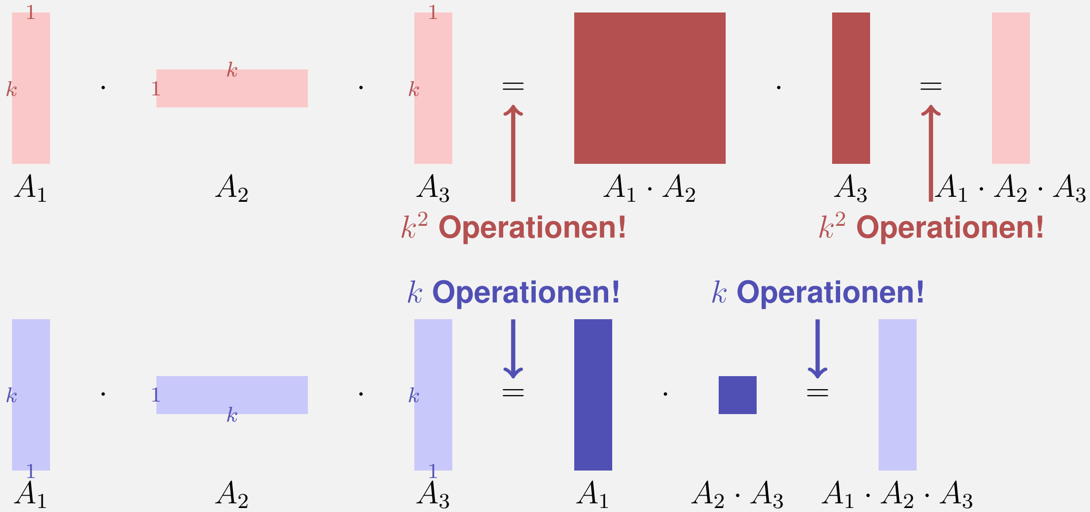
Does it matter?



Does it matter?



Does it matter?



Recursion

- Assume that the best possible computation of $(A_1 \cdot A_2 \cdots A_i)$ and $(A_{i+1} \cdot A_{i+2} \cdots A_n)$ is known for each i.
- Compute best i, done.

$n \times n$-table M. entry $M[p, q]$ provides costs of the best possible bracketing $(A_p \cdot A_{p+1} \cdots A_q)$.

$$M[p, q] \leftarrow \min_{p \le i < q} \left(M[p, i] + M[i + 1, q] + \text{costs of the last multiplication}\right)$$

Computation of the DP-table

- Base cases $M[p,p] \leftarrow 0$ for all $1 \leq p \leq n$.
- Computation of $M[p,q]$ depends on $M[i,j]$ with $p \leq i \leq j \leq q$, $(i,j) \neq (p,q)$.
  In particular $M[p,q]$ depends at most from entries $M[i,j]$ with $i - j < q - p$.
  Consequence: fill the table from the diagonal.

Analysis

DP-table has n^2 entries. Computation of an entry requires considering up to $n-1$ other entries.

Overal runtime $\mathcal{O}(n^3)$.

Readout the order from M: exercise!

Digression: matrix multiplication

Consider the mutliplicaiton of two $n \times n$ matrices.

Let

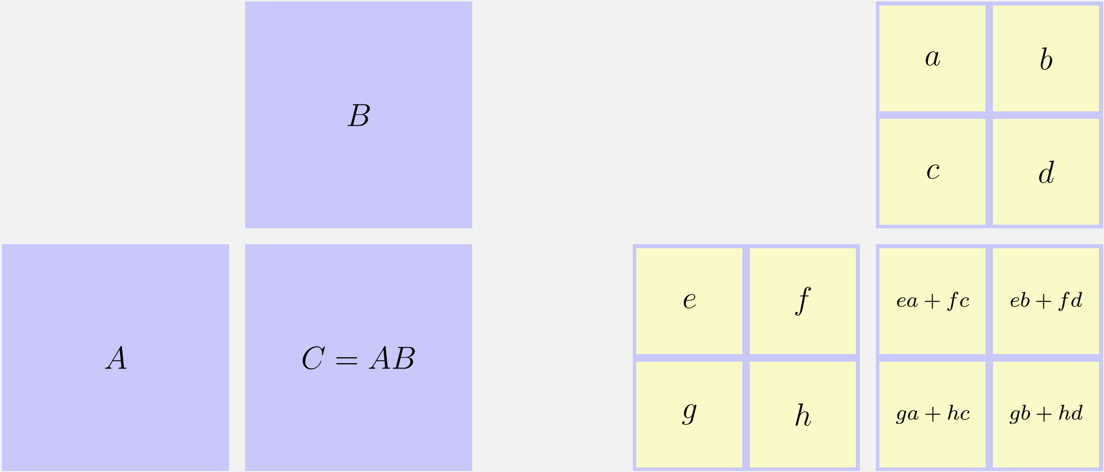
$$A = (a_{ij})_{1 \le i,j \le n}, B = (b_{ij})_{1 \le i,j \le n}, C = (c_{ij})_{1 \le i,j \le n},$$
$$C = A \cdot B$$

then

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

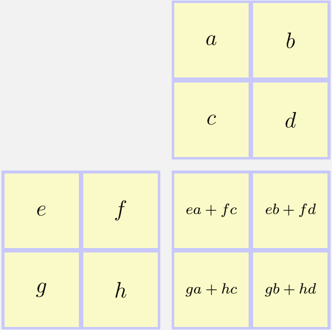
Naive algorithm requires $\Theta(n^3)$ elementary multiplications.

Divide and Conquer



Divide and Conquer

- Assumption $n = 2^k$.
- Number of elementary multiplications: $M(n) = 8M(n/2)$, $M(1) = 1$.
- yields $M(n) = 8^{\log_2 n} = n^{\log_2 8} = n^3$. No advantage



Strassen's Matrix Multiplication

- Nontrivial observation by Strassen (1969):

  It suffices to compute the seven products

  $A = (e+h)\cdot(a+d)$, $B = (g+h)\cdot a$,

  $C = e\cdot(b-d)$, $D = h\cdot(c-a)$, $E = (e+f)\cdot d$,

  $F = (g-e)\cdot(a+b)$, $G = (f-h)\cdot(c+d)$. Denn:

  $ea+fc = A+D-E+G$, $eb+fd = C+E$,

  $ga+hc = B+D$, $gb+hd = A-B+C+F$.

- This yields $M'(n) = 7M(n/2)$, $M'(1) = 1$.

  Thus $M'(n) = 7^{\log_2 n} = n^{\log_2 7} \approx n^{2.807}$.

- Fastest currently known algorithm:

  $\mathcal{O}(n^{2.37})$