# Data Structures and Algorithms

Course at D-MATH (CSE) of ETH Zurich

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# 1. Introduction

Algorithms and Data Structures, Three Examples



## Goals of the course

- Understand the design and analysis of fundamental algorithms and data structures.
- An advanced insight into a modern programming model (with C++).
- Knowledge about chances, problems and limits of the parallel and concurrent computing.



## Goals of the course

On the one hand

- Essential basic knowlegde from computer science.

Andererseits

- Preparation for your further course of studies and practical considerations.

# Contents





# 1.2 Algorithms

[Cormen et al, Kap. 1;Ottman/Widmayer, Kap. 1.1]



## Algorithm

Algorithm: well defined computing procedure to compute *output* data from *input* data



## example problem

**Input** : A sequence of $n$ numbers $(a_1, a_2, \dots, a_n)$

**Output** : Permutation $(a'_1, a'_2, \dots, a'_n)$ of the sequence $(a_i)_{1 \le i \le n}$, such that $a'_1 \le a'_2 \le \cdots \le a'_n$

**Possible input**

$(1, 7, 3)$, $(15, 13, 12, -0.5)$, $(1)$ ...

Every example represents a *problem instance*

## Examples for algorithmic problems

- Tables and statistis: sorting, selection and searching
- routing: shortest path algorithm, heap data structure
- DNA matching: Dynamic Programming
- fabrication pipeline: Topological Sorting
- autocomletion and spell-checking: Dictionaries / Trees
- Symboltables (compiler) : Hash-Tables
- The travelling Salesman: Dynamic Programming, Minimum Spanning Tree, Simulated Annealing
- Drawing at the computer: Digitizing lines and circles, filling polygons
- Page-Rank: (Markov-Chain) Monte Carlo ...



## Characteristics

- Extremely large number of potential solutions
- Practical applicability



## Darta Structures

- Organisation of the data tailored towards the algorithms that operate on the data.
- Programs = algorithms + data structures.



## Very hard problems.

- NP-compleete problems: no known efficient solution (but the non-existence of such a solution is not proven yet!)
- Example: travelling salesman problem

## A dream

- If computers were infinitely fast and had an infinite amount of memory ...
- ... then we would still need the theory of algorithms (only) for statements about correctness (and termination).



## The reality

Resources are bounded and not free:

- Computing time $\rightarrow$ Efficiency
- Storage space $\rightarrow$ Efficiency



# 1.3 Ancient Egyptian Multiplication

Ancient Egyptian Multiplication



## Ancient Egyptian Multiplication[1]

Compute $11 \cdot 9$

| | |
|---|---|
| 11 | 9 |
| ~~22~~ | ~~4~~ |
| ~~44~~ | ~~2~~ |
| 88 | 1 |
| 99 | – |

| | |
|---|---|
| 9 | 11 |
| 18 | 5 |
| ~~36~~ | ~~2~~ |
| 72 | 1 |
| 99 | |

1. Double left, integer division by 2 on the right
2. Even number on the right $\Rightarrow$ eliminate row.
3. Add remaining rows on the left.

[1] Also known as russian multiplication

## Advantages

- Short description, easy to grasp
- Efficient to implement on a computer: double = left shift, divide by 2 = right shift

**Beispiel**

| | |
|---|---|
| *left shift* | $9 = 01001_2 \to 10010_2 = 18$ |
| *right shift* | $9 = 01001_2 \to 00100_2 = 4$ |



## Questions

- Does this always work (negative numbers?)?
- If not, when does it work?
- How do you prove correctness?
- Is it better than the school method?
- What does “good” mean at all?
- How to write this down precisely?



## Observation

If $b > 1$, $a \in \mathbb{Z}$, then:

$$a \cdot b = \begin{cases} 2a \cdot \frac{b}{2} & \text{falls } b \text{ gerade,} \\ a + 2a \cdot \frac{b-1}{2} & \text{falls } b \text{ ungerade.} \end{cases}$$



## Termination

$$a \cdot b = \begin{cases} a & \text{falls } b = 1, \\ 2a \cdot \frac{b}{2} & \text{falls } b \text{ gerade,} \\ a + 2a \cdot \frac{b-1}{2} & \text{falls } b \text{ ungerade.} \end{cases}$$

## Recursively, Functional

$$f(a,b) = \begin{cases} a & \text{falls } b = 1, \\ f(2a, \frac{b}{2}) & \text{falls } b \text{ gerade}, \\ a + f(2a, \frac{b-1}{2}) & \text{falls } b \text{ ungerade}. \end{cases}$$



## Implemented

```
// pre: b>0
// post: return a∗b
int f(int a, int b){
    if(b==1)
        return a;
    else if (b%2 == 0)
        return f(2∗a, b/2);
    else
        return a + f(2∗a, (b−1)/2);
}
```



## Correctnes

$$f(a,b) = \begin{cases} a & \text{if } b = 1, \\ f(2a, \frac{b}{2}) & \text{if } b \text{ even}, \\ a + f(2a \cdot \frac{b-1}{2}) & \text{if } b \text{ odd}. \end{cases}$$

Remaining to show: $f(a,b) = a \cdot b$ for $a \in \mathbb{Z}$, $b \in \mathbb{N}^+$.



## Proof by induction

Base clause: $b = 1 \Rightarrow f(a,b) = a = a \cdot 1$.
Hypothesis: $f(a,b') = a \cdot b'$ für $0 < b' \leq b$
Step: $f(a,b+1) \stackrel{!}{=} a \cdot (b+1)$

$$f(a,b+1) = \begin{cases} f(2a, \frac{\overbrace{b+1}^{\leq b}}{2}) = a \cdot (b+1) & \text{if } b \text{ odd}, \\ a + f(2a, \underbrace{\frac{b}{2}}_{\leq b}) = a + a \cdot b & \text{if } b \text{ even}. \end{cases}$$

$\blacksquare$

## End Recursion

The recursion can be writen as *end recursion*

```
// pre: b>0
// post: return a*b
int f(int a, int b){
  if(b==1)
    return a;
  else if (b%2 == 0)
    return f(2*a, b/2);
  else
    return a + f(2*a, (b-1)/2);
}
```

→

```
// pre: b>0
// post: return a*b
int f(int a, int b){
  if(b==1)
    return a;
  int z=0;
  if (b%2 != 0){
    --b;
    z=a;
  }
  return z + f(2*a, b/2);
}
```



## End-Recursion ⇒ Iteration

```
// pre: b>0
// post: return a*b
int f(int a, int b){
  if(b==1)
    return a;
  int z=0;
  if (b%2 != 0){
    --b;
    z=a;
  }
  return z + f(2*a, b/2);
}
```

→

```
int f(int a, int b) {
  int res = 0;
  while (b != 1) {
    int z = 0;
    if (b % 2 != 0){
      --b;
      z = a;
    }
    res += z;
    a *= 2; // neues a
    b /= 2; // neues b
  }
  res += a; // Basisfall b=1
  return res;
}
```



## Simplify

```
int f(int a, int b) {
  int res = 0;
  while (b != 1) {
    int z = 0;
    if (b % 2 != 0){
      --b; ⟶ Teil der Division
      z = a;⟶ Direkt in res
    }
    res += z;
    a *= 2;
    b /= 2;
  }
  res += a; ⟶ in den Loop
  return res;
}
```

→

```
// pre: b>0
// post: return a*b
int f(int a, int b) {
  int res = 0;
  while (b > 0) {
    if (b % 2 != 0)
      res += a;
    a *= 2;
    b /= 2;
  }
  return res;
}
```



## Invariants!

```
// pre: b>0
// post: return a*b
int f(int a, int b) {
  int res = 0;
  ----------------------------
  while (b > 0) {
    if (b % 2 != 0){
    ----------------------------
      res += a;
      --b;
    ----------------------------
    }
    ----------------------------
    a *= 2;
    b /= 2;
  ----------------------------
  }
  ----------------------------
  return res;
}
```

Sei $x := a \cdot b$.

here: $x = \boxed{a \cdot b + res}$

if here $x = a \cdot b + res$ ...

... then also here $x = a \cdot b + res$
$b$ even

here: $x = a \cdot b + res$
here: $x = a \cdot b + res$ und $b = 0$
Also $res = x$.

## Conclusion

The expression $a \cdot b + res$ is an *invariant*

- Values of $a$, $b$, $res$ change but the invariant remains basically unchanged
- The invariant is only temporarily discarded by some statement but then re-established
- If such short statement sequences are considered atomiv, the value remains indeed invariant
- In particular the loop contains an invariant, called *loop invariant* and operates there like the induction step in induction proofs.
- Invariants are obviously powerful tools for proofs!



## Further simplification

```
// pre: b>0
// post: return a*b
int f(int a, int b) {
  int res = 0;
  while (b > 0) {
    if (b % 2 != 0){
      res += a;
      --b;
    }
    a *= 2;
    b /= 2;
  }
  return res;
}
```

→

```
// pre: b>0
// post: return a*b
int f(int a, int b) {
  int res = 0;
  while (b > 0) {
    res += a * (b%2);
    a *= 2;
    b /= 2;
  }
  return res;
}
```



## Analysis

```
// pre: b>0
// post: return a*b
int f(int a, int b) {
  int res = 0;
  while (b > 0) {
    res += a * (b%2);
    a *= 2;
    b /= 2;
  }
  return res;
}
```

Ancient Egyptian Multiplication corresponds to the school method with radix $2$.

$$
\begin{array}{cccccccccc}
1 & 0 & 0 & 1 & \times & 1 & 0 & 1 & 1 & \\
\hline
 & & & & & 1 & 0 & 0 & 1 & (9) \\
 & & & & 1 & 0 & 0 & 1 & & (18) \\
\hline
 & & & & 1 & 1 & 0 & 1 & 1 & \\
 & & 1 & 0 & 0 & 1 & & & & (72) \\
\hline
 & & 1 & 1 & 0 & 0 & 0 & 1 & 1 & (99)
\end{array}
$$



## Efficiency

Question: how long does a multiplication of $a$ and $b$ take?

- Measure for efficiency
  - Total number of fundamental operations: double, divide by 2, shift, test for “even”, addition
  - In the recursive and recursive code: maximally 6 operations per call or iteration, respectively
- Essential criterion:
  - Number of recursion calls or
  - Number iterations (in the iterative case)
- $\frac{b}{2^n} \leq 1$ holds for $n \geq \log_2 b$. Consequently not more than $6\lceil \log_2 b \rceil$ fundamental operations.

## 1.4 Fast Integer Multiplication

[Ottman/Widmayer, Kap. 1.2.3]



## Example 2: Multiplication of large Numbers

Primary school:

$$
\begin{array}{cccccc|c}
 & a & b & & c & d & \\
 & 6 & 2 & \cdot & 3 & 7 & \\
\hline
 & & & & 1 & 4 & d \cdot b \\
 & & & 4 & 2 & & d \cdot a \\
 & & & & 6 & & c \cdot b \\
 & & 1 & 8 & & & c \cdot a \\
\hline
= & & 2 & 2 & 9 & 4 &
\end{array}
$$

$2 \cdot 2 = 4$ single-digit multiplications. $\Rightarrow$ Multiplication of two $n$-digit numbers: $n^2$ single-digit multiplications



## Observation

$$
\begin{aligned}
ab \cdot cd &= (10 \cdot a + b) \cdot (10 \cdot c + d) \\
&= 100 \cdot a \cdot c + 10 \cdot a \cdot c \\
&\quad + 10 \cdot b \cdot d + b \cdot d \\
&\quad + 10 \cdot (a - b) \cdot (d - c)
\end{aligned}
$$



## Improvement?

$$
\begin{array}{cccccc|c}
 & a & b & & c & d & \\
 & 6 & 2 & \cdot & 3 & 7 & \\
\hline
 & & & & 1 & 4 & d \cdot b \\
 & & & 1 & 4 & & d \cdot b \\
 & & & 1 & 6 & & (a-b) \cdot (d-c) \\
 & & & 1 & 8 & & c \cdot a \\
 & & 1 & 8 & & & c \cdot a \\
\hline
= & & 2 & 2 & 9 & 4 &
\end{array}
$$

$\rightarrow 3$ single-digit multiplications.

## Large Numbers

$$6237 \cdot 5898 = \underbrace{62}_{a'}\underbrace{37}_{b'} \cdot \underbrace{58}_{c'}\underbrace{98}_{d'}$$

Recursive / inductive application: compute $a' \cdot c'$, $a' \cdot d'$, $b' \cdot c'$ and $c' \cdot d'$ as shown above.

$\rightarrow 3 \cdot 3 = 9$ instead of $16$ single-digit multiplications.



## Generalization

Assumption: two numbers with $n$ digits each, $n = 2^k$ for some $k$.

$$\begin{aligned}(10^{n/2}a + b) \cdot (10^{n/2}c + d) &= 10^n \cdot a \cdot c + 10^{n/2} \cdot a \cdot c \\ &\quad + 10^{n/2} \cdot b \cdot d + b \cdot d \\ &\quad + 10^{n/2} \cdot (a - b) \cdot (d - c)\end{aligned}$$

Recursive application of this formula: algorithm by Karatsuba and Ofman (1962).



## Analysis

$M(n)$: Number of single-digit multiplications.

Recursive application of the algorithm from above $\Rightarrow$ recursion equality:

$$M(2^k) = \begin{cases} 1 & \text{if } k = 0, \\ 3 \cdot M(2^{k-1}) & \text{if } k > 0. \end{cases}$$



## Iterative Substition

Iterative substition of the recursion formula in order to guess a solution of the recursion formula:

$$\begin{aligned}M(2^k) &= 3 \cdot M(2^{k-1}) = 3 \cdot 3 \cdot M(2^{k-2}) = 3^2 \cdot M(2^{k-2}) \\ &= \ldots \\ &\overset{!}{=} 3^k \cdot M(2^0) = 3^k.\end{aligned}$$

## Proof: induction

*Hypothesis H*:

$$M(2^k) = 3^k.$$

*Base clause ($k = 0$)*:

$$M(2^0) = 3^0 = 1. \quad \checkmark$$

*Induction step ($k \to k+1$)*:

$$M(2^{k+1}) \overset{\text{def}}{=} 3 \cdot M(2^k) \overset{\text{H}}{=} 3 \cdot 3^k = 3^{k+1}.$$

$\blacksquare$



## Comparison

Traditionally $n^2$ single-digit multiplications.

Karatsuba/Ofman:

$$M(n) = 3^{\log_2 n} = (2^{\log_2 3})^{\log_2 n} = 2^{\log_2 3 \log_2 n} = n^{\log_2 3} \approx n^{1.58}.$$

Example: number with $1000$ digits: $1000^2/1000^{1.58} \approx 18$.



## Best possible algorithm?

We only know the upper bound $n^{\log_2 3}$.

There are (for large $n$) practically relevant algorithms that are faster. The best upper bound is not known.

Lower bound: $n/2$ (each digit has to be considered at at least once)



# 1.5 Finde den Star

## Is this constructive?

Exercise: find a faster multiplication algorithm.
Unsystematic search for a solution $\Rightarrow$ ☹.

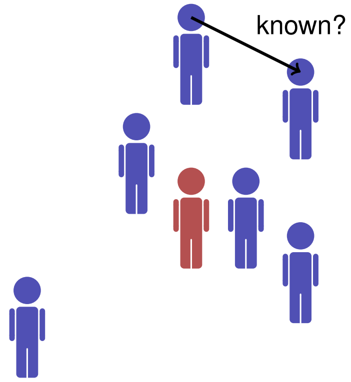
Let us consider a more constructive example.



## Example 3: find the star!

Room with $n > 1$ people.

- *Star:* Person that does not know anyone but is known by everyone.
- *Fundamental operation:* Only allowed question to a person $A$: ”Do you know $B$?” ($B \neq A$)

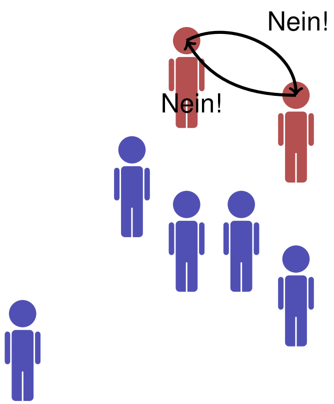





## Problemeigenschaften

- Possible: no star present
- Possible: one star present
- More than one star possible?

Assumption: two stars $S_1$, $S_2$.
$S_1$ knows $S_2$ $\Rightarrow$ $S_1$ no star.
$S_1$ does not know $S_2$ $\Rightarrow$ $S_2$ no star. $\perp$






## Naive solution

Ask everyone about everyone

Result:

| | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | - | yes | no | no |
| 2 | no | - | no | no |
| 3 | yes | yes | - | no |
| 4 | yes | yes | yes | - |

Star is $2$.

Numer operations (questions): $n \cdot (n-1)$.

## Better approach?

Induction: partition the problem into smaller pieces.

- $n = 2$: Two questions suffice
- $n > 2$: Send one person out. Find the star within $n-1$ people. Then check $A$ with $2 \cdot (n-1)$ questions.

Overal
$F(n) = 2(n-1) + F(n-1) = 2(n-1) + 2(n-2) + \cdots + 2 = n(n-1).$

No benefit. ☹



## Improvement

Idea: avoid to send the star out.

- Ask an arbitrary person $A$ if she knows $B$.
- If yes: $A$ is no star.
- If no: $B$ is no star.
- At the end 2 people remain that might contain a star. We check the potential star $X$ with any person that is out.



## Analyse

$$F(n) = \begin{cases} 2 & \text{for } n = 2, \\ 1 + F(n-1) + 2 & \text{for } n > 2. \end{cases}$$

Iterative substitution:

$F(n) = 3 + F(n-1) = 2 \cdot 3 + F(n-2) = \cdots = 3 \cdot (n-2) + 2 = 3n - 4.$

Proof: exercise!



## Moral

With many problems an inductive or recursive pattern can be developed that is based on the piecewise simplification of the problem. Next example in the next lecture.