Data Structures and Algorithms

Course at D-MATH (CSE) of ETH Zurich

Felix Friedrich

FS 2018

Welcome!

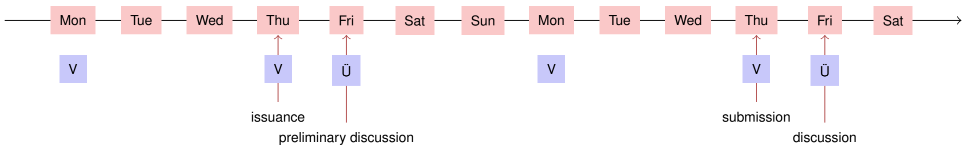
Course homepage

`http://lec.inf.ethz.ch/DA/2018`

The team:

Chefassistent	Alexander Pilz
Assistenten	Marija Kranjcevic
	Anian Ruoss
	Philippe Schlattner
	Friedrich Ginnold
Dozent	Felix Friedrich

Exercises



- Exercises availabe at lectures.
- Preliminary discussion in the following recitation session
- Solution of the exercise until the day before the next recitation session.
- Dicussion of the exercise in the next recitation session.

Exercises

- The solution of the weekly exercises is thus voluntary but *stronly* recommended.

Lacking Resources are no Excuse!

For the exercises we use an online development environment that requires only a browser, internet connection and your ETH login.

If you do not have access to a computer: there are a a lot of computers publicly accessible at ETH.

literature

Algorithmen und Datenstrukturen, *T. Ottmann, P. Widmayer*, Spektrum-Verlag, 5. Auflage, 2011

Algorithmen - Eine Einführung, *T. Cormen, C. Leiserson, R. Rivest, C. Stein*, Oldenbourg, 2010

Introduction to Algorithms, *T. Cormen, C. Leiserson, R. Rivest, C. Stein* , 3rd ed., MIT Press, 2009

The C++ Programming Language, *B. Stroustrup*, 4th ed., Addison-Wesley, 2013.

The Art of Multiprocessor Programming, *M. Herlihy, N. Shavit*, Elsevier, 2012.

Relevant for the exam

Material for the exam comprises

- Course content (lectures, handout)
- Exercises content (exercise sheets, recitation hours)

Written exam (120 min). Examination aids: four A4 pages (or two sheets of 2 A4 pages double sided) either hand written or with font size minimally 11 pt.

Offer

- Doing the weekly exercise series $\rightarrow$ bonus of maximally 0.25 of a grade points for the exam.
- The bonus is proportional to the achieved points of **specially marked bonus-task**. The full number of points corresponds to a bonus of 0.25 of a grade point.
- The **admission** to the specially marked bonus tasks can depend on the successul completion of other exercise tasks. The achieved grade bonus expires as soon as the course has been given again.

Academic integrity

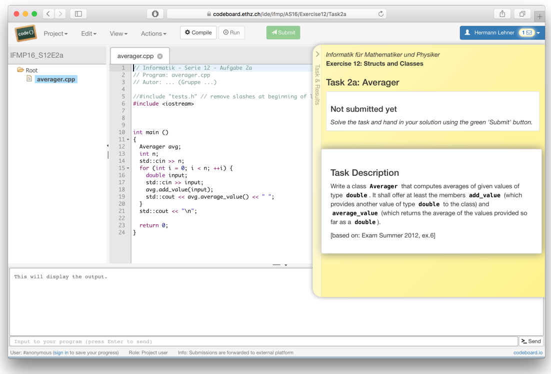
Rule: You submit solutions that you have written yourself and that you have understood.

We check this (partially automatically) and reserve our rights to adopt disciplinary measures.

Codeboard

Codeboard is an online IDE: programming in the browser!

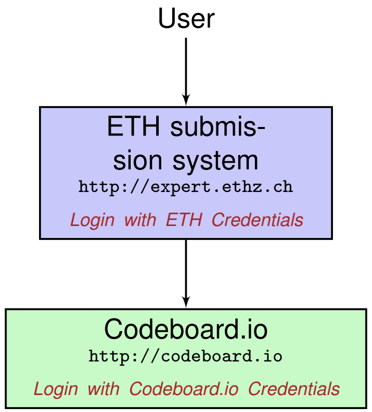
- Bring your laptop / tablet / ... along, if available.
- You can try out examples in class without having to install any tools.



Expert

Our exercise system consists of two independent systems that communicate with each other:

- **The ETH submission system:** Allows us to evaluate your tasks.
- **The online IDE:** The programming environment



Exercise Registration

Codeboard.io Registration

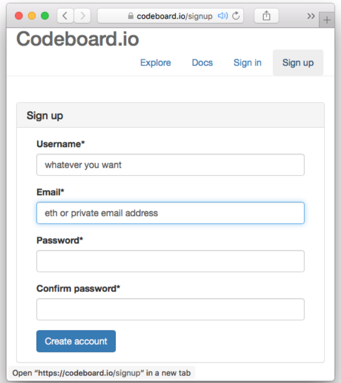
Go to `http://codeboard.io` and create an account, stay logged in.

Registration for exercises

Go to `http://expert.ethz.ch/da2018` and inscribe for one of the exercise groups there.

Codeboard.io Registration

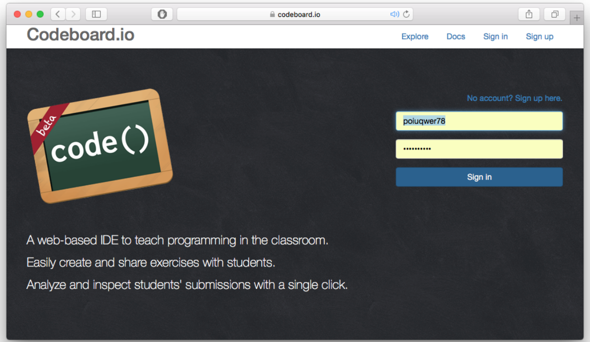
If you do not yet have an **Codeboard.io** account ...



- We use the online IDE **Codeboard.io**
- Create an account to store your progress and be able to review submissions later on
- Credentials can be chose arbitrarily *Do not use the ETH password.*

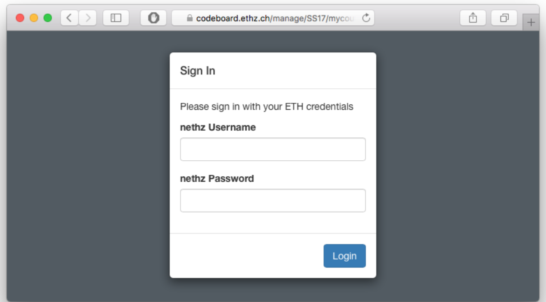
Codeboard.io Login

If you have an account, log in:



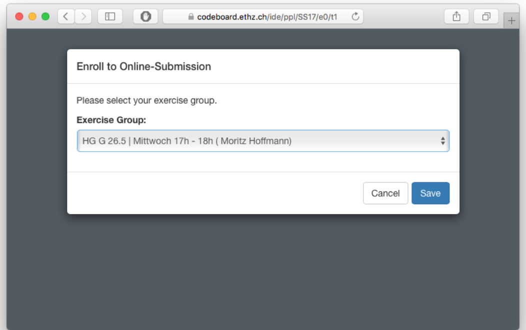
Exercise group registration I

- Visit `http://expert.ethz.ch/da2018`
- Log in with your nethz account.



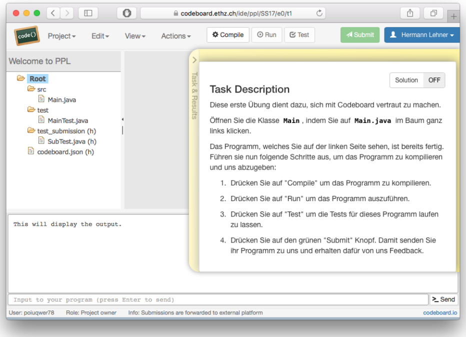
Exercise group registration II

Register with this dialog for an exercise group.



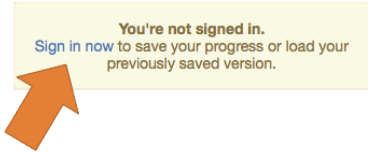
The first exercise.

You are now registered and the first exercise is loaded. Follow the instructions in the yellow box.



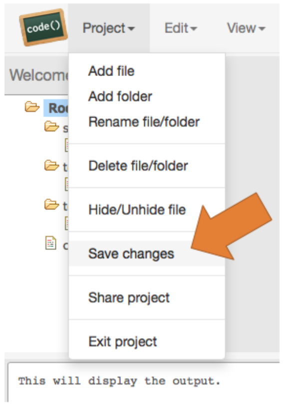
The first exercise – codeboard.io login

Attention If you see this message, click on Sign in now and register with you **codeboard.io** account.



The first exercise – store progress

Attention! Store your progress regularly. So you can continue working at any different location.



In your and our Interest

Please let us know as soon as possible when you see problems, if

- the lectures are too fast, too hard, too slow, ...
- you have problems with the exercises ...
- you do not feel well looked after ...

Briefly: if there is anything that we can fix

Should there be any Problems ...

- with the course content
  - definitely attend all recitation sessions
  - ask questions there
  - request a meeting with the assistant
- other problems
  - Email to the head TA (Alexander Pilz) or
  - Email to lecturer (Felix Friedrich)
- We are willing to help.

1. Introduction

Algorithms and Data Structures, Three Examples

Goals of the course

- Understand the design and analysis of fundamental algorithms and data structures.
- An advanced insight into a modern programming model (with C++).
- Knowledge about chances, problems and limits of the parallel and concurrent computing.

Goals of the course

On the one hand

- Essential basic knowlegde from computer science.

Andererseits

- Preparation for your further course of studies and practical considerations.

Contents

data structures / algorithms

The notion invariant, cost model, Landau notation
algorithms design, induction
searching, selection and sorting
dynamic programming
dictionaries: hashing and search trees
graphs, shortest paths, backtracking, flow
geometric algorithms, high peformance LA
Randomized algorithms (Gibbs/SA), multiscale approach
sorting networks, parallel algorithms

prorgamming with C++

RAII, Move Konstruktion, Smart Pointers,
Templates and generic programming
Exceptions
functors and lambdas
Constexpr, user defined literals
threads, mutex and monitors
promises and futures

parallel programming

parallelism vs. concurrency, speedup (Amdahl/-Gustavson), races, memory reordering, atomir registers, RMW (CAS,TAS), deadlock/starvation

1.2 Algorithms

[Cormen et al, Kap. 1;Ottman/Widmayer, Kap. 1.1]

Algorithm

Algorithm: well defined computing procedure to compute *output* data from *input* data

example problem

Input : A sequence of n numbers $(a_1, a_2, \ldots, a_n)$
Output : Permutation $(a'_1, a'_2, \ldots, a'_n)$ of the sequence $(a_i)_{1\leq i\leq n}$, such that $a'_1 \leq a'_2 \leq \cdots \leq a'_n$

Possible input

$(1, 7, 3)$, $(15, 13, 12, -0.5)$, (1) ...

Every example represents a *problem instance*

Examples for algorithmic problems

- Tables and statistis: sorting, selection and searching
- routing: shortest path algorithm, heap data structure
- DNA matching: Dynamic Programming
- fabrication pipeline: Topological Sorting
- autocomletion and spell-checking: Dictionaries / Trees
- Symboltables (compiler) : Hash-Tables
- The travelling Salesman: Dynamic Programming, Minimum Spanning Tree, Simulated Annealing
- Drawing at the computer: Digitizing lines and circles, filling polygons
- Page-Rank: (Markov-Chain) Monte Carlo ...

Characteristics

- Extremely large number of potential solutions
- Practical applicability

Darta Structures

- Organisation of the data tailored towards the algorithms that operate on the data.
- Programs = algorithms + data structures.

Very hard problems.

- NP-compleete problems: no known efficient solution (but the non-existence of such a solution is not proven yet!)
- Example: travelling salesman problem

A dream

- If computers were infinitely fast and had an infinite amount of memory ...
- ... then we would still need the theory of algorithms (only) for statements about correctness (and termination).

The reality

Resources are bounded and not free:

- Computing time $\rightarrow$ Efficiency
- Storage space $\rightarrow$ Efficiency

1.3 Ancient Egyptian Multiplication

Ancient Egyptian Multiplication

Ancient Egyptian Multiplication[1]

Compute $11 \cdot 9$

$$\begin{array}{r|r} 11 & 9 \\ \cancel{22} & \cancel{4} \\ \cancel{44} & \cancel{2} \\ 88 & 1 \\ \hline 99 & - \end{array} \qquad \begin{array}{r|r} 9 & 11 \\ 18 & 5 \\ \cancel{36} & \cancel{2} \\ 72 & 1 \\ \hline 99 & \end{array}$$

1. Double left, integer division by 2 on the right
2. Even number on the right $\Rightarrow$ eliminate row.
3. Add remaining rows on the left.

[1] Also known as russian multiplication

Advantages

- Short description, easy to grasp
- Efficient to implement on a computer: double = left shift, divide by 2 = right shift

Beispiel

left shift $9 = 01001_2 \to 10010_2 = 18$
right shift $9 = 01001_2 \to 00100_2 = 4$

Questions

- Does this always work (negative numbers?)?
- If not, when does it work?
- How do you prove correctness?
- Is it better than the school method?
- What does “good” mean at all?
- How to write this down precisely?

Observation

If $b > 1$, $a \in \mathbb{Z}$, then:

$$a \cdot b = \begin{cases} 2a \cdot \frac{b}{2} & \text{falls } b \text{ gerade,} \\ a + 2a \cdot \frac{b-1}{2} & \text{falls } b \text{ ungerade.} \end{cases}$$

Termination

$$a \cdot b = \begin{cases} a & \text{falls } b = 1, \\ 2a \cdot \frac{b}{2} & \text{falls } b \text{ gerade,} \\ a + 2a \cdot \frac{b-1}{2} & \text{falls } b \text{ ungerade.} \end{cases}$$

Recursively, Functional

$$f(a,b) = \begin{cases} a & \text{falls } b = 1, \\ f(2a, \frac{b}{2}) & \text{falls } b \text{ gerade}, \\ a + f(2a, \frac{b-1}{2}) & \text{falls } b \text{ ungerade.} \end{cases}$$

Implemented

```
// pre: b>0
// post: return a*b
int f(int a, int b){
    if(b==1)
        return a;
    else if (b%2 == 0)
        return f(2*a, b/2);
    else
        return a + f(2*a, (b-1)/2);
}
```


Correctnes

$$f(a,b) = \begin{cases} a & \text{if } b = 1, \\ f(2a, \frac{b}{2}) & \text{if } b \text{ even}, \\ a + f(2a \cdot \frac{b-1}{2}) & \text{if } b \text{ odd.} \end{cases}$$

Remaining to show: $f(a,b) = a \cdot b$ for $a \in \mathbb{Z}$, $b \in \mathbb{N}^+$.

Proof by induction

Base clause: $b = 1 \Rightarrow f(a,b) = a = a \cdot 1$.
Hypothesis: $f(a,b') = a \cdot b'$ für $0 < b' \leq b$
Step: $f(a,b+1) \stackrel{!}{=} a \cdot (b+1)$

$$f(a,b+1) = \begin{cases} f(2a, \frac{\overbrace{b+1}^{\leq b}}{2}) = a \cdot (b+1) & \text{if } b \text{ odd}, \\ a + f(2a, \underbrace{\frac{b}{2}}_{\leq b}) = a + a \cdot b & \text{if } b \text{ even.} \end{cases}$$

■

End Recursion

The recursion can be writen as *end recursion*

```
// pre: b>0
// post: return a*b
int f(int a, int b){
  if(b==1)
    return a;
  else if (b%2 == 0)
    return f(2*a, b/2);
  else
    return a + f(2*a, (b-1)/2);
}
```

→

```
// pre: b>0
// post: return a*b
int f(int a, int b){
  if(b==1)
    return a;
  int z=0;
  if (b%2 != 0){
    --b;
    z=a;
  }
  return z + f(2*a, b/2);
}
```


End-Recursion ⇒ Iteration

```
// pre: b>0
// post: return a*b
int f(int a, int b){
  if(b==1)
    return a;
  int z=0;
  if (b%2 != 0){
    --b;
    z=a;
  }
  return z + f(2*a, b/2);
}
```

→

```
int f(int a, int b) {
  int res = 0;
  while (b != 1) {
    int z = 0;
    if (b % 2 != 0){
      --b;
      z = a;
    }
    res += z;
    a *= 2; // neues a
    b /= 2; // neues b
  }
  res += a; // Basisfall b=1
  return res;
}
```


Simplify

```
int f(int a, int b) {
  int res = 0;
  while (b != 1) {
    int z = 0;
    if (b % 2 != 0){
      --b;     → Teil der Division
      z = a;   → Direkt in res
    }
    res += z;
    a *= 2;
    b /= 2;
  }
  res += a;    → in den Loop
  return res;
}
```

→

```
// pre: b>0
// post: return a*b
int f(int a, int b) {
  int res = 0;
  while (b > 0) {
    if (b % 2 != 0)
      res += a;
    a *= 2;
    b /= 2;
  }
  return res;
}
```


Invariants!

```
// pre: b>0
// post: return a*b
int f(int a, int b) {
  int res = 0;
  while (b > 0) {
    if (b % 2 != 0){
      res += a;
      --b;
    }
    a *= 2;
    b /= 2;
  }
  return res;
}
```

Sei $x := a \cdot b$.

here: $x = \boxed{a \cdot b + res}$

if here $x = a \cdot b + res$ ...

... then also here $x = a \cdot b + res$
b even

here: $x = a \cdot b + res$
here: $x = a \cdot b + res$ und $b = 0$
Also $res = x$.

Conclusion

The expression $a \cdot b + res$ is an *invariant*

- Values of a, b, res change but the invariant remains basically unchanged
- The invariant is only temporarily discarded by some statement but then re-established
- If such short statement sequences are considered atomiv, the value remains indeed invariant
- In particular the loop contains an invariant, called *loop invariant* and operates there like the induction step in induction proofs.
- Invariants are obviously powerful tools for proofs!

Further simplification

```
// pre: b>0
// post: return a*b
int f(int a, int b) {
  int res = 0;
  while (b > 0) {
    if (b % 2 != 0){
      res += a;
      --b;
    }
    a *= 2;
    b /= 2;
  }
  return res;
}
```

$\longrightarrow$

```
// pre: b>0
// post: return a*b
int f(int a, int b) {
  int res = 0;
  while (b > 0) {
    res += a * (b%2);
    a *= 2;
    b /= 2;
  }
  return res;
}
```


Analysis

```
// pre: b>0
// post: return a*b
int f(int a, int b) {
  int res = 0;
  while (b > 0) {
    res += a * (b%2);
    a *= 2;
    b /= 2;
  }
  return res;
}
```

Ancient Egyptian Multiplication corresponds to the school method with radix 2.

$$
\begin{array}{cccccccccc}
1 & 0 & 0 & 1 & \times & 1 & 0 & 1 & 1 & \\
\hline
 & & & & & 1 & 0 & 0 & 1 & (9) \\
 & & & & 1 & 0 & 0 & 1 & & (18) \\
\hline
 & & & & 1 & 1 & 0 & 1 & 1 & \\
 & & 1 & 0 & 0 & 1 & & & & (72) \\
\hline
 & & 1 & 1 & 0 & 0 & 0 & 1 & 1 & (99)
\end{array}
$$

Efficiency

Question: how long does a multiplication of a and b take?

- Measure for efficiency
  - Total number of fundamental operations: double, divide by 2, shift, test for “even”, addition
  - In the recursive and recursive code: maximally 6 operations per call or iteration, respectively
- Essential criterion:
  - Number of recursion calls or
  - Number iterations (in the iterative case)
- $\frac{b}{2^n} \leq 1$ holds for $n \geq \log_2 b$. Consequently not more than $6\lceil \log_2 b \rceil$ fundamental operations.

1.4 Fast Integer Multiplication

[Ottman/Widmayer, Kap. 1.2.3]

Example 2: Multiplication of large Numbers

Primary school:

$$
\begin{array}{cccccc|c}
 & a & b & & c & d & \\
 & 6 & 2 & \cdot & 3 & 7 & \\
\hline
 & & & & 1 & 4 & d\cdot b \\
 & & & 4 & 2 & & d\cdot a \\
 & & & & 6 & & c\cdot b \\
 & & 1 & 8 & & & c\cdot a \\
\hline
= & & 2 & 2 & 9 & 4 &
\end{array}
$$

$2 \cdot 2 = 4$ single-digit multiplications. $\Rightarrow$ Multiplication of two n-digit numbers: n^2 single-digit multiplications

Observation

$$
\begin{aligned}
ab \cdot cd &= (10 \cdot a + b) \cdot (10 \cdot c + d) \\
&= 100 \cdot a \cdot c + 10 \cdot a \cdot c \\
&\quad + 10 \cdot b \cdot d + b \cdot d \\
&\quad + 10 \cdot (a - b) \cdot (d - c)
\end{aligned}
$$

Improvement?

$$
\begin{array}{cccccc|c}
 & a & b & & c & d & \\
 & 6 & 2 & \cdot & 3 & 7 & \\
\hline
 & & & & 1 & 4 & d\cdot b \\
 & & & 1 & 4 & & d\cdot b \\
 & & & 1 & 6 & & (a-b)\cdot(d-c) \\
 & & & 1 & 8 & & c\cdot a \\
 & & 1 & 8 & & & c\cdot a \\
\hline
= & & 2 & 2 & 9 & 4 &
\end{array}
$$

$\rightarrow$ 3 single-digit multiplications.

Large Numbers

$$6237 \cdot 5898 = \underbrace{62}_{a'}\underbrace{37}_{b'} \cdot \underbrace{58}_{c'}\underbrace{98}_{d'}$$

Recursive / inductive application: compute $a' \cdot c'$, $a' \cdot d'$, $b' \cdot c'$ and $c' \cdot d'$ as shown above.

$\rightarrow 3 \cdot 3 = 9$ instead of 16 single-digit multiplications.

Generalization

Assumption: two numbers with n digits each, $n = 2^k$ for some k.

$$\begin{aligned}(10^{n/2}a + b) \cdot (10^{n/2}c + d) &= 10^n \cdot a \cdot c + 10^{n/2} \cdot a \cdot c \\ &\quad + 10^{n/2} \cdot b \cdot d + b \cdot d \\ &\quad + 10^{n/2} \cdot (a - b) \cdot (d - c)\end{aligned}$$

Recursive application of this formula: algorithm by Karatsuba and Ofman (1962).

Analysis

$M(n)$: Number of single-digit multiplications.

Recursive application of the algorithm from above $\Rightarrow$ recursion equality:

$$M(2^k) = \begin{cases} 1 & \text{if } k = 0, \\ 3 \cdot M(2^{k-1}) & \text{if } k > 0. \end{cases}$$

Iterative Substition

Iterative substition of the recursion formula in order to guess a solution of the recursion formula:

$$\begin{aligned} M(2^k) &= 3 \cdot M(2^{k-1}) = 3 \cdot 3 \cdot M(2^{k-2}) = 3^2 \cdot M(2^{k-2}) \\ &= \dots \\ &\overset{!}{=} 3^k \cdot M(2^0) = 3^k. \end{aligned}$$

Proof: induction

Hypothesis H:

$$M(2^k) = 3^k.$$

Base clause ($k = 0$):

$$M(2^0) = 3^0 = 1. \quad \checkmark$$

Induction step ($k \to k+1$):

$$M(2^{k+1}) \stackrel{\text{def}}{=} 3 \cdot M(2^k) \stackrel{\text{H}}{=} 3 \cdot 3^k = 3^{k+1}.$$

$\blacksquare$

Comparison

Traditionally n^2 single-digit multiplications.

Karatsuba/Ofman:

$$M(n) = 3^{\log_2 n} = (2^{\log_2 3})^{\log_2 n} = 2^{\log_2 3 \log_2 n} = n^{\log_2 3} \approx n^{1.58}.$$

Example: number with 1000 digits: $1000^2/1000^{1.58} \approx 18$.

Best possible algorithm?

We only know the upper bound $n^{\log_2 3}$.

There are (for large n) practically relevant algorithms that are faster. The best upper bound is not known.

Lower bound: $n/2$ (each digit has to be considered at at least once)

1.5 Finde den Star

Is this constructive?

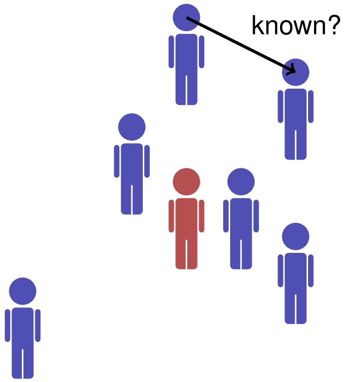
Exercise: find a faster multiplication algorithm.
Unsystematic search for a solution $\Rightarrow$ ☹.

Let us consider a more constructive example.

Example 3: find the star!

Room with $n > 1$ people.

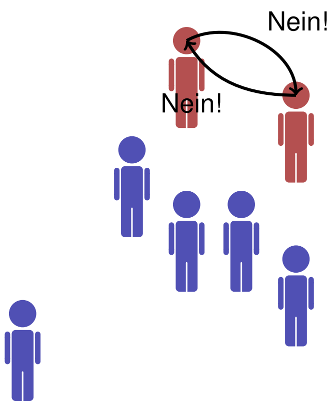
- *Star:* Person that does not know anyone but is known by everyone.
- *Fundamental operation:* Only allowed question to a person A: "Do you know B?" ($B \neq A$)



Problemeigenschaften

- Possible: no star present
- Possible: one star present
- More than one star possible?

Assumption: two stars S_1, S_2.
S_1 knows $S_2 \Rightarrow S_1$ no star.
S_1 does not know $S_2 \Rightarrow S_2$ no star. $\bot$



Naive solution

Ask everyone about everyone

Result:

	1	2	3	4
1	-	yes	no	no
2	no	-	no	no
3	yes	yes	-	no
4	yes	yes	yes	-

Star is 2.

Numer operations (questions): $n \cdot (n-1)$.

Better approach?

Induction: partition the problem into smaller pieces.

- $n = 2$: Two questions suffice
- $n > 2$: Send one person out. Find the star within $n - 1$ people. Then check A with $2 \cdot (n - 1)$ questions.

Overal
$F(n) = 2(n-1)+F(n-1) = 2(n-1)+2(n-2)+\cdots+2 = n(n-1).$

No benefit.

Improvement

Idea: avoid to send the star out.

- Ask an arbitrary person A if she knows B.
- If yes: A is no star.
- If no: B is no star.
- At the end 2 people remain that might contain a star. We check the potential star X with any person that is out.

Analyse

$$F(n) = \begin{cases} 2 & \text{for } n = 2, \\ 1 + F(n-1) + 2 & \text{for } n > 2. \end{cases}$$

Iterative substitution:

$$F(n) = 3+F(n-1) = 2\cdot 3+F(n-2) = \cdots = 3\cdot(n-2)+2 = 3n-4.$$

Proof: exercise!

Moral

With many problems an inductive or recursive pattern can be developed that is based on the piecewise simplification of the problem. Next example in the next lecture.

2. Efficiency of algorithms

Efficiency of Algorithms, Random Access Machine Model, Function Growth, Asymptotics [Cormen et al, Kap. 2.2,3,4.2-4.4 | Ottman/Widmayer, Kap. 1.1]

Efficiency of Algorithms

Goals

- Quantify the runtime behavior of an algorithm independent of the machine.
- Compare efficiency of algorithms.
- Understand dependece on the input size.

Technology Model

Random Access Machine (RAM)

- Execution model: instructions are executed one after the other (on one processor core).
- Memory model: constant access time.
- Fundamental operations: computations ($+$,$-$,$\cdot$,...) comparisons, assignment / copy, flow control (jumps)
- Unit cost model: fundamental operations provide a cost of 1.
- Data types: fundamental types like size-limited integer or floating point number.

Size of the Input Data

Typical: number of input objects (of fundamental type).

Sometimes: number bits for a *reasonable / cost-effective* representation of the data.

Asymptotic behavior

An exact running time can normally not be predicted even for small input data.

- We consider the asymptotic behavior of the algorithm.
- And ignore all constant factors.

Example

An operation with cost 20 is no worse than one with cost 1
Linear growth with gradient 5 is as good as linear growth with gradient 1.

2.2 Function growth

$\mathcal{O}$, Θ, Ω [Cormen et al, Kap. 3; Ottman/Widmayer, Kap. 1.1]

Superficially

Use the asymptotic notation to specify the execution time of algorithms.

We write $\Theta(n^2)$ and mean that the algorithm behaves for large n like n^2: when the problem size is doubled, the execution time multiplies by four.

More precise: asymptotic upper bound

provided: a function $g : \mathbb{N} \to \mathbb{R}$.

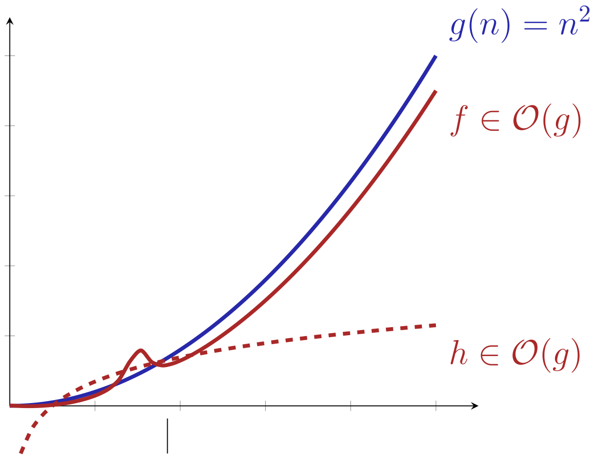
Definition:

$$\mathcal{O}(g) = \{f : \mathbb{N} \to \mathbb{R} | \\ \exists c > 0, n_0 \in \mathbb{N} : 0 \le f(n) \le c \cdot g(n) \ \forall n \ge n_0\}$$

Notation:

$$\mathcal{O}(g(n)) := \mathcal{O}(g(\cdot)) = \mathcal{O}(g).$$

Graphic



Examples

$$\mathcal{O}(g) = \{f : \mathbb{N} \to \mathbb{R} |\ \ \exists c > 0, n_0 \in \mathbb{N} : 0 \le f(n) \le c \cdot g(n)\ \forall n \ge n_0\}$$

$f(n)$	$f \in \mathcal{O}(?)$	Example
$3n + 4$	$\mathcal{O}(n)$	$c = 4, n_0 = 4$
$2n$	$\mathcal{O}(n)$	$c = 2, n_0 = 0$
$n^2 + 100n$	$\mathcal{O}(n^2)$	$c = 2, n_0 = 100$
$n + \sqrt{n}$	$\mathcal{O}(n)$	$c = 2, n_0 = 1$

Property

$$f_1 \in \mathcal{O}(g), f_2 \in \mathcal{O}(g) \Rightarrow f_1 + f_2 \in \mathcal{O}(g)$$

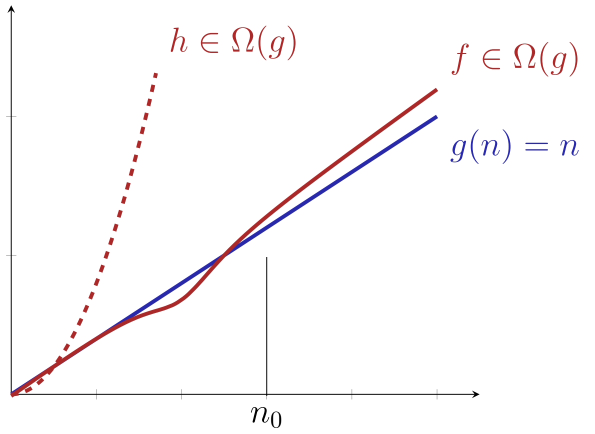
Converse: asymptotic lower bound

Given: a function $g : \mathbb{N} \to \mathbb{R}$.

Definition:

$$\Omega(g) = \{f : \mathbb{N} \to \mathbb{R} | \\ \exists c > 0, n_0 \in \mathbb{N} : 0 \le c \cdot g(n) \le f(n)\ \forall n \ge n_0\}$$

Example



Asymptotic tight bound

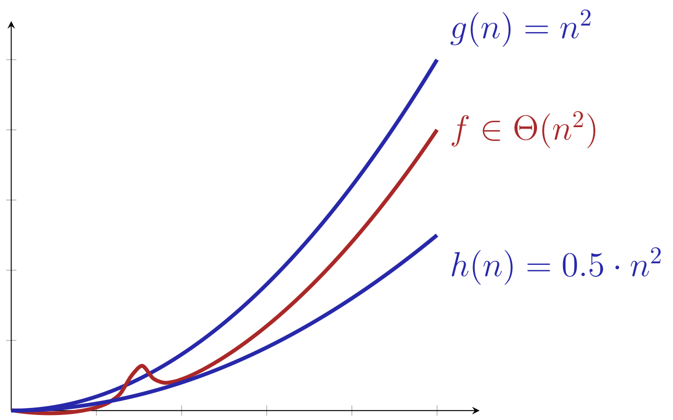
Given: function $g : \mathbb{N} \rightarrow \mathbb{R}$.

Definition:

$$\Theta(g) := \Omega(g) \cap \mathcal{O}(g).$$

Simple, closed form: exercise.

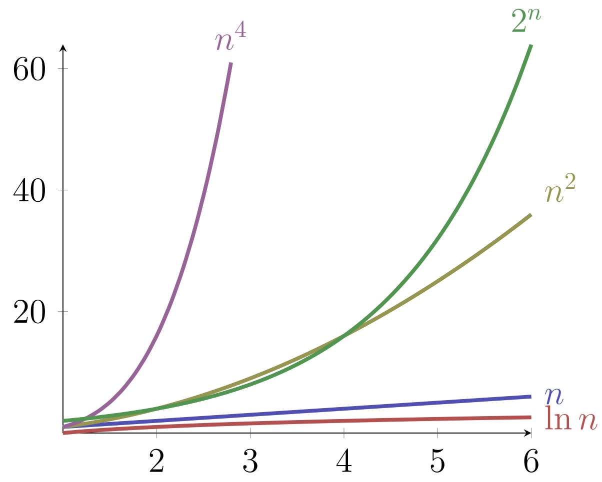
Example



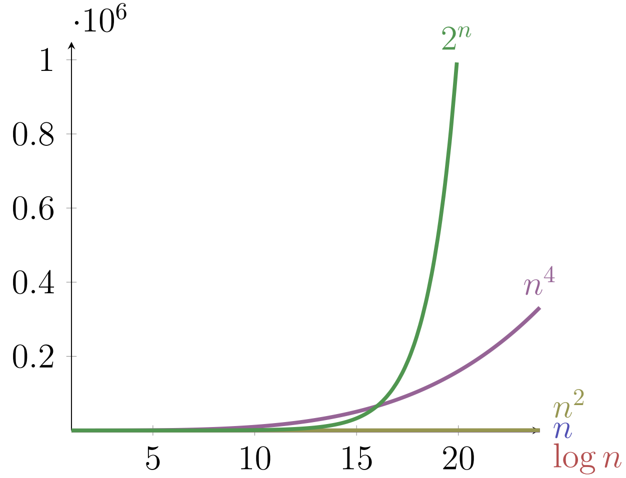
Notions of Growth

$\mathcal{O}(1)$	bounded	array access
$\mathcal{O}(\log \log n)$	double logarithmic	interpolated binary sorted sort
$\mathcal{O}(\log n)$	logarithmic	binary sorted search
$\mathcal{O}(\sqrt{n})$	like the square root	naive prime number test
$\mathcal{O}(n)$	linear	unsorted naive search
$\mathcal{O}(n \log n)$	superlinear / loglinear	good sorting algorithms
$\mathcal{O}(n^2)$	quadratic	simple sort algorithms
$\mathcal{O}(n^c)$	polynomial	matrix multiply
$\mathcal{O}(2^n)$	exponential	Travelling Salesman Dynamic Programming
$\mathcal{O}(n!)$	factorial	Travelling Salesman naively

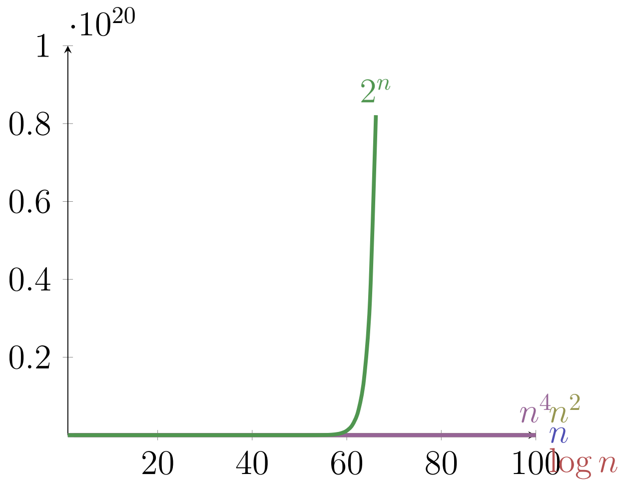
Small n



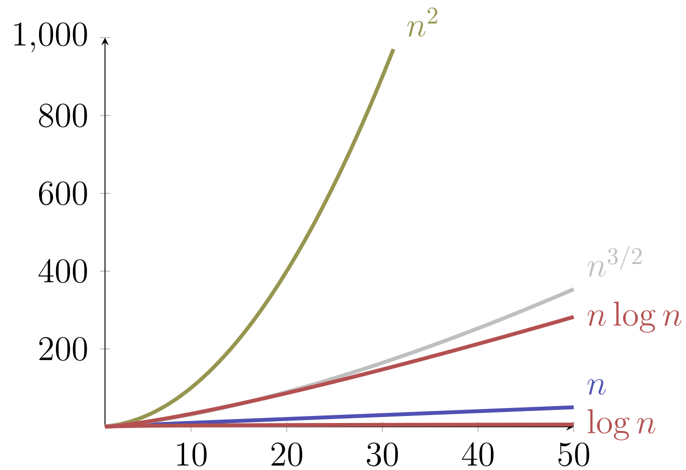
Larger n



“Large” n



Logarithms



Time Consumption

Assumption 1 Operation = $1\mu s$.

problem size	1	100	10000	10^6	10^9
$\log_2 n$	$1\mu s$	$7\mu s$	$13\mu s$	$20\mu s$	$30\mu s$
n	$1\mu s$	$100\mu s$	$1/100s$	$1s$	17 minutes
$n \log_2 n$	$1\mu s$	$700\mu s$	$13/100\mu s$	$20s$	8.5 hours
n^2	$1\mu s$	$1/100s$	1.7 minutes	11.5 days	317 centuries
2^n	$1\mu s$	10^{14} centuries	$\approx \infty$	$\approx \infty$	$\approx \infty$

A good strategy?

... Then I simply buy a new machine If today I can solve a problem of size n, then with a 10 or 100 times faster machine I can solve ...

Komplexität	(speed $\times 10$)	(speed $\times 100$)
$\log_2 n$	$n \to n^{10}$	$n \to n^{100}$
n	$n \to 10 \cdot n$	$n \to 100 \cdot n$
n^2	$n \to 3.16 \cdot n$	$n \to 10 \cdot n$
2^n	$n \to n + 3.32$	$n \to n + 6.64$

Examples

- $n \in \mathcal{O}(n^2)$ correct, but too imprecise: $n \in \mathcal{O}(n)$ and even $n \in \Theta(n)$.
- $3n^2 \in \mathcal{O}(2n^2)$ correct but uncommon: Omit constants: $3n^2 \in \mathcal{O}(n^2)$.
- $2n^2 \in \mathcal{O}(n)$ is wrong: $\frac{2n^2}{cn} = \frac{2}{c}n \underset{n\to\infty}{\to} \infty$!
- $\mathcal{O}(n) \subseteq \mathcal{O}(n^2)$ is correct
- $\Theta(n) \subseteq \Theta(n^2)$ is wrong $n \notin \Omega(n^2) \supset \Theta(n^2)$

Useful Tool

Theorem

Let $f, g : \mathbb{N} \to \mathbb{R}^+$ be two functions, then it holds that

1. $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0 \Rightarrow f \in \mathcal{O}(g)$, $\mathcal{O}(f) \subsetneq \mathcal{O}(g)$.
2. $\lim_{n\to\infty} \frac{f(n)}{g(n)} = C > 0$ (C constant) $\Rightarrow f \in \Theta(g)$.
3. $\frac{f(n)}{g(n)} \underset{n\to\infty}{\to} \infty \Rightarrow g \in \mathcal{O}(f)$, $\mathcal{O}(g) \subsetneq \mathcal{O}(f)$.

About the Notation

Common notation

$$f = \mathcal{O}(g)$$

should be read as $f \in \mathcal{O}(g)$.

Clearly it holds that

$$f_1 = \mathcal{O}(g), f_2 = \mathcal{O}(g) \not\Rightarrow f_1 = f_2!$$

Beispiel

$n = \mathcal{O}(n^2), n^2 = \mathcal{O}(n^2)$ but naturally $n \neq n^2$.

Algorithms, Programs and Execution Time

Program: concrete implementation of an algorithm.

Execution time of the program: measurable value on a concrete machine. Can be bounded from above and below.

Beispiel

3GHz computer. Maximal number of operations per cycle (e.g. 8). $\Rightarrow$ lower bound.
A single operations does never take longer than a day $\Rightarrow$ upper bound.

From an *asymptotic* point of view the bounds coincide.

Complexity

Complexity of a problem P: minimal (asymptotic) costs over all algorithms A that solve P.

Complexity of the single-digit multiplication of two numbers with n digits is $\Omega(n)$ and $\mathcal{O}(n^{\log_3 2})$ (Karatsuba Ofman).

Example:

Problem	Complexity	$\mathcal{O}(n)$	$\mathcal{O}(n)$	$\mathcal{O}(n^2)$
		$\uparrow$	$\uparrow$	$\uparrow$
Algorithm	Costs[2]	$3n-4$	$\mathcal{O}(n)$	$\Theta(n^2)$
		$\downarrow$	$\updownarrow$	$\updownarrow$
Program	Execution time	$\Theta(n)$	$\mathcal{O}(n)$	$\Theta(n^2)$

3. Design of Algorithms

Maximum Subarray Problem [Ottman/Widmayer, Kap. 1.3]
Divide and Conquer [Ottman/Widmayer, Kap. 1.2.2. S.9; Cormen et al, Kap. 4-4.1]

Algorithm Design

Inductive development of an algorithm: partition into subproblems, use solutions for the subproblems to find the overal solution.

Goal: development of the asymptotically most efficient (correct) algorithm.

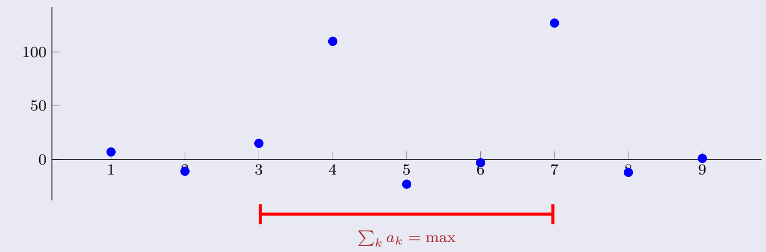
Efficiency towards run time costs (# fundamental operations) or /and memory consumption.

Maximum Subarray Problem

Given: an array of n rational numbers $(a_1, \ldots, a_n)$.

Wanted: interval $[i, j]$, $1 \leq i \leq j \leq n$ with maximal positive sum $\sum_{k=i}^{j} a_k$.

Example: $a = (7, -11, 15, 110, -23, -3, 127, -12, 1)$



Naive Maximum Subarray Algorithm

Input : A sequence of n numbers $(a_1, a_2, \ldots, a_n)$
Output : I, J such that $\sum_{k=I}^{J} a_k$ maximal.

$M \leftarrow 0$; $I \leftarrow 1$; $J \leftarrow 0$
for $i \in \{1, \ldots, n\}$ **do**
  for $j \in \{i, \ldots, n\}$ **do**
    $m = \sum_{k=i}^{j} a_k$
    if $m > M$ **then**
      $M \leftarrow m$; $I \leftarrow i$; $J \leftarrow j$

return I, J

Analysis

Theorem

The naive algorithm for the Maximum Subarray problem executes $\Theta(n^3)$ additions.

Beweis:

$$\begin{aligned}
\sum_{i=1}^{n} \sum_{j=i}^{n} (j - i + 1) &= \sum_{i=1}^{n} \sum_{j=0}^{n-i} (j+1) = \sum_{i=1}^{n} \sum_{j=1}^{n-i+1} j = \sum_{i=1}^{n} \frac{(n-i+1)(n-i+2)}{2} \\
&= \sum_{i=0}^{n} \frac{i \cdot (i+1)}{2} = \frac{1}{2} \left(\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i \right) \\
&= \frac{1}{2} \left(\frac{n(2n+1)(n+1)}{6} + \frac{n(n+1)}{2} \right) = \frac{n^3 + 3n^2 + 2n}{6} = \Theta(n^3).
\end{aligned}$$

■

Observation

$$\sum_{k=i}^{j} a_k = \underbrace{\left(\sum_{k=1}^{j} a_k\right)}_{S_j} - \underbrace{\left(\sum_{k=1}^{i-1} a_k\right)}_{S_{i-1}}$$

Prefix sums

$$S_i := \sum_{k=1}^{i} a_k.$$

Maximum Subarray Algorithm with Prefix Sums

Input : A sequence of n numbers $(a_1, a_2, \ldots, a_n)$
Output : I, J such that $\sum_{k=J}^{J} a_k$ maximal.

$\mathcal{S}_0 \leftarrow 0$
for $i \in \{1, \ldots, n\}$ **do** // prefix sum
  $\mathcal{S}_i \leftarrow \mathcal{S}_{i-1} + a_i$
$M \leftarrow 0$; $I \leftarrow 1$; $J \leftarrow 0$
for $i \in \{1, \ldots, n\}$ **do**
  for $j \in \{i, \ldots, n\}$ **do**
    $m = \mathcal{S}_j - \mathcal{S}_{i-1}$
    if $m > M$ **then**
      $M \leftarrow m$; $I \leftarrow i$; $J \leftarrow j$

Analysis

Theorem

The prefix sum algorithm for the Maximum Subarray problem conducts $\Theta(n^2)$ additions and subtractions.

Beweis:

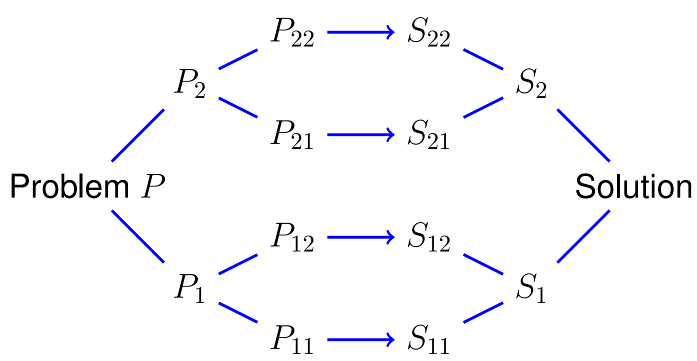
$$\sum_{i=1}^{n} 1 + \sum_{i=1}^{n}\sum_{j=i}^{n} 1 = n + \sum_{i=1}^{n}(n-i+1) = n + \sum_{i=1}^{n} i = \Theta(n^2)$$

■

divide et impera

Divide and Conquer

Divide the problem into subproblems that contribute to the simplified computation of the overal problem.



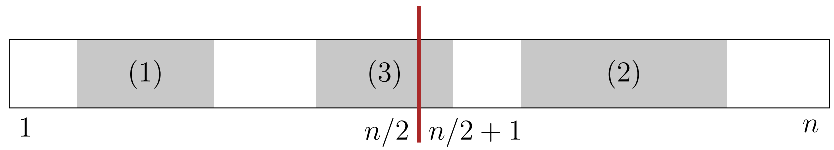
Maximum Subarray – Divide

- Divide: Divide the problem into two (roughly) equally sized halves:
  $(a_1, \ldots, a_n) = (a_1, \ldots, a_{\lfloor n/2 \rfloor}, \quad a_{\lfloor n/2 \rfloor + 1}, \ldots, a_1)$
- Simplifying assumption: $n = 2^k$ for some $k \in \mathbb{N}$.

Maximum Subarray – Conquer

If i and j are indices of a solution $\Rightarrow$ case by case analysis:

1. Solution in left half $1 \leq i \leq j \leq n/2 \Rightarrow$ Recursion (left half)
2. Solution in right half $n/2 < i \leq j \leq n \Rightarrow$ Recursion (right half)
3. Solution in the middle $1 \leq i \leq n/2 < j \leq n \Rightarrow$ Subsequent observation



Maximum Subarray – Observation

Assumption: solution in the middle $1 \leq i \leq n/2 < j \leq n$

$$
\begin{aligned}
S_{\max} &= \max_{\substack{1 \leq i \leq n/2 \\ n/2 < j \leq n}} \sum_{k=i}^{j} a_k = \max_{\substack{1 \leq i \leq n/2 \\ n/2 < j \leq n}} \left(\sum_{k=i}^{n/2} a_k + \sum_{k=n/2+1}^{j} a_k \right) \\
&= \max_{1 \leq i \leq n/2} \sum_{k=i}^{n/2} a_k + \max_{n/2 < j \leq n} \sum_{k=n/2+1}^{j} a_k \\
&= \max_{1 \leq i \leq n/2} \underbrace{S_{n/2} - S_{i-1}}_{\text{suffix sum}} + \max_{n/2 < j \leq n} \underbrace{S_j - S_{n/2}}_{\text{prefix sum}}
\end{aligned}
$$

Maximum Subarray Divide and Conquer Algorithm

Input : A sequence of n numbers $(a_1, a_2, \ldots, a_n)$
Output : Maximal $\sum_{k=i'}^{j'} a_k$.

if $n = 1$ **then**
    return $\max\{a_1, 0\}$
else
    Divide $a = (a_1, \ldots, a_n)$ in $A_1 = (a_1, \ldots, a_{n/2})$ und $A_2 = (a_{n/2+1}, \ldots, a_n)$
    Recursively compute best solution W_1 in A_1
    Recursively compute best solution W_2 in A_2
    Compute greatest suffix sum S in A_1
    Compute greatest prefix sum P in A_2
    Let $W_3 \leftarrow S + P$
    return $\max\{W_1, W_2, W_3\}$

Analysis

Theorem

The divide and conquer algorithm for the maximum subarray sum problem conducts a number of $\Theta(n \log n)$ additions and comparisons.

Analysis

Input : A sequence of n numbers $(a_1, a_2, \dots, a_n)$

Output : Maximal $\sum_{k=i'}^{j'} a_k$.

if $n = 1$ **then**

$\Theta(1)$ **return** $\max\{a_1, 0\}$

else

$\Theta(1)$ Divide $a = (a_1, \dots, a_n)$ in $A_1 = (a_1, \dots, a_{n/2})$ und $A_2 = (a_{n/2+1}, \dots, a_n)$

$T(n/2)$ Recursively compute best solution W_1 in A_1

$T(n/2)$ Recursively compute best solution W_2 in A_2

$\Theta(n)$ Compute greatest suffix sum S in A_1

$\Theta(n)$ Compute greatest prefix sum P in A_2

$\Theta(1)$ Let $W_3 \leftarrow S + P$

$\Theta(1)$ **return** $\max\{W_1, W_2, W_3\}$

Analysis

Recursion equation

$$T(n) = \begin{cases} c & \text{if } n = 1 \\ 2T(\frac{n}{2}) + a \cdot n & \text{if } n > 1 \end{cases}$$

Analysis

Mit $n = 2^k$:

$$\overline{T}(k) = \begin{cases} c & \text{if } k = 0 \\ 2\overline{T}(k-1) + a \cdot 2^k & \text{if } k > 0 \end{cases}$$

Solution:

$$\overline{T}(k) = 2^k \cdot c + \sum_{i=0}^{k-1} 2^i \cdot a \cdot 2^{k-i} = c \cdot 2^k + a \cdot k \cdot 2^k = \Theta(k \cdot 2^k)$$

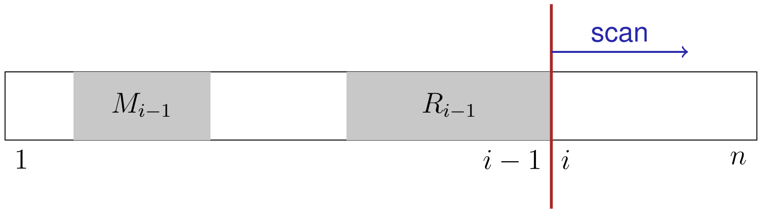
also

$$T(n) = \Theta(n \log n)$$

■

Maximum Subarray Sum Problem – Inductively

Assumption: maximal value M_{i-1} of the subarray sum is known for $(a_1, \dots, a_{i-1})$ $(1 < i \le n)$.



a_i: generates at most a better interval at the right bound (prefix sum).

$R_{i-1} \Rightarrow R_i = \max\{R_{i-1} + a_i, 0\}$

Inductive Maximum Subarray Algorithm

Input : A sequence of n numbers $(a_1, a_2, \dots, a_n)$.
Output : $\max\{0, \max_{i,j} \sum_{k=i}^{j} a_k\}$.

$M \leftarrow 0$
$R \leftarrow 0$
for $i = 1 \dots n$ **do**
    $R \leftarrow R + a_i$
    if $R < 0$ **then**
        $R \leftarrow 0$
    if $R > M$ **then**
        $M \leftarrow R$
return M;

Analysis

Theorem

The inductive algorithm for the Maximum Subarray problem conducts a number of $\Theta(n)$ additions and comparisons.

Complexity of the problem?

Can we improve over $\Theta(n)$?

Every correct algorithm for the Maximum Subarray Sum problem must consider each element in the algorithm.

Assumption: the algorithm does not consider a_i.

1. The algorithm provides a solution including a_i. Repeat the algorithm with a_i so small that the solution must not have contained the point in the first place.
2. The algorithm provides a solution not including a_i. Repeat the algorithm with a_i so large that the solution must have contained the point in the first place.

Complexity of the maximum Subarray Sum Problem

Theorem

The Maximum Subarray Sum Problem has Complexity $\Theta(n)$.

Beweis: Inductive algorithm with asymptotic execution time $\mathcal{O}(n)$.
Every algorithm has execution time $\Omega(n)$.
Thus the complexity of the problem is $\Omega(n) \cap \mathcal{O}(n) = \Theta(n)$. ■

4. Searching

Linear Search, Binary Search, Interpolation Search, Lower Bounds
[Ottman/Widmayer, Kap. 3.2, Cormen et al, Kap. 2: Problems 2.1-3,2.2-3,2.3-5]

The Search Problem

Provided

- A set of data sets

examples

telephone book, dictionary, symbol table

- Each dataset has a key k.
- Keys are comparable: unique answer to the question $k_1 \le k_2$ for keys k_1, k_2.

Task: find data set by key k.

The Selection Problem

Provided

- Set of data sets with comparable keys k.

Wanted: data set with smallest, largest, middle key value. Generally: find a data set with i-smallest key.

Search in Array

Provided

- Array A with n elements $(A[1], \dots, A[n])$.
- Key b

Wanted: index k, $1 \leq k \leq n$ with $A[k] = b$ or "not found".

22	20	32	10	35	24	42	38	28	41
1	2	3	4	5	6	7	8	9	10

Linear Search

Traverse the array from $A[1]$ to $A[n]$.

- *Best case:* 1 comparison.
- *Worst case:* n comparisons.
- Assumption: each permutation of the n keys with same probability. *Expected* number of comparisons:

$$\frac{1}{n}\sum_{i=1}^{n} i = \frac{n+1}{2}.$$

Search in a Sorted Array

Provided

- Sorted array A with n elements $(A[1], \dots, A[n])$ with $A[1] \leq A[2] \leq \dots \leq A[n]$.
- Key b

Wanted: index k, $1 \leq k \leq n$ with $A[k] = b$ or "not found".

10	20	22	24	28	32	35	38	41	42
1	2	3	4	5	6	7	8	9	10

Divide and Conquer!

Search $b = 23$.

10	20	22	24	28	32	35	38	41	42	$b < 28$
1	2	3	4	5	6	7	8	9	10	
10	20	22	24	28	32	35	38	41	42	$b > 20$
1	2	3	4	5	6	7	8	9	10	
10	20	22	24	28	32	35	38	41	42	$b > 22$
1	2	3	4	5	6	7	8	9	10	
10	20	22	24	28	32	35	38	41	42	$b < 24$
1	2	3	4	5	6	7	8	9	10	
10	20	22	24	28	32	35	38	41	42	erfolglos
1	2	3	4	5	6	7	8	9	10	

Binary Search Algorithm BSearch($A[l..r], b$)

Input : Sorted array A of n keys. Key b. Bounds $1 \le l \le r \le n$ or $l > r$ beliebig.
Output : Index of the found element. 0, if not found.

```
m ← ⌊(l + r)/2⌋
if l > r then // Unsuccessful search
    return NotFound
else if b = A[m] then// found
    return m
else if b < A[m] then// element to the left
    return BSearch(A[l..m − 1], b)
else // b > A[m]: element to the right
    return BSearch(A[m + 1..r], b)
```


Analysis (worst case)

Recurrence ($n = 2^k$)

$$T(n) = \begin{cases} d & \text{falls } n = 1, \\ T(n/2) + c & \text{falls } n > 1. \end{cases}$$

Compute:

$$\begin{aligned} T(n) &= T\left(\frac{n}{2}\right) + c = T\left(\frac{n}{4}\right) + 2c \\ &= T\left(\frac{n}{2^i}\right) + i \cdot c \\ &= T\left(\frac{n}{n}\right) + c \cdot \log_2 n = d + c \cdot \log_2 n \end{aligned}$$

$\Rightarrow$ Assumption: $T(n) = d + c \log_2 n$

Analysis (worst case)

$$T(n) = \begin{cases} d & \text{if } n = 1, \\ T(n/2) + c & \text{if } n > 1. \end{cases}$$

Guess : $T(n) = d + c \cdot \log_2 n$

Proof by induction:

- Base clause: $T(1) = d$.
- Hypothesis: $T(n/2) = d + c \cdot \log_2 n/2$
- Step: $(n/2 \to n)$

$$T(n) = T(n/2) + c = d + c \cdot (\log_2 n - 1) + c = d + c \log_2 n.$$

Result

Theorem

The binary sorted search algorithm requires $\Theta(\log n)$ fundamental operations.

Iterative Binary Search Algorithm

Input : Sorted array A of n keys. Key b.
Output : Index of the found element. 0, if unsuccessful.

$l \leftarrow 1$; $r \leftarrow n$
while $l \leq r$ **do**
  $m \leftarrow \lfloor (l+r)/2 \rfloor$
  if $A[m] = b$ **then**
    return m
  else if $A[m] < b$ **then**
    $l \leftarrow m+1$
  else
    $r \leftarrow m-1$
return *NotFound*;

Correctness

Algorithm terminates only if A is empty or b is found.

Invariant: If b is in A then b is in domain $A[l..r]$

Proof by induction

- Base clause $b \in A[1..n]$ (oder nicht)
- Hypothesis: invariant holds after i steps.
- Step:
  $b < A[m] \Rightarrow b \in A[l..m-1]$
  $b > A[m] \Rightarrow b \in A[m+1..r]$

Can this be improved?

Assumption: *values* of the array are uniformly distributed.

Example
Search for "Becker" at the very beginning of a telephone book while search for "Wawrinka" rather close to the end.
Binary search always starts in the middle.

Binary search always takes $m = \left\lfloor l + \frac{r-l}{2} \right\rfloor$.

Interpolation search

Expected relative position of b in the search interval $[l, r]$

$$\rho = \frac{b - A[l]}{A[r] - A[l]} \in [0, 1].$$

New 'middle': $l + \rho \cdot (r - l)$

Expected number of comparisons $\mathcal{O}(\log \log n)$ (without proof).

? Would you always prefer interpolation search?

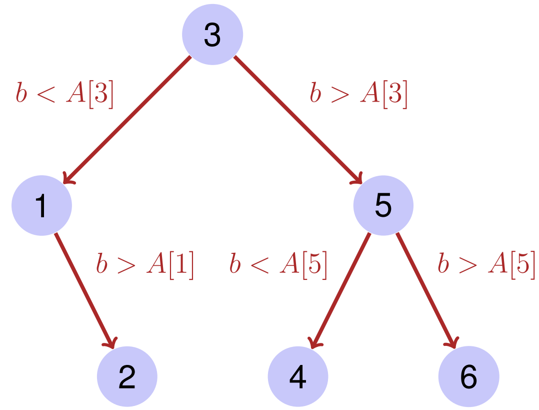
! No: worst case number of comparisons $\Omega(n)$.

Lower Bounds

Binary Search (worst case): $\Theta(\log n)$ comparisons.

Does for *any* search algorithm in a sorted array (worst case) hold that number comparisons = $\Omega(\log n)$?

Decision tree



- For any input $b = A[i]$ the algorithm must succeed $\Rightarrow$ decision tree comprises at least n nodes.
- Number comparisons in worst case = height of the tree = maximum number nodes from root to leaf.

Decision Tree

Binary tree with height h has at most $2^0 + 2^1 + \cdots + 2^{h-1} = 2^h - 1 < 2^h$ nodes.

$$2^h > n \Rightarrow h > \log_2 n$$

Decision tree with n node has at least height $\log_2 n$.

Number decisions = $\Omega(\log n)$.

Theorem

Any search algorithm on sorted data with length n requires in the worst case $\Omega(\log n)$ comparisons.

Lower bound for Search in Unsorted Array

Theorem

Any search algorithm with unsorted data of length n requires in the worst case $\Omega(n)$ comparisons.

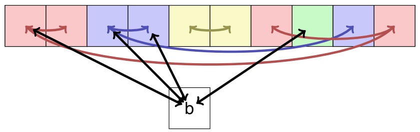
Attempt

? Correct?

"Proof": to find b in A, b must be compared with each of the n elements $A[i]$ $(1 \leq i \leq n)$.

! Wrong argument! It is still possible to compare elements within A.

Better Argument



- Different comparisons: Number comparisons with b: e Number comparisons without b: i
- Comparisons induce g groups. Initially $g = n$.
- To connect two groups at least one comparison is needed: $n - g \leq i$.
- At least one element per group must be compared with b.
- Number comparisons $i + e \geq n - g + g = n$. ■

5. Selection

The Selection Problem, Randomised Selection, Linear Worst-Case Selection [Ottman/Widmayer, Kap. 3.1, Cormen et al, Kap. 9]

Min and Max

? To separately find minimum an maximum in $(A[1], \dots, A[n])$, $2n$ comparisons are required. (How) can an algorithm with less than $2n$ comparisons for both values at a time can be found?

! Possible with $\frac{3}{2}n$ comparisons: compare 2 elemetns each and then the smaller one with min and the greater one with max.

The Problem of Selection

Input

- unsorted array $A = (A_1, \dots, A_n)$ with pairwise different values
- Number $1 \le k \le n$.

Output $A[i]$ with $|\{j : A[j] < A[i]\}| = k - 1$

Special cases

$k = 1$: Minimum: Algorithm with n comparison operations trivial.
$k = n$: Maximum: Algorithm with n comparison operations trivial.
$k = \lfloor n/2 \rfloor$: Median.

Approaches

- Repeatedly find and remove the minimum $\mathcal{O}(k \cdot n)$.
  Median: $\mathcal{O}(n^2)$
- Sorting (covered soon): $\mathcal{O}(n \log n)$
- Use a pivot $\mathcal{O}(n)$!

Use a pivot

1. Choose a *pivot* p
2. Partition A in two parts, thereby determining the rank of p.
3. Recursion on the relevant part. If $k = r$ then found.

$\le$	$\le$	$\le$	$\le$	$\le$	p	$>$	$>$	$>$	$>$
1					r				n

Algorithmus Partition($A[l..r], p$)

Input : Array A, that contains the pivot p in the interval $[l, r]$ at least once.
Output : Array A partitioned in $[l..r]$ around p. Returns position of p.
while $l \le r$ **do**
  while $A[l] < p$ **do**
    $l \leftarrow l + 1$
  while $A[r] > p$ **do**
    $r \leftarrow r - 1$
  swap($A[l]$, $A[r]$)
  if $A[l] = A[r]$ **then**
    $l \leftarrow l + 1$
return l-1

Correctness: Invariant

Invariant I: $A_i \leq p \; \forall i \in [0, l)$, $A_i \geq p \; \forall i \in (r, n]$, $\exists k \in [l, r] : A_k = p$.

while $l \leq r$ **do**
- I
- **while** $A[l] < p$ **do**
  - $l \leftarrow l + 1$
- I und $A[l] \geq p$
- **while** $A[r] > p$ **do**
  - $r \leftarrow r - 1$
- I und $A[r] \leq p$
- swap($A[l]$, $A[r]$)
- I und $A[l] \leq p \leq A[r]$
- **if** $A[l] = A[r]$ **then**
  - $l \leftarrow l + 1$
- I

return l-1

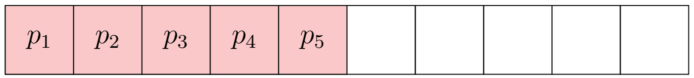
Correctness: progress

while $l \leq r$ **do**
- **while** $A[l] < p$ **do** — progress if $A[l] < p$
  - $l \leftarrow l + 1$
- **while** $A[r] > p$ **do** — progress if $A[r] > p$
  - $r \leftarrow r - 1$
- swap($A[l]$, $A[r]$) — progress if $A[l] > p$ oder $A[r] < p$
- **if** $A[l] = A[r]$ **then** — progress if $A[l] = A[r] = p$
  - $l \leftarrow l + 1$

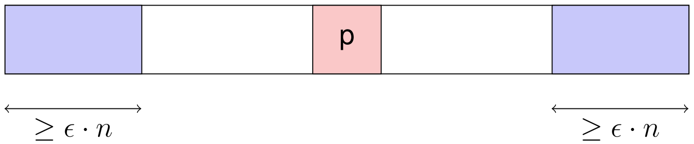
return l-1

Choice of the pivot.

The minimum is a bad pivot: worst case $\Theta(n^2)$



A good pivot has a linear number of elements on both sides.



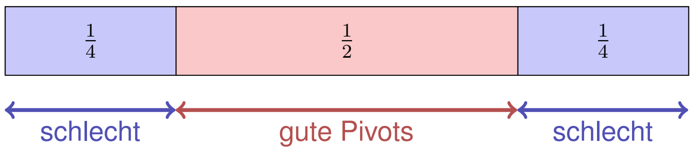
Analysis

Partitioning with factor q ($0 < q < 1$): two groups with $q \cdot n$ and $(1 - q) \cdot n$ elements (without loss of generality $g \geq 1 - q$).

$$
\begin{aligned}
T(n) &\leq T(q \cdot n) + c \cdot n \\
&= c \cdot n + q \cdot c \cdot n + T(q^2 \cdot n) = \ldots = c \cdot n \sum_{i=0}^{\log_q(n)-1} q^i + T(1) \\
&\leq c \cdot n \underbrace{\sum_{i=0}^{\infty} q^i}_{\text{geom. Reihe}} + d = c \cdot n \cdot \frac{1}{1-q} + d = \mathcal{O}(n)
\end{aligned}
$$

How can we achieve this?

Randomness to our rescue (Tony Hoare, 1961). In each step choose a random pivot.



Probability for a good pivot in one trial: $\frac{1}{2} =: \rho$.

Probability for a good pivot after k trials: $(1-\rho)^{k-1} \cdot \rho$.

Expected value of the geometric distribution: $1/\rho = 2$

[Expected value of the Geometric Distribution]

Random variable $X \in \mathbb{N}^+$ with $\mathbb{P}(X = k) = (1-p)^{k-1} \cdot p$.

Expected value

$$\begin{aligned}
\mathbb{E}(X) &= \sum_{k=1}^{\infty} k \cdot (1-p)^{k-1} \cdot p = \sum_{k=1}^{\infty} k \cdot q^{k-1} \cdot (1-q) \\
&= \sum_{k=1}^{\infty} k \cdot q^{k-1} - k \cdot q^k = \sum_{k=0}^{\infty} (k+1) \cdot q^k - k \cdot q^k \\
&= \sum_{k=0}^{\infty} q^k = \frac{1}{1-q} = \frac{1}{p}.
\end{aligned}$$

Algorithm Quickselect ($A[l..r], k$)

Input : Array A with length n. Indices $1 \leq l \leq k \leq r \leq n$, such that for all $x \in A[l..r] : |\{j|A[j] \leq x\}| \geq l$ and $|\{j|A[j] \leq x\}| \leq r$.
Output : Value $x \in A[l..r]$ with $|\{j|A[j] \leq x\}| \geq k$ and $|\{j|x \leq A[j]\}| \geq n-k+1$
if l=r **then**
  return $A[l]$;
$x \leftarrow$ RandomPivot$(A[l..r])$
$m \leftarrow$ Partition$(A[l..r], x)$
if $k < m$ **then**
  return QuickSelect$(A[l..m-1], k)$
else if $k > m$ **then**
  return QuickSelect$(A[m+1..r], k)$
else
  return $A[k]$

Algorithm RandomPivot ($A[l..r]$)

Input : Array A with length n. Indices $1 \leq l \leq i \leq r \leq n$
Output : Random "good" pivot $x \in A[l..r]$
repeat
  choose a random pivot $x \in A[l..r]$
  $p \leftarrow l$
  for $j = l$ **to** r **do**
    if $A[j] \leq x$ **then** $p \leftarrow p+1$
until $\left\lfloor \frac{3l+r}{4} \right\rfloor \leq p \leq \left\lceil \frac{l+3r}{4} \right\rceil$
return x

This algorithm is only of theoretical interest and delivers a good pivot in 2 expected iterations. Practically, in algorithm QuickSelect a uniformly chosen random pivot can be chosen or a deterministic one such as the median of three elements.

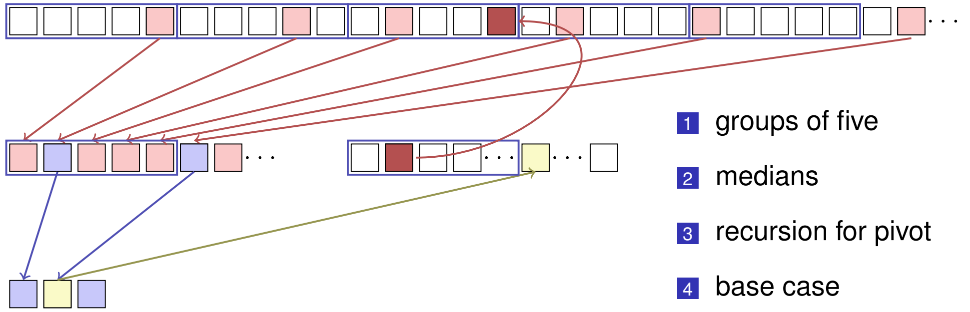
Median of medians

Goal: find an algorithm that even in worst case requires only linearly many steps.

Algorithm Select (k-smallest)

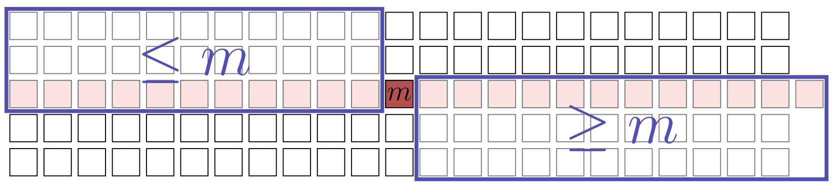
- Consider groups of five elements.
- Compute the median of each group (straighforward)
- Apply Select recursively on the group medians.
- Partition the array around the found median of medians. Result: i
- If $i = k$ then result. Otherwise: select recursively on the proper side.

Median of medians



1. groups of five
2. medians
3. recursion for pivot
4. base case
5. pivot (level 1)
6. partition (level 1)
7. median = pivot level 0
8. 2. recursion starts

How good is this?



Number points left / right of the median of medians (without median group and the rest group) $\geq 3 \cdot (\lceil \frac{1}{2} \lceil \frac{n}{5} \rceil \rceil - 2) \geq \frac{3n}{10} - 6$

Second call with maximally $\lceil \frac{7n}{10} + 6 \rceil$ elements.

Analysis

Recursion inequality:

$$T(n) \leq T\left(\left\lceil \frac{n}{5} \right\rceil\right) + T\left(\left\lceil \frac{7n}{10} + 6 \right\rceil\right) + d \cdot n.$$

with some constant d.

Claim:

$$T(n) = \mathcal{O}(n).$$

Proof

Base clause: choose c large enough such that

$$T(n) \le c \cdot n \text{ für alle } n \le n_0.$$

Induction hypothesis:

$$T(i) \le c \cdot i \text{ für alle } i < n.$$

Induction step:

$$\begin{aligned} T(n) &\le T\left(\left\lceil \frac{n}{5} \right\rceil\right) + T\left(\left\lceil \frac{7n}{10} + 6 \right\rceil\right) + d \cdot n \\ &= c \cdot \left\lceil \frac{n}{5} \right\rceil + c \cdot \left\lceil \frac{7n}{10} + 6 \right\rceil + d \cdot n. \end{aligned}$$

Proof

Induction step:

$$\begin{aligned} T(n) &\le c \cdot \left\lceil \frac{n}{5} \right\rceil + c \cdot \left\lceil \frac{7n}{10} + 6 \right\rceil + d \cdot n \\ &\le c \cdot \frac{n}{5} + c + c \cdot \frac{7n}{10} + 6c + c + d \cdot n = \frac{9}{10} \cdot c \cdot n + 8c + d \cdot n. \end{aligned}$$

Choose $c \ge 80 \cdot d$ and $n_0 = 91$.

$$T(n) \le \frac{72}{80} \cdot c \cdot n + 8c + \frac{1}{80} \cdot c \cdot n = c \cdot \underbrace{\left(\frac{73}{80} n + 8\right)}_{\le n \text{ für } n > n_0} \le c \cdot n.$$

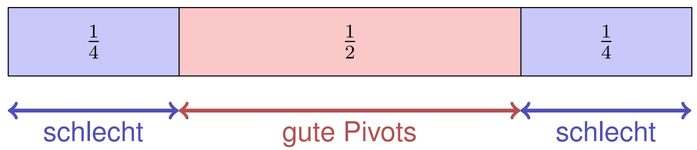
Result

Theorem

The k-the element of a sequence of n elements can be found in at most $\mathcal{O}(n)$ steps.

Overview

1.	Repeatedly find minimum	$\mathcal{O}(n^2)$
2.	Sorting and choosing $A[i]$	$\mathcal{O}(n \log n)$
3.	Quickselect with random pivot	$\mathcal{O}(n)$ expected
4.	Median of Medians (Blum)	$\mathcal{O}(n)$ worst case



6. C++ advanced (I)

Repetition: vectors, pointers and iterators, range for, keyword auto, a class for vectors, subscript-operator, move-construction, iterators

We look back...

```
#include <iostream>
#include <vector>

int main(){
  // Vector of length 10
  std::vector<int> v(10,0);
  // Input
  for (int i = 0; i < v.length(); ++i)
    std::cin >> v[i];
  // Output
  for (std::vector::iterator it = v.begin(); it != v.end(); ++it)
    std::cout << *it << " ";
}
```

We want to understand this in depth!

At least this is too pedestrian

Useful tools (1): auto (C++11)

The keyword auto:

The type of a variable is inferred from the initializer.

Examples

```
int x = 10;
auto y = x; // int
auto z = 3; // int
std::vector<double> v(5);
auto i = v[3]; // double
```


Etwas besser...

```
#include <iostream>
#include <vector>

int main(){
  std::vector<int> v(10,0); // Vector of length 10

  for (int i = 0; i < v.length(); ++i)
    std::cin >> v[i];

  for (auto it = x.begin(); it != x.end(); ++it){
    std::cout << *it << " ";
  }
}
```

Useful tools (2): range `for` (C++11)

```
for (range-declaration : range-expression)
    statement;
```

range-declaration: named variable of element type specified via the sequence in range-expression

range-expression: Expression that represents a sequence of elements via iterator pair `begin()`, `end()` or in the form of an intializer list.

Examples

```
std::vector<double> v(5);
for (double x: v) std::cout << x; // 00000
for (int x: {1,2,5}) std::cout << x; // 125
for (double& x: v) x=5;
```


That is indeed cool!

```
#include <iostream>
#include <vector>

int main(){
  std::vector<int> v(10,0); // Vector of length 10

  for (auto& x: v)
    std::cin >> x;

  for (const auto i: x)
    std::cout << i << " ";
}
```


For our detailed understanding

We build a vector class with the same capabilities ourselves!

On the way we learn about

- RAII (Resource Acquisition is Initialization) and move construction
- Index operators and other utilities
- Templates
- Exception Handling
- Functors and lambda expressions

A class for vectors

```
class vector{
  int size;
  double* elem;
public:
    // constructors
    vector(): size{0}, elem{nullptr} {};

    vector(int s):size{s}, elem{new double[s]} {}
    // destructor
    ~vector(){
        delete[] elem;
    }
    // something is missing here
}
```

Element access

```
class vector{
  ...
  // getter. pre: 0 <= i < size;
  double get(int i) const{
      return elem[i];
  }
  // setter. pre: 0 <= i < size;
  void set(int i, double d){  // setter
      elem[i] = d;
  }
  // length property
  int length() const {
      return size;
  }
}
```

```
class vector{
public:
    vector ();
    vector( int s);
    ~vector ();
    double get( int i ) const;
    void set( int i , double d);
    int length() const;
}
```


What's the problem here?

```
int main(){
  vector v(32);
  for (int i = 0; i<v.length(); ++i)
    v.set(i,i);
  vector w = v;
  for (int i = 0; i<w.length(); ++i)
    w.set(i,i*i);
  return 0;
}
```

```
class vector{
public:
  vector ();
  vector( int s);
  ~vector ();
  double get( int i );
  void set( int i , double d);
  int length() const;
}
```

```
*** Error in 'vector1': double free or corruption
(!prev): 0x0000000000d23c20 ***
======= Backtrace: =========
/lib/x86_64-linux-gnu/libc.so.6(+0x777e5)[0x7fe5a5ac97e5]
```


Rule of Three!

```
class vector{
...
  public:
  // Copy constructor
  vector(const vector &v):
    size{v.size}, elem{new double[v.size]} {
    std::copy(v.elem, v.elem+v.size, elem);
  }
}
```

```
class vector{
public:
  vector ();
  vector( int s);
  ~vector ();
  vector(const vector &v);
  double get( int i );
  void set( int i , double d);
  int length() const;
}
```


Rule of Three!

```
class vector{
...
  // Assignment operator
  vector& operator=(const vector&v){
    if (v.elem == elem) return *this;
    if (elem != nullptr) delete[] elem;
    size = v.size;
    elem = new double[size];
    std::copy(v.elem, v.elem+v.size, elem);
    return *this;
  }
}
```

```
class vector{
public:
  vector ();
  vector( int s);
  ~vector ();
  vector(const vector &v);
  vector& operator=(const vector&v);
  double get( int i );
  void set( int i , double d);
  int length() const;
}
```

Now it is correct, but cumbersome.

More elegant this way:

```
class vector{
...
  // Assignment operator
  vector& operator= (const vector&v){
    vector cpy(v);
    swap(cpy);
    return *this;
  }
private:
  // helper function
  void swap(vector& v){
    std::swap(size, v.size);
    std::swap(elem, v.elem);
  }
}
```

```
class vector{
public:
  vector();
  vector( int s);
  ~vector();
  vector(const vector &v);
  vector& operator=(const vector&v);
  double get(int i);
  void set(int i, double d);
  int length() const;
}
```


Syntactic sugar.

Getters and setters are poor. We want an index operator.

Overloading! So?

```
class vector{
...
  double operator[] (int pos) const{
    return elem[pos];
  }

  void operator[] (int pos, double value){
    elem[pos] = double;
  }
}
```

Nein!

Reference types!

```
class vector{
...
  // for const objects
  double operator[] (int pos) const{
    return elem[pos];
  }
  // for non−const objects
  double& operator[] (int pos){
    return elem[pos]; // return by reference!
  }
}
```

```
class vector{
public:
  vector();
  vector( int s);
  ~vector();
  vector(const vector &v);
  vector& operator=(const vector&v);
  double operator[] (int pos) const;
  double& operator[] (int pos);
  int length() const;
}
```


So far so good.

```
int main(){
  vector v(32); // Constructor
  for (int i = 0; i<v.length(); ++i)
    v[i] = i; // Index−Operator (Referenz!)

  vector w = v; // Copy Constructor
  for (int i = 0; i<w.length(); ++i)
    w[i] = i*i;

  const auto u = w;
  for (int i = 0; i<u.length(); ++i)
    std::cout << v[i] << ":" << u[i] << " "; // 0:0 1:1 2:4 ...
  return 0;
}
```

```
class vector{
public:
  vector();
  vector( int s);
  ~vector();
  vector(const vector &v);
  vector& operator=(const vector&v);
  double operator[] (int pos) const;
  double& operator[] (int pos);
  int length() const;
}
```

Number copies

How often is `v` being copied?

```
vector operator+ (const vector& l, double r){
    vector result (l);  // Kopie von l nach result
    for (int i = 0; i < l.length(); ++i) result[i] = l[i] + r;
    return result; // Dekonstruktion von result nach Zuweisung
}

int main(){
    vector v(16); // allocation of elems[16]
    v = v + 1;   // copy when assigned!
    return 0;    // deconstruction of v
}
```

`v` is copied twice

Move construction and move assignment

```
class vector{
...
    // move constructor
    vector (vector&& v): size(0), elem{nullptr}{
        swap(v);
    };
    // move assignment
    vector& operator=(vector&& v){
        swap(v);
        return *this;
    };
}
```

```
class vector{
public:
  vector ();
  vector( int s);
  ~vector ();
  vector(const vector &v);
  vector& operator=(const vector&v);
  vector (vector&& v);
  vector& operator=(vector&& v);
  double operator[] (int pos) const;
  double& operator[] (int pos);
  int length() const;
}
```


Explanation

When the source object of an assignment will not continue existing after an assignment the compiler can use the move assignment instead of the assignment operator.[3] A potentially expensive copy operations is avoided this way.

Number of copies in the previous example goes down to 1.

[3] Analogously so for the copy-constructor and the move constructor

Illustration of the Move-Semantics

```
// nonsense implementation of a "vector" for demonstration purposes
class vec{
public:
  vec () {
        std::cout << "default constructor\n";}
  vec (const vec&) {
        std::cout << "copy constructor\n";}
  vec& operator = (const vec&) {
        std::cout << "copy assignment\n"; return *this;}
  ~vec() {}
};
```

How many Copy Operations?

```
vec operator + (const vec& a, const vec& b){
    vec tmp = a;
    // add b to tmp
    return tmp;
}

int main (){
    vec f;
    f = f + f + f + f;
}
```

Output
default constructor
copy constructor
copy constructor
copy constructor
copy assignment

4 copies of the vector

Illustration of the Move-Semantics

```
// nonsense implementation of a "vector" for demonstration purposes
class vec{
public:
  vec () { std::cout << "default constructor\n";}
  vec (const vec&) { std::cout << "copy constructor\n";}
  vec& operator = (const vec&) {
        std::cout << "copy assignment\n"; return *this;}
  ~vec() {}
  // new: move constructor and assignment
  vec (vec&&) {
        std::cout << "move constructor\n";}
  vec& operator = (vec&&) {
        std::cout << "move assignment\n"; return *this;}
};
```


How many Copy Operations?

```
vec operator + (const vec& a, const vec& b){
    vec tmp = a;
    // add b to tmp
    return tmp;
}

int main (){
    vec f;
    f = f + f + f + f;
}
```

Output
default constructor
copy constructor
copy constructor
copy constructor
move assignment

3 copies of the vector

How many Copy Operations?

```
vec operator + (vec a, const vec& b){
    // add b to a
    return a;
}

int main (){
    vec f;
    f = f + f + f + f;
}
```

Output
default constructor
copy constructor
move constructor
move constructor
move constructor
move assignment

1 copy of the vector

Explanation: move semantics are applied when an x-value (expired value) is assigned. R-value return values of a function are examples of x-values.

http://en.cppreference.com/w/cpp/language/value_category

How many Copy Operations?

```
void swap(vec& a, vec& b){
    vec tmp = a;
    a=b;
    b=tmp;
}

int main (){
    vec f;
    vec g;
    swap(f,g);
}
```

Output
default constructor
default constructor
copy constructor
copy assignment
copy assignment

3 copies of the vector

Forcing x-values

```
void swap(vec& a, vec& b){
    vec tmp = std::move(a);
    a=std::move(b);
    b=std::move(tmp);
}
int main (){
    vec f;
    vec g;
    swap(f,g);
}
```

Output
default constructor
default constructor
move constructor
move assignment
move assignment

0 copies of the vector

Explanation: With std::move an l-value expression can be transformed into an x-value. Then move-semantics are applied. http://en.cppreference.com/w/cpp/utility/move

Range for

We wanted this:

```
vector v = ...;
for (auto x: v)
  std::cout << x << " ";
```

In order to support this, an iterator must be provided via begin and end .

Iterator for the vector

```
class vector{
...
    // Iterator
    double* begin(){
        return elem;
    }
    double* end(){
        return elem+size;
    }
}
```

```
class vector{
public:
  vector ();
  vector( int s);
  ~vector();
  vector(const vector &v);
  vector& operator=(const vector&v);
  vector (vector&& v);
  vector& operator=(vector&& v);
  double operator[] (int pos) const;
  double& operator[] (int pos);
  int length() const;
  double* begin();
  double* end();
}
```

Const Iterator for the vector

```
class vector{
...
        // Const-Iterator
    const double* begin() const{
        return elem;
    }
    const double* end() const{
        return elem+size;
    }

}
```

```
class vector{
public:
  vector();
  vector(int s);
  ~vector();
  vector(const vector &v);
  vector& operator=(const vector&v);
  vector (vector&& v);
  vector& operator=(vector&& v);
  double operator[] (int pos) const;
  double& operator[] (int pos);
  int length() const;
  double* begin();
  double* end();
  const double* begin() const;
  const double* end() const;
}
```


Intermediate result

```
vector Natural(int from, int to){
  vector v(to-from+1);
  for (auto& x: v) x = from++;
  return v;
}

int main(){
  vector v = Natural(5,12);
  for (auto x: v)
    std::cout << x << " "; // 5 6 7 8 9 10 11 12
  std::cout << "\n";
  std::cout << "sum="
            << std::accumulate(v.begin(), v.end(),0); // sum = 68
  return 0;
}
```


Useful tools (3): `using` (C++11)

`using` replaces in C++11 the old `typedef`.

```
using identifier = type-id;
```

Beispiel

```
using element_t = double;
class vector{
      std::size_t size;
      element_t* elem;
...
}
```


7. Sorting I

Simple Sorting

7.1 Simple Sorting

Selection Sort, Insertion Sort, Bubblesort [Ottman/Widmayer, Kap. 2.1, Cormen et al, Kap. 2.1, 2.2, Exercise 2.2-2, Problem 2-2

Problem

Input: An array $A = (A[1], ..., A[n])$ with length n.

Output: a permutation A' of A, that is sorted: $A'[i] \leq A'[j]$ for all $1 \leq i \leq j \leq n$.

Algorithm: IsSorted(A)

Input : Array $A = (A[1], ..., A[n])$ with length n.
Output : Boolean decision "sorted" or "not sorted"

```
for i ← 1 to n − 1 do
    if A[i] > A[i + 1] then
        return "not sorted";
return "sorted";
```


Observation

IsSorted(A):"not sorted", if $A[i] > A[i+1]$ for an i.

⇒ idea:

```
for j ← 1 to n − 1 do
    if A[j] > A[j + 1] then
        swap(A[j], A[j + 1]);
```

Give it a try

5	6	2	8	4	1	$(j = 1)$
5	6	2	8	4	1	$(j = 2)$
5	2	6	8	4	1	$(j = 3)$
5	2	6	8	4	1	$(j = 4)$
5	2	6	4	8	1	$(j = 5)$
5	2	6	4	1	8	

- Not sorted! ☹.
- But the greatest element moves to the right
  $\Rightarrow$ new idea! ☺

Try it out

5	6	2	8	4	1	$(j = 1, i = 1)$
5	6	2	8	4	1	$(j = 2)$
5	2	6	8	4	1	$(j = 3)$
5	2	6	8	4	1	$(j = 4)$
5	2	6	4	8	1	$(j = 5)$
5	2	6	4	1	8	$(j = 1, i = 2)$
2	5	6	4	1	8	$(j = 2)$
2	5	6	4	1	8	$(j = 3)$
2	5	4	6	1	8	$(j = 4)$
2	5	4	1	6	8	$(j = 1, i = 3)$
2	5	4	1	6	8	$(j = 2)$
2	4	5	1	6	8	$(j = 3)$
2	4	1	5	6	8	$(j = 1, i = 4)$
2	4	1	5	6	8	$(j = 2)$
2	1	4	5	6	8	$(i = 1, j = 5)$
1	2	4	5	6	8	

- Apply the procedure iteratively.
- For $A[1, \dots, n]$, then $A[1, \dots, n-1]$, then $A[1, \dots, n-2]$, etc.

Algorithm: Bubblesort

Input : Array $A = (A[1], \dots, A[n])$, $n \geq 0$.
Output : Sorted Array A

for $i \leftarrow 1$ **to** $n - 1$ **do**
  for $j \leftarrow 1$ **to** $n - i$ **do**
    if $A[j] > A[j+1]$ **then**
      swap$(A[j], A[j+1])$;

Analysis

Number key comparisons $\sum_{i=1}^{n-1}(n-i) = \frac{n(n-1)}{2} = \Theta(n^2)$.
Number swaps in the worst case: $\Theta(n^2)$

? What is the worst case?

! If A is sorted in decreasing order.

? Algorithm can be adapted such that it terminates when the array is sorted. Key comparisons and swaps of the modified algorithm in the best case?

! Key comparisons = $n - 1$. Swaps = 0.

Selection Sort

5	6	2	8	4	1	$(i = 1)$
1	6	2	8	4	5	$(i = 2)$
1	2	6	8	4	5	$(i = 3)$
1	2	4	8	6	5	$(i = 4)$
1	2	4	5	6	8	$(i = 5)$
1	2	4	5	6	8	$(i = 6)$
1	2	4	5	6	8	

- Iterative procedure as for Bubblesort.
- Selection of the smallest (or largest) element by immediate search.

Algorithm: Selection Sort

Input : Array $A = (A[1], \dots, A[n])$, $n \geq 0$.
Output : Sorted Array A
for $i \leftarrow 1$ **to** $n - 1$ **do**
  $p \leftarrow i$
  for $j \leftarrow i + 1$ **to** n **do**
    if $A[j] < A[p]$ **then**
      $p \leftarrow j$;
  swap$(A[i], A[p])$

Analysis

Number comparisons in worst case: $\Theta(n^2)$.

Number swaps in the worst case: $n - 1 = \Theta(n)$

Best case number comparisons: $\Theta(n^2)$.

Insertion Sort

5 |	6	2	8	4	1	$(i = 1)$
5	6 |	2	8	4	1	$(i = 2)$
5	6	2 |	8	4	1	$(i = 3)$
2	5	6	8 |	4	1	$(i = 4)$
2	5	6	8	4 |	1	$(i = 5)$
2	4	5	6	8	1 |	$(i = 6)$
1	2	4	5	6	8	

- Iterative procedure: $i = 1...n$
- Determine insertion position for element i.
- Insert element i array block movement potentially required

Insertion Sort

? What is the disadvantage of this algorithm compared to sorting by selection?

! Many element movements in the worst case.

? What is the advantage of this algorithm compared to selection sort?

! The search domain (insertion interval) is already sorted. Consequently: binary search possible.

Algorithm: Insertion Sort

Input : Array $A = (A[1], \dots, A[n])$, $n \geq 0$.
Output : Sorted Array A
for $i \leftarrow 2$ **to** n **do**
  $x \leftarrow A[i]$
  $p \leftarrow$ BinarySearch($A[1...i-1], x$); // Smallest $p \in [1, i]$ with $A[p] \geq x$
  for $j \leftarrow i-1$ **downto** p **do**
    $A[j+1] \leftarrow A[j]$
  $A[p] \leftarrow x$

Analysis

Number comparisons in the worst case:
$\sum_{k=1}^{n-1} a \cdot \log k = a \log((n-1)!) \in \mathcal{O}(n \log n)$.

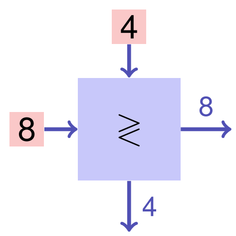
Number comparisons in the best case $\Theta(n \log n)$.[4]

Number swaps in the worst case $\sum_{k=2}^{n} (k-1) \in \Theta(n^2)$

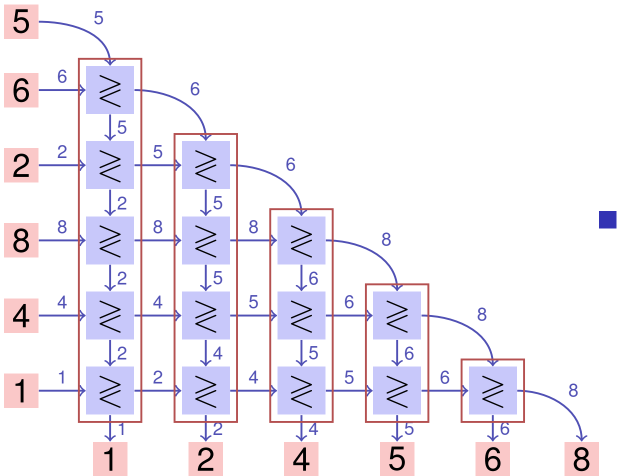
[4] With slight modification of the function BinarySearch for the minimum / maximum: $\Theta(n)$

Different point of view

Sorting node:

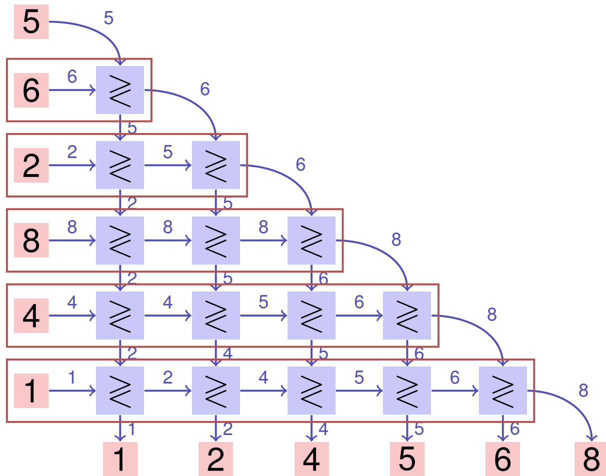


Different point of view



- Like selection sort
  [and like Bubblesort]

Different point of view



- Like insertion sort

Conclusion

In a certain sense, Selection Sort, Bubble Sort and Insertion Sort provide the same kind of sort strategy. Will be made more precise. [5]

[5] In the part about parallel sorting networks. For the sequential code of course the observations as described above still hold.

Shellsort

Insertion sort on subsequences of the form $(A_{k\cdot i})$ $(i \in \mathbb{N})$ with decreasing distances k. Last considered distance must be $k = 1$.

Good sequences: for example sequences with distances $k \in \{2^i 3^j | 0 \leq i, j\}$.

Shellsort

9	8	7	6	5	4	3	2	1	0	
1	8	7	6	5	4	3	2	9	0	insertion sort, $k = 4$
1	0	7	6	5	4	3	2	9	8	
1	0	3	6	5	4	7	2	9	8	
1	0	3	2	5	4	7	6	9	8	
1	0	3	2	5	4	7	6	9	8	insertion sort, $k = 2$
1	0	3	2	5	4	7	6	9	8	
0	1	2	3	4	5	6	7	8	9	insertion sort, $k = 1$

8. Sorting II

Heapsort, Quicksort, Mergesort

8.1 Heapsort

[Ottman/Widmayer, Kap. 2.3, Cormen et al, Kap. 6]

Heapsort

Inspiration from selectsort: fast insertion

Inspiration from insertion sort: fast determination of position

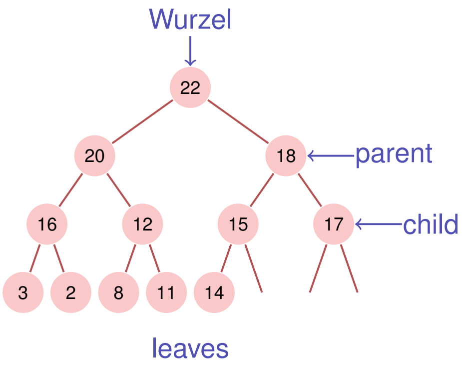
? Can we have the best of both worlds?

! Yes, but it requires some more thinking...

[Max-]Heap[6]

Binary tree with the following properties

1. complete up to the lowest level
2. Gaps (if any) of the tree in the last level to the right
3. *Heap-Condition:* Max-(Min-)Heap: key of a child smaller (greater) thant that of the parent node

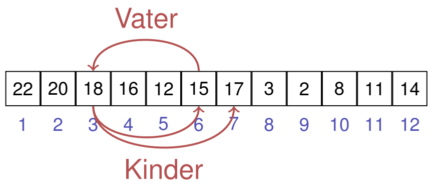


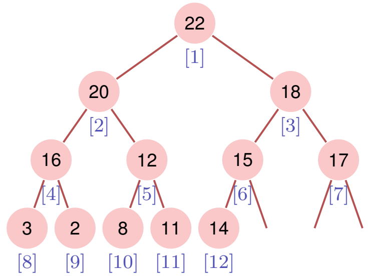
[6]Heap(data structure), not: as in “heap and stack” (memory allocation)

Heap and Array

Tree $\to$ Array:

- children$(i) = \{2i, 2i+1\}$
- parent$(i) = \lfloor i/2 \rfloor$



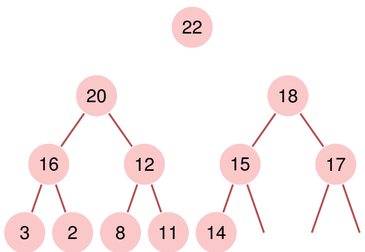


Depends on the starting index[7]

[7]For array that start at 0: $\{2i, 2i+1\} \to \{2i+1, 2i+2\}$, $\lfloor i/2 \rfloor \to \lfloor (i-1)/2 \rfloor$

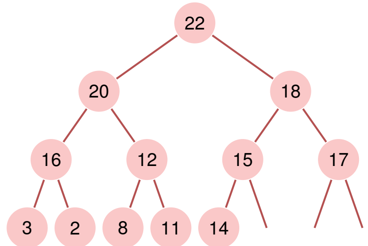
Recursive heap structure

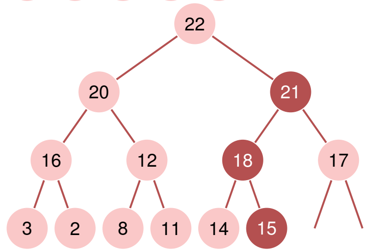
A heap consists of two heaps:



Insert

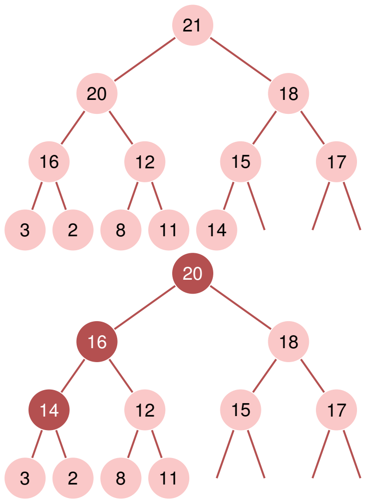
- Insert new element at the first free position. Potentially violates the heap property.
- Reestablish heap property: climb successively
- Worst case number of operations: $\mathcal{O}(\log n)$





Remove the maximum

- Replace the maximum by the lower right element
- Reestablish heap property: sink successively (in the direction of the greater child)
- Worst case number of operations: $\mathcal{O}(\log n)$



Algorithm Sink(A, i, m)

```
Input :   Array A with heap structure for the children of i. Last element m.
Output :  Array A with heap structure for i with last element m.
while 2i ≤ m do
    j ← 2i; // j left child
    if j < m and A[j] < A[j + 1] then
        j ← j + 1; // j right child with greater key
    if A[i] < A[j] then
        swap(A[i], A[j])
        i ← j; // keep sinking
    else
        i ← m; // sinking finished
```


Sort heap

$A[1, ..., n]$ is a Heap.
While $n > 1$

- swap($A[1]$, $A[n]$)
- Sink($A, 1, n-1$);
- $n \leftarrow n - 1$

		7	6	4	5	1	2
swap	$\Rightarrow$	2	6	4	5	1	7
sink	$\Rightarrow$	6	5	4	2	1	7
swap	$\Rightarrow$	1	5	4	2	6	7
sink	$\Rightarrow$	5	4	2	1	6	7
swap	$\Rightarrow$	1	4	2	5	6	7
sink	$\Rightarrow$	4	1	2	5	6	7
swap	$\Rightarrow$	2	1	4	5	6	7
sink	$\Rightarrow$	2	1	4	5	6	7
swap	$\Rightarrow$	1	2	4	5	6	7

Heap creation

Observation: Every leaf of a heap is trivially a correct heap.

Consequence: Induction from below!

Algorithm HeapSort(A, n)

Input : Array A with length n.
Output : A sorted.
// Build the heap.
for $i \leftarrow n/2$ **downto** 1 **do**
    Sink(A, i, n);
// Now A is a heap.
for $i \leftarrow n$ **downto** 2 **do**
    swap($A[1], A[i]$)
    Sink($A, 1, i-1$)
// Now A is sorted.

Analysis: sorting a heap

Sink traverses at most $\log n$ nodes. For each node 2 key comparisons. $\Rightarrow$ sorting a heap costs in the worst case $2 \log n$ comparisons.

Number of memory movements of sorting a heap also $\mathcal{O}(n \log n)$.

Analysis: creating a heap

Calls to sink: $n/2$. Thus number of comparisons and movements: $v(n) \in \mathcal{O}(n \log n)$.

But mean length of sinking paths is much smaller:

$$v(n) = \sum_{h=0}^{\lfloor \log n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil \cdot c \cdot h \in \mathcal{O}(n \sum_{h=0}^{\lfloor \log n \rfloor} \frac{h}{2^h})$$

with $s(x) := \sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2} \quad (0 < x < 1)$ [8] and $s(\frac{1}{2}) = 2$:

$$v(n) \in \mathcal{O}(n).$$

[8] $f(x) = \frac{1}{1-x} = 1 + x + x^2 ... \Rightarrow f'(x) = \frac{1}{(1-x)^2} = 1 + 2x + ...$

8.2 Mergesort

[Ottman/Widmayer, Kap. 2.4, Cormen et al, Kap. 2.3],

Intermediate result

Heapsort: $\mathcal{O}(n \log n)$ Comparisons and movements.

? Disadvantages of heapsort?

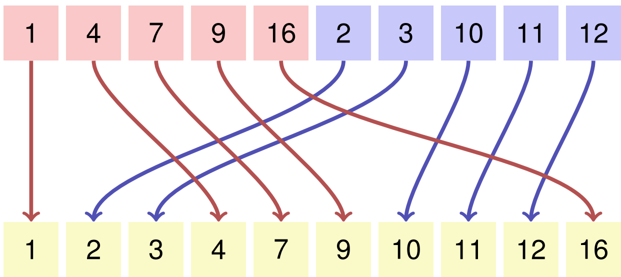
- ! Missing locality: heapsort jumps around in the sorted array (negative cache effect).
- ! Two comparisons required before each necessary memory movement.

Mergesort

Divide and Conquer!

- Assumption: two halves of the array A are already sorted.
- Minimum of A can be evaluated with two comparisons.
- Iteratively: sort the pre-sorted array A in $\mathcal{O}(n)$.

Merge



Algorithm Merge(A, l, m, r)

Input : Array A with length n, indexes $1 \leq l \leq m \leq r \leq n$. $A[l, \dots, m]$, $A[m+1, \dots, r]$ sorted
Output : $A[l, \dots, r]$ sorted

$B \leftarrow$ new Array$(r - l + 1)$
$i \leftarrow l$; $j \leftarrow m + 1$; $k \leftarrow 1$
while $i \leq m$ and $j \leq r$ **do**
  if $A[i] \leq A[j]$ **then** $B[k] \leftarrow A[i]$; $i \leftarrow i + 1$
  else $B[k] \leftarrow A[j]$; $j \leftarrow j + 1$
  $k \leftarrow k + 1$;
while $i \leq m$ **do** $B[k] \leftarrow A[i]$; $i \leftarrow i + 1$; $k \leftarrow k + 1$
while $j \leq r$ **do** $B[k] \leftarrow A[j]$; $j \leftarrow j + 1$; $k \leftarrow k + 1$
for $k \leftarrow l$ **to** r **do** $A[k] \leftarrow B[k - l + 1]$

Correctness

Hypothesis: after k iterations of the loop in line 3 $B[1,\dots,k]$ is sorted and $B[k] \le A[i]$, if $i \le m$ and $B[k] \le A[j]$ if $j \le r$.

Proof by induction:
Base case: the empty array $B[1,\dots,0]$ is trivially sorted.
Induction step $(k \to k+1)$:

- wlog $A[i] \le A[j]$, $i \le m, j \le r$.
- $B[1,\dots,k]$ is sorted by hypothesis and $B[k] \le A[i]$.
- After $B[k+1] \leftarrow A[i]$ $B[1,\dots,k+1]$ is sorted.
- $B[k+1] = A[i] \le A[i+1]$ (if $i+1 \le m$) and $B[k+1] \le A[j]$ if $j \le r$.
- $k \leftarrow k+1, i \leftarrow i+1$: Statement holds again.

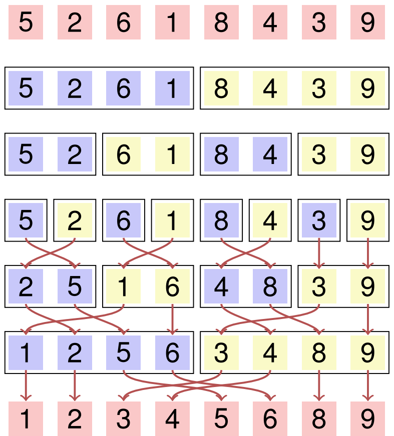
Analysis (Merge)

Lemma

If: array A with length n, indexes $1 \le l < r \le n$. $m = \lfloor (l+r)/2 \rfloor$ and $A[l,\dots,m]$, $A[m+1,\dots,r]$ sorted.
Then: in the call of Merge(A,l,m,r) a number of $\Theta(r-l)$ key movements and comparisons are executed.

Proof: straightforward(Inspect the algorithm and count the operations.)

Mergesort



Split

Split

Split

Merge

Merge

Merge

Algorithm recursive 2-way Mergesort(A, l, r)

Input : Array A with length n. $1 \le l \le r \le n$
Output : Array $A[l,\dots,r]$ sorted.
if $l < r$ **then**
  $m \leftarrow \lfloor (l+r)/2 \rfloor$ // middle position
  Mergesort(A, l, m) // sort lower half
  Mergesort$(A, m+1, r)$ // sort higher half
  Merge(A, l, m, r) // Merge subsequences

Analysis

Recursion equation for the number of comparisons and key movements:

$$C(n) = C(\left\lceil \frac{n}{2} \right\rceil) + C(\left\lfloor \frac{n}{2} \right\rfloor) + \Theta(n) \in \Theta(n \log n)$$

Algorithm StraightMergesort(A)

Avoid recursion: merge sequences of length $1, 2, 4, ...$ directly

Input : Array A with length n
Output : Array A sorted

```
length ← 1
while length < n do                     // Iterate over lengths n
    r ← 0
    while r + length < n do             // Iterate over subsequences
        l ← r + 1
        m ← l + length − 1
        r ← min(m + length, n)
        Merge(A, l, m, r)
    length ← length · 2
```


Analysis

Like the recursive variant, the straight 2-way mergesort always executes a number of $\Theta(n \log n)$ key comparisons and key movements.

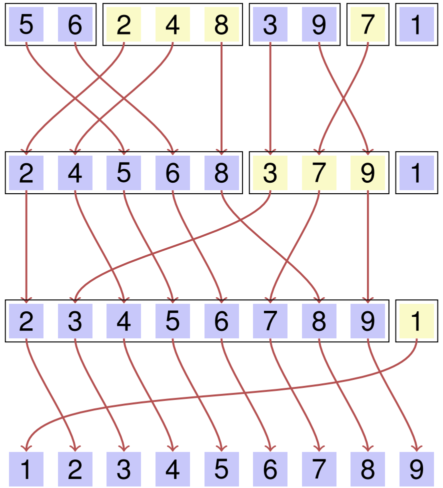
Natural 2-way mergesort

Observation: the variants above do not make use of any presorting and always execute $\Theta(n \log n)$ memory movements.

? How can partially presorted arrays be sorted better?

! Recursive merging of previously sorted parts (*runs*) of A.

Natural 2-way mergesort



Algorithm NaturalMergesort(A)

Input : Array A with length $n > 0$
Output : Array A sorted
repeat
  $r \leftarrow 0$
  while $r < n$ **do**
    $l \leftarrow r + 1$
    $m \leftarrow l$; **while** $m < n$ **and** $A[m+1] \geq A[m]$ **do** $m \leftarrow m + 1$
    if $m < n$ **then**
      $r \leftarrow m + 1$; **while** $r < n$ **and** $A[r+1] \geq A[r]$ **do** $r \leftarrow r + 1$
      Merge(A, l, m, r);
    else
      $r \leftarrow n$
until $l = 1$

Analysis

In the best case, natural merge sort requires only $n - 1$ comparisons.

? Is it also asymptotically better than StraightMergesort on average?

! No. Given the assumption of pairwise distinct keys, on average there are $n/2$ positions i with $k_i > k_{i+1}$, i.e. $n/2$ runs. Only one iteration is saved on average.

Natural mergesort executes in the worst case and on average a number of $\Theta(n \log n)$ comparisons and memory movements.

8.3 Quicksort

[Ottman/Widmayer, Kap. 2.2, Cormen et al, Kap. 7]

Quicksort

? What is the disadvantage of Mergesort?

! Requires $\Theta(n)$ storage for merging.

? How could we reduce the merge costs?

! Make sure that the left part contains only smaller elements than the right part.

? How?

! Pivot and Partition!

Quicksort (arbitrary pivot)

2	4	5	6	8	$\boxed{3}$	7	9	1
$\boxed{2}$	1	3	6	8	$\boxed{5}$	7	9	4
1	2	3	4	5	8	$\boxed{7}$	9	6
1	2	3	4	5	6	7	9	$\boxed{8}$
1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9

Algorithm Quicksort($A[l, \dots, r]$

Input : Array A with length n. $1 \leq l \leq r \leq n$.
Output : Array A, sorted between l and r.
if $l < r$ **then**
    Choose pivot $p \in A[l, \dots, r]$
    $k \leftarrow$ Partition$(A[l, \dots, r], p)$
    Quicksort$(A[l, \dots, k-1])$
    Quicksort$(A[k+1, \dots, r])$

Reminder: algorithm Partition($A[l, \dots, r], p$)

Input : Array A, that contains the pivot p in $[l, r]$ at least once.
Output : Array A partitioned around p. Returns the position of p.
while $l \leq r$ **do**
    while $A[l] < p$ **do**
        $l \leftarrow l + 1$
    while $A[r] > p$ **do**
        $r \leftarrow r - 1$
    swap($A[l]$, $A[r]$)
    if $A[l] = A[r]$ **then** // Only for keys that are not pairwise different
        $l \leftarrow l + 1$
return l-1

Analysis: number comparisons

Best case. Pivot = median; number comparisons:

$$T(n) = 2T(n/2) + c \cdot n, \; T(1) = 0 \quad \Rightarrow \quad T(n) \in \mathcal{O}(n \log n)$$

Worst case. Pivot = min or max; number comparisons:

$$T(n) = T(n-1) + c \cdot n, \; T(1) = 0 \quad \Rightarrow \quad T(n) \in \Theta(n^2)$$

Analysis: number swaps

Result of a call to partition (pivot 3):

2	1	3	6	8	5	7	9	4

? How many swaps have taken place?

! 2. The maximum number of swaps is given by the number of keys in the smaller part.

Analysis: number swaps

Intellectual game

- Each key from the smaller part pay a coin when swapped.
- When a key has paid a coin then the domain containing the key is less than or equal to half the previous size.
- Every key needs to pay at most $\log n$ coins. But there are only n keys.

Consequence: there are $\mathcal{O}(n \log n)$ key swaps in the worst case.

Randomized Quicksort

Despite the worst case running time of $\Theta(n^2)$, quicksort is used practically very often.

Reason: quadratic running time unlikely provided that the choice of the pivot and the pre-sorting are not very disadvantageous.

Avoidance: randomly choose pivot. Draw uniformly from $[l, r]$.

Analysis (randomized quicksort)

Expected number of compared keys with input length n:

$$T(n) = (n-1) + \frac{1}{n}\sum_{k=1}^{n}\left(T(k-1) + T(n-k)\right),\ T(0) = T(1) = 0$$

Claim $T(n) \leq 4n \log n$.

Proof by induction:

Base case straightforward for $n = 0$ (with $0 \log 0 := 0$) and for $n = 1$.

Hypothesis: $T(n) \leq 4n \log n$ for some n.

Induction step: $(n-1 \to n)$

Analysis (randomized quicksort)

$$\begin{aligned} T(n) &= n - 1 + \frac{2}{n}\sum_{k=0}^{n-1} T(k) \overset{\mathsf{H}}{\leq} n - 1 + \frac{2}{n}\sum_{k=0}^{n-1} 4k \log k \\ &= n - 1 + \sum_{k=1}^{n/2} 4k \underbrace{\log k}_{\leq \log n - 1} + \sum_{k=n/2+1}^{n-1} 4k \underbrace{\log k}_{\leq \log n} \\ &\leq n - 1 + \frac{8}{n}\left((\log n - 1)\sum_{k=1}^{n/2} k + \log n \sum_{k=n/2+1}^{n-1} k\right) \\ &= n - 1 + \frac{8}{n}\left((\log n) \cdot \frac{n(n-1)}{2} - \frac{n}{4}\left(\frac{n}{2} + 1\right)\right) \\ &= 4n \log n - 4 \log n - 3 \leq 4n \log n \end{aligned}$$

Analysis (randomized quicksort)

Theorem

On average randomized quicksort requires $\mathcal{O}(n \cdot \log n)$ comparisons.

Practical considerations

Worst case recursion depth $n - 1$[9]. Then also a memory consumption of $\mathcal{O}(n)$.

Can be avoided: recursion only on the smaller part. Then guaranteed $\mathcal{O}(\log n)$ worst case recursion depth and memory consumption.

[9] stack overflow possible!

Quicksort with logarithmic memory consumption

Input : Array A with length n. $1 \leq l \leq r \leq n$.
Output : Array A, sorted between l and r.

while $l < r$ **do**
    Choose pivot $p \in A[l, \ldots, r]$
    $k \leftarrow$ Partition$(A[l, \ldots, r], p)$
    if $k - l < r - k$ **then**
        Quicksort$(A[l, \ldots, k-1])$
        $l \leftarrow k + 1$
    else
        Quicksort$(A[k+1, \ldots, r])$
        $r \leftarrow k - 1$

The call of Quicksort$(A[l, \ldots, r])$ in the original algorithm has moved to iteration (tail recursion!): the if-statement became a while-statement.

Practical considerations.

Practically the pivot is often the median of three elements. For example: Median3$(A[l], A[r], A[\lfloor l + r/2 \rfloor])$.

There is a variant of quicksort that requires only constant storage. Idea: store the old pivot at the position of the new pivot.

9. C++ advanced (II): Templates

Motivation

Goal: generic vector class and functionality.

Examples

```
vector<double> vd(10);
vector<int> vi(10);
vector<char> vi(20);

auto nd = vd * vd; // norm (vector of double)
auto ni = vi * vi; // norm (vector of int)
```

Types as Template Parameters

1. In the concrete implementation of a class replace the type that should become generic (in our example: `double`) by a representative element, e.g. `T`.
2. Put in front of the class the construct `template<typename T>`[10] Replace `T` by the representative name).

The construct `template<typename T>` can be understood as "for all types T".

[10] equally: `template<class T>`

Types as Template Parameters

```
template <typename ElementType>
class vector{
    size_t size;
    ElementType* elem;
public:
    ...
    vector(size_t s):
        size{s},
        elem{new ElementType[s]}{}
    ...
    ElementType& operator[](size_t pos){
        return elem[pos];
    }
    ...
}
```


Template Instances

`vector<typeName>` generates a type instance `vector` with `ElementType=typeName`.
Notation: Instantiation

Examples

```
vector<double> x;       // vector of double
vector<int> y;          // vector of int
vector<vector<double>> x; // vector of vector of double
```


Type-checking

Templates are basically replacement rules at instantiation time and applied compilation. It is checked as little as necessary and as much as possible.

Example

```
template <typename T>
class vector{
...
  // pre: vector contains at least one element, elements comparable
  // post: return minimum of contained elements
  T min() const{
    auto min = elem[0];
    for (auto x=elem+1; x<elem+size; ++x){
      if (*x<min) min = *x;
    }
    return min;
  }
...
}
```

```
vector<int> a(10); // ok
auto m = a.min(); // ok
vector<vector<int>> b(10); // ok;
auto n = b.min(); no match for operator< !
```


Generic Programming

Generic components should be developed rather as a generalization of one or more examples than from first principles.

```
using size_t=std::size_t;
template <typename T>
class vector{
public:
  vector ();
  vector(size_t s);
  ~vector();
  vector(const vector &v);
  vector& operator=(const vector&v);
  vector (vector&& v);
  vector& operator=(vector&& v);
  T operator[] (size_t pos) const;
  T& operator[] (size_t pos);
  int length() const;
  T* begin();
  T* end();
  const T* begin() const;
  const T* end() const;
}
```


Function Templates

1. In a concrete implementation of a function replace the type that should become generic by a replacement, .e.g `T`,
2. Put in front of the function the construct `template<typename T>`[11] (Replace `T` by the replacement name)

[11] equally: `template<class T>`

Function Templates

```
template <typename T>
void swap(T& x, T&y){
    T temp = x;
    x = y;
    y = temp;
}
```

Types of the parameter determine the version of the function that is (compiled) and used:

```
int x=5;
int y=6;
swap(x,y); // calls swap with T=int
```

Limits of Magic

```
template <typename T>
void swap(T& x, T&y){
    T temp = x;
    x = y;
    y = temp;
}
```

An inadmissible version of the function is not generated:

```
int x=5;
double y=6;
swap(x,y); // error:  no matching function for ...
```


Limits of Magic

Separation of declaration and definition is possible ...

```
template <typename T>
class Pair{
    T left; T right;
public:
    Pair(T l, T r):left{l}, right{r}{}
    T Min();
    //...
};

template <typename T>
T Pair<T>::Min(){
    return left < right ? left : right;
}
```


Limits of Magic

Hiding implementations common in OOP is limited. The definition cannot be provided in separate, non-included file.

```
template <typename T>
class Pair{
    T left; T right;
public:
    Pair(T l, T r):left{l}, right{r}{}
    T Min();
    //...
};

template <typename T> // cannot be hidden from the user of Pair !
T Pair<T>::Min(){
    return left < right ? left : right;
}
```


Useful!

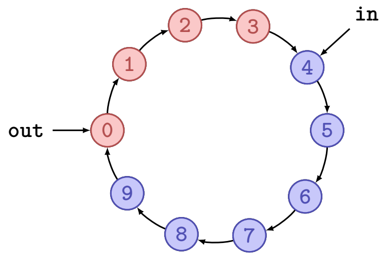
```
// Output of an arbitrary container
template <typename T>
void output(const T& t){
    for (auto x: t)
        std::cout << x << " ";
    std::cout << "\n";
}

int main(){
  std::vector<int> v={1,2,3};
  output(v); // 1 2 3
}
```

Powerful!

```
template <typename T> // square number
T sq(T x){
    return x*x;
}
template <typename Container, typename F>
void apply(Container& c, F f){ // x <- f(x) forall x in c
    for(auto& x: c)
        x = f(x);
}
int main(){
  std::vector<int> v={1,2,3};
  apply(v,sq<int>);
  output(v); // 1 4 9
}
```

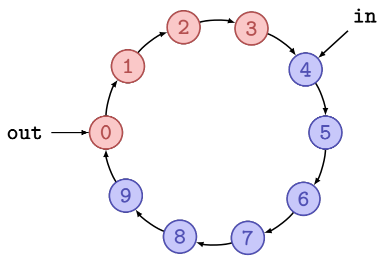

Template Parameterization with Values

```
template <typename T, int size>
class CircularBuffer{
  T buf[size] ;
  int in; int out;
public:
  CircularBuffer():in{0},out{0}{};
  bool empty(){
    return in == out;
  }
  bool full(){
    return (in + 1) % size == out;
  }
  void put(T x); // declaration
  T get();       // declaration
};
```



Template Parameterization with Values

```
template <typename T, int size>
void CircularBuffer<T,size>::put(T x){
    assert(!full());
    buf[in] = x;
    in = (in + 1) % size;
}

template <typename T, int size>
T CircularBuffer<T,size>::get(){
    assert(!empty());
    T x = buf[out];
    out = (out + 1) % size;   ←—— Potential for optimization if size = 2^k.
    return x;
}
```



10. Sorting III

Lower bounds for the comparison based sorting, radix- and bucket-sort

10.1 Lower bounds for comparison based sorting

[Ottman/Widmayer, Kap. 2.8, Cormen et al, Kap. 8.1]

Lower bound for sorting

Up to here: worst case sorting takes $\Omega(n \log n)$ steps.

Is there a better way? No:

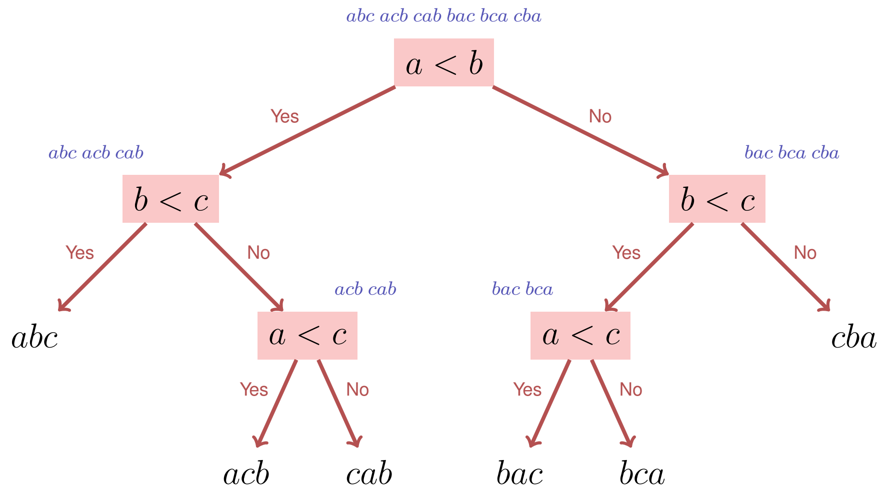
Theorem

Sorting procedures that are based on comparison require in the worst case and on average at least $\Omega(n \log n)$ key comparisons.

Comparison based sorting

- An algorithm must identify the correct one of $n!$ permutations of an array $(A_i)_{i=1,\ldots,n}$.
- At the beginning the algorithm know nothing about the array structure.
- We consider the knowledge gain of the algorithm in the form of a decision tree:
  - Nodes contain the remaining possibilities.
  - Edges contain the decisions.

Decision tree



Decision tree

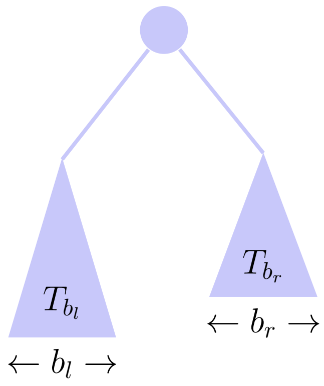
The height of a binary tree with L leaves is at least $\log_2 L$. $\Rightarrow$ The heigh of the decision tree $h \geq \log n! \in \Omega(n \log n)$.[12]

Thus the length of the longest path in the decision tree $\in \Omega(n \log n)$.

Remaining to show: mean length $M(n)$ of a path $M(n) \in \Omega(n \log n)$.

[12] $\log n! \in \Theta(n \log n)$:
$\log n! = \sum_{k=1}^{n} \log k \leq n \log n$.
$\log n! = \sum_{k=1}^{n} \log k \geq \sum_{k=n/2}^{n} \log k \geq \frac{n}{2} \cdot \log \frac{n}{2}$.

Average lower bound



- Decision tree T_n with n leaves, average height of a leaf $m(T_n)$
- Assumption $m(T_n) \geq \log n$ not for all n.
- Choose smalles b with $m(T_b) < \log n \Rightarrow b \geq 2$
- $b_l + b_r = b$, wlog $b_l > 0$ und $b_r > 0 \Rightarrow$ $b_l < b, b_r < b \Rightarrow m(T_{b_l}) \geq \log b_l$ und $m(T_{b_r}) \geq \log b_r$

Average lower bound

Average height of a leaf:

$$\begin{aligned} m(T_b) &= \frac{b_l}{b}(m(T_{b_l}) + 1) + \frac{b_r}{b}(m(T_{b_r}) + 1) \\ &\geq \frac{1}{b}(b_l(\log b_l + 1) + b_r(\log b_r + 1)) = \frac{1}{b}(b_l \log 2b_l + b_r \log 2b_r) \\ &\geq \frac{1}{b}(b \log b) = \log b. \end{aligned}$$

Contradiction. ■

The last inequality holds because $f(x) = x \log x$ is convex and for a convex function it holds that $f((x+y)/2) \leq 1/2 f(x) + 1/2 f(y)$ ($x = 2b_l$, $y = 2b_r$).[13] Enter $x = 2b_l$, $y = 2b_r$, and $b_l + b_r = b$.

[13] generally $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$ for $0 \leq \lambda \leq 1$.

10.2 Radixsort and Bucketsort

Radixsort, Bucketsort [Ottman/Widmayer, Kap. 2.5, Cormen et al, Kap. 8.3]

Radix Sort

Sorting based on comparison: comparable keys ($<$ or $>$, often $=$). No further assumptions.

Different idea: use more information about the keys.

Annahmen

Assumption: keys representable as words from an alphabet containing m elements.

Examples

$m = 10$	decimal numbers	$183 = 183_{10}$
$m = 2$	dual numbers	101_2
$m = 16$	hexadecimal numbers	$A0_{16}$
$m = 26$	words	‘‘INFORMATIK’’

m is called the radix of the representation.

Assumptions

- keys = m-adic numbers with same length.
- Procedure z for the extraction of digit k in $\mathcal{O}(1)$ steps.

Example

$z_{10}(0, 85) = 5$
$z_{10}(1, 85) = 8$
$z_{10}(2, 85) = 0$

Radix-Exchange-Sort

Keys with radix 2.

Observation: if $k \geq 0$,

$$z_2(i, x) = z_2(i, y) \text{ for all } i > k$$

and

$$z_2(k, x) < z_2(k, y),$$

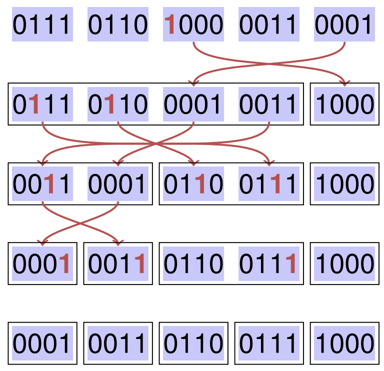
then $x < y$.

Radix-Exchange-Sort

Idea:

- Start with a maximal k.
- Binary partition the data sets with $z_2(k,\cdot) = 0$ vs. $z_2(k,\cdot) = 1$ like with quicksort.
- $k \leftarrow k - 1$.

Radix-Exchange-Sort



Algorithm RadixExchangeSort(A, l, r, b)

Input : Array A with length n, left and right bounds $1 \leq l \leq r \leq n$, bit position b

Output : Array A, sorted in the domain $[l, r]$ by bits $[0, \ldots, b]$.

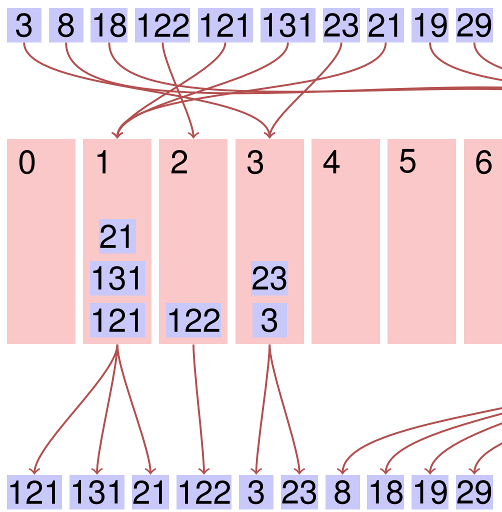
if $l > r$ **and** $b \geq 0$ **then**
    $i \leftarrow l - 1$
    $j \leftarrow r + 1$
    repeat
        repeat $i \leftarrow i + 1$ **until** $z_2(b, A[i]) = 1$ **and** $i \geq j$
        repeat $j \leftarrow j + 1$ **until** $z_2(b, A[j]) = 0$ **and** $i \geq j$
        if $i < j$ **then** swap$(A[i], A[j])$
    until $i \geq j$
    RadixExchangeSort$(A, l, i - 1, b - 1)$
    RadixExchangeSort$(A, i, r, b - 1)$

Analysis

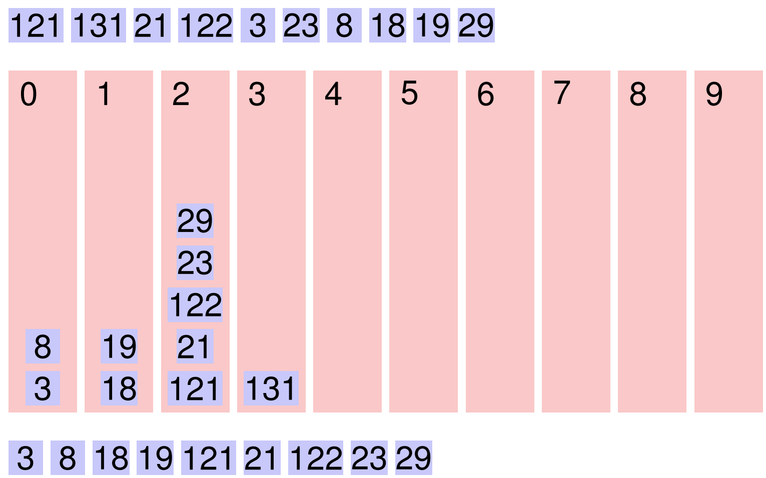
RadixExchangeSort provide recursion with maximal recursion depth = maximal number of digits p.

Worst case run time $\mathcal{O}(p \cdot n)$.

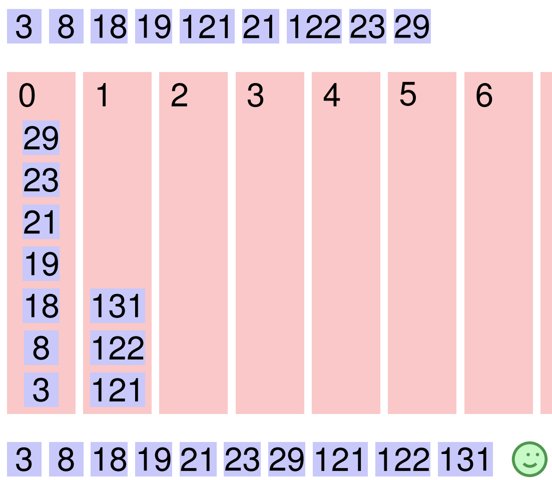
Bucket Sort



Bucket Sort



Bucket Sort



implementation details

Bucket size varies greatly. Two possibilities

- Linked list for each digit.
- One array of length n. compute offsets for each digit in the first iteration.

11. Fundamental Data Structures

Abstract data types stack, queue, implementation variants for linked lists, amortized analysis [Ottman/Widmayer, Kap. 1.5.1-1.5.2, Cormen et al, Kap. 10.1.-10.2,17.1-17.3]

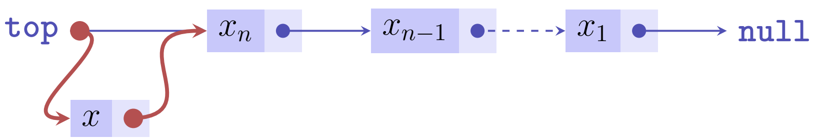
Abstract Data Types

We recall

A *stack* is an abstract data type (ADR) with operations

- `push`(x, S): Puts element x on the stack S.
- `pop`(S): Removes and returns top most element of S or `null`
- `top`(S): Returns top most element of S or `null`.
- `isEmpty`(S): Returns `true` if stack is empty, `false` otherwise.
- `emptyStack`$()$: Returns an empty stack.

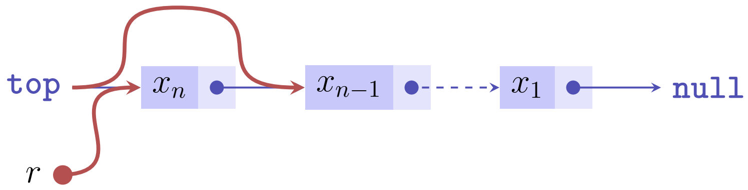
Implementation Push



`push`(x, S):

1. Create new list element with x and pointer to the value of `top`.
2. Assign the node with x to `top`.

Implementation Pop



`pop`(S):

1. If `top`=`null`, then return `null`
2. otherwise memorize pointer p of `top` in r.
3. Set `top` to $p.next$ and return r

Analysis

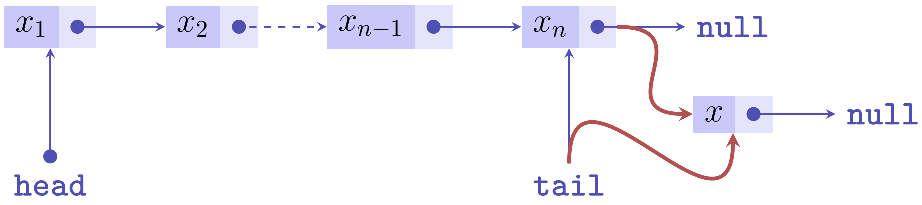
Each of the operations `push`, `pop`, `top` and `isEmpty` on a stack can be executed in $\mathcal{O}(1)$ steps.

Queue (fifo)

A queue is an ADT with the following operations

- `enqueue`(x, Q): adds x to the tail (=end) of the queue.
- `dequeue`(Q): removes x from the head of the queue and returns x (`null` otherwise)
- `head`(Q): returns the object from the head of the queue (`null` otherwise)
- `isEmpty`(Q): return `true` if the queue is empty, otherwise `false`
- `emptyQueue`$()$: returns empty queue.

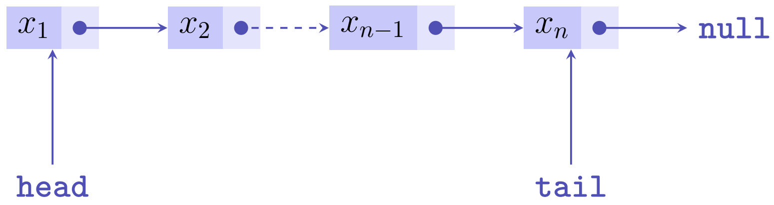
Implementation Queue



`enqueue`(x, S):

1. Create a new list element with x and pointer to `null`.
2. If `tail` $\neq$ `null`, then set `tail.next` to the node with x.
3. Set `tail` to the node with x.
4. If `head` $=$ `null`, then set `head` to `tail`.

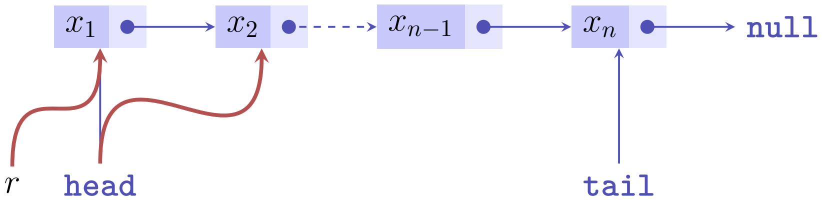
Invariants



With this implementation it holds that

- either `head` $=$ `tail` $=$ `null`,
- or `head` $=$ `tail` $\neq$ `null` and `head.next` $=$ `null`
- or `head` $\neq$ `null` and `tail` $\neq$ `null` and `head` $\neq$ `tail` and `head.next` $\neq$ `null`.

Implementation Queue



dequeue(S):

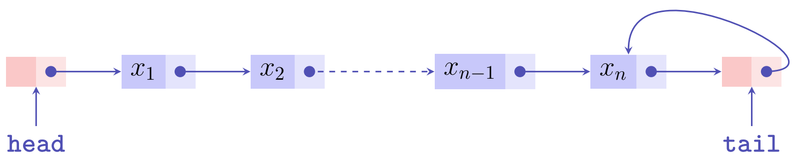
1. Store pointer to head in r. If $r =$ null, then return r .
2. Set the pointer of head to head.next.
3. Is now head $=$ null then set tail to null.
4. Return the value of r.

Analysis

Each of the operations enqueue, dequeue, head and isEmpty on the queue can be executed in $\mathcal{O}(1)$ steps.

Implementation Variants of Linked Lists

List with dummy elements (sentinels).

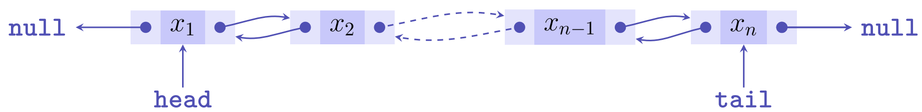


Advantage: less special cases

Variant: like this with pointer of an element stored singly indirect. (Example: pointer to x_3 points to x_2.)

Implementation Variants of Linked Lists

Doubly linked list



Overview

	enqueue	delete	search	concat
(A)	$\Theta(1)$	$\Theta(n)$	$\Theta(n)$	$\Theta(n)$
(B)	$\Theta(1)$	$\Theta(n)$	$\Theta(n)$	$\Theta(1)$
(C)	$\Theta(1)$	$\Theta(1)$	$\Theta(n)$	$\Theta(1)$
(D)	$\Theta(1)$	$\Theta(1)$	$\Theta(n)$	$\Theta(1)$

(A) = singly linked
(B) = Singly linked with dummy element at the beginning and the end
(C) = Singly linked with indirect element addressing
(D) = doubly linked

priority queue

Priority Queue

Operations

- `insert(x,p,Q)`: Enter object x with priority p.
- `extractMax(Q)`: Remove and return object x with highest priority.

Implementation Priority Queue

With a Max Heap

Thus

- `insert` in $\mathcal{O}(\log n)$ and
- `extractMax` in $\mathcal{O}(\log n)$.

Multistack

Multistack adds to the stack operations below

`multipop(s,S)`: remove the $\min(\text{size}(S), k)$ most recently inserted objects and return them.

Implementation as with the stack. Runtime of `multipop` is $\mathcal{O}(k)$.

Academic Question

If we execute on a stack with n elements a number of n times `multipop(k,S)` then this costs $\mathcal{O}(n^2)$?

Certainly correct because each `multipop` may take $\mathcal{O}(n)$ steps.

How to make a better estimation?

Idea (accounting)

Introduction of a cost model:

- Each call of `push` costs 1 CHF and additional 1 CHF will be put to account.
- Each call to `pop` costs 1 CHF and will be paid from the account.

Account will never have a negative balance. Thus: maximal costs = number of `push` operations times two.

More Formal

Let t_i denote the real costs of the operation i. Potential function $\Phi_i \geq 0$ for the "account balance" after i operations. $\Phi_i \geq \Phi_0 \; \forall i$.

Amortized costs of the ith operation:

$$a_i := t_i + \Phi_i - \Phi_{i-1}.$$

It holds

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} (t_i + \Phi_i - \Phi_{i-1}) = \left(\sum_{i=1}^{n} t_i \right) + \Phi_n - \Phi_0 \geq \sum_{i=1}^{n} t_i.$$

Goal: find potential function that evens out expensive operations.

Example stack

Potential function Φ_i = number element on the stack.

- `push`(x, S): real costs $t_i = 1$. $\Phi_i - \Phi_{i-1} = 1$. Amortized costs $a_i = 2$.
- `pop`(S): real costs $t_i = 1$. $\Phi_i - \Phi_{i-1} = -1$. Amortized costs $a_i = 0$.
- `multipop`(k, S): real costs $t_i = k$. $\Phi_i - \Phi_{i-1} = -k$. amortized costs $a_i = 0$.

All operations have *constant amortized cost*! Therefore, on average Multipop requires a constant amount of time. [14]

[14] Note that we are not talking about the probabilistic mean but the (worst-case) average of the costs.

Example Binary Counter

Binary counter with k bits. In the worst case for each count operation maximally k bitflips. Thus $\mathcal{O}(n \cdot k)$ bitflips for counting from 1 to n. Better estimation?

Real costs t_i = number bit flips from 0 to 1 plus number of bit-flips from 1 to 0.

$$...0\underbrace{1111111}_{l \text{ Einsen}}+1 = ...1\underbrace{0000000}_{l \text{ Zeroes}}.$$

$$\Rightarrow t_i = l + 1$$

Example Binary Counter

$$...0\underbrace{1111111}_{l \text{ Einsen}}+1 = ...1\underbrace{0000000}_{l \text{ Nullen}}$$

potential function Φ_i: number of 1-bits of x_i.

$$\Rightarrow \Phi_i - \Phi_{i-1} = 1 - l,$$
$$\Rightarrow a_i = t_i + \Phi_i - \Phi_{i-1} = l + 1 + (1 - l) = 2.$$

Amortized constant cost for each count operation. ☺

12. Dictionaries

Dictionary, Self-ordering List, Implementation of Dictionaries with Array / List /Skip lists. [Ottman/Widmayer, Kap. 3.3,1.7, Cormen et al, Kap. Problem 17-5]

Dictionary

ADT to manage keys from a set $\mathcal{K}$ with operations

- `insert`(k, D): Insert $k \in \mathcal{K}$ to the dictionary D. Already exists $\Rightarrow$ error messsage.
- `delete`(k, D): Delete k from the dictionary D. Not existing $\Rightarrow$ error message.
- `search`(k, D): Returns `true` if $k \in D$, otherwise `false`

Idea

Implement dictionary as sorted array

Worst case number of fundamental operations

Search	$\mathcal{O}(\log n)$
Insert	$\mathcal{O}(n)$
Delete	$\mathcal{O}(n)$

Other idea

Implement dictionary as a linked list

Worst case number of fundamental operations

Search	$\mathcal{O}(n)$
Insert	$\mathcal{O}(1)$[15]
Delete	$\mathcal{O}(n)$

[15] Provided that we do not have to check existence.

Self Ordered Lists

Problematic with the adoption of a linked list: linear search time

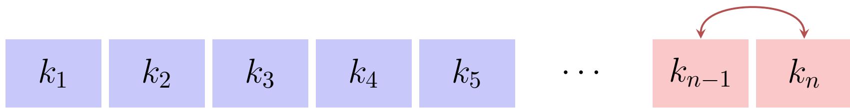
Idea: Try to order the list elements such that accesses over time are possible in a faster way

For example

- Transpose: For each access to a key, the key is moved one position closer to the front.
- Move-to-Front (MTF): For each access to a key, the key is moved to the front of the list.

Transpose

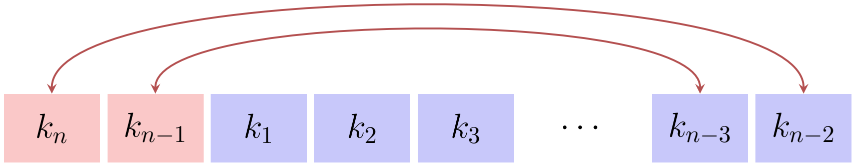
Transpose:



Worst case: Alternating sequence of n accesses to k_{n-1} and k_n.
Runtime: $\Theta(n^2)$

Move-to-Front

Move-to-Front:



Alternating sequence of n accesses to k_{n-1} and k_n. Runtime: $\Theta(n)$

Also here we can provide a sequence of accesses with quadratic runtime, e.g. access to the last element. But there is no obvious strategy to counteract much better than MTF..

Analysis

Compare MTF with the best-possible competitor (algorithm) A. How much better can A be?

Assumptions:

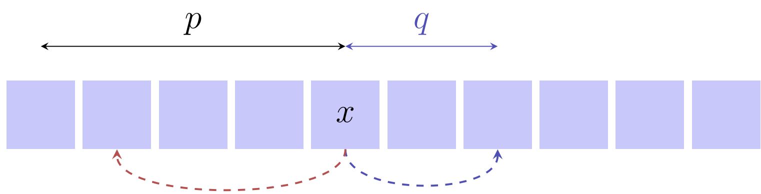
- MTF and A may only move the accessed element.
- MTF and A start with the same list.

Let M_k and A_k designate the lists after the kth step. $M_0 = A_0$.

Analysis

Costs:

- Access to x: position p of x in the list.
- No further costs, if x is moved before p
- Further costs q for each element that x is moved back starting from p.



Amortized Analysis

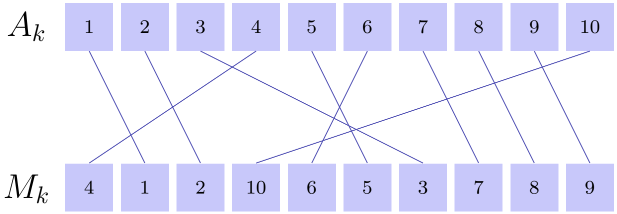
Let an arbitrary sequence of search requests be given and let $G_k^{(M)}$ and $G_k^{(A)}$ the costs in step k for Move-to-Front and A, respectively. Want estimation of $\sum_k G_k^{(M)}$ compared with $\sum_k G_k^{(A)}$.

$\Rightarrow$ Amortized analysis with potential function Φ.

Potential Function

Potential function Φ = Number of inversions of A vs. MTF.

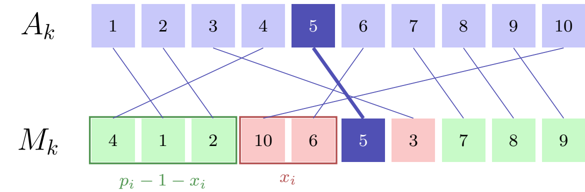
Inversion = Pair x, y such that for the positions of a and y
$\left(p^{(A)}(x) < p^{(A)}(y)\right) \neq \left(p^{(M)}(x) < p^{(M)}(y)\right)$

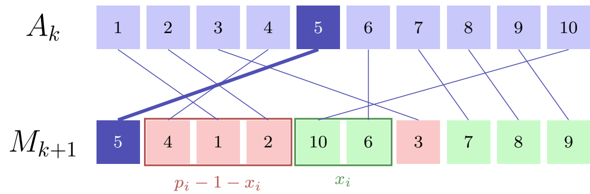


#inversion = #crossings

Estimating the Potential Function: MTF

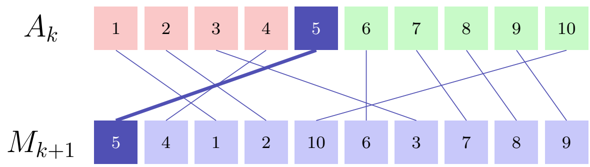
- Element i at position $p_i := p^{(M)}(i)$.
- access costs $C_k^{(M)} = p_i$.
- x_i: Number elements that are in M before p_i and in A after i .
- MTF removes x_i inversions.
- $p_i - x_i - 1$: Number elements that in M are before p_i and in A are before i.
- MTF generates $p_i - 1 - x_i$ inversions.

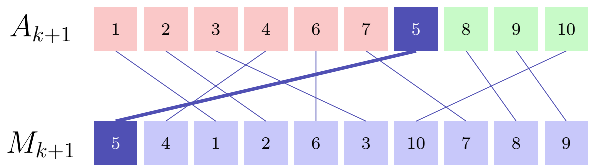




Estimating the Potential Function: A

- Wlog element i at position $p^{(A)}(i)$.
- $X_k^{(A)}$: number movements to the back (otherwise 0).
- access costs for i: $C_k^{(A)} = p^{(A)}(i) \geq p^{(M)}(i) - x_i$.
- A increases the number of inversions maximally by $X_k^{(A)}$.





Estimation

$$\Phi_{k+1} - \Phi_k \leq -x_i + (p_i - 1 - x_i) + X_k^{(A)}$$

Amortized costs of MTF in step k:

$$\begin{aligned}
a_k^{(M)} &= C_k^{(M)} + \Phi_{k+1} - \Phi_k \\
&\leq p_i - x_i + (p_i - 1 - x_i) + X_k^{(A)} \\
&= (p_i - x_i) + (p_i - x_i) - 1 + X_k^{(A)} \\
&\leq C_k^{(A)} + C_k^{(A)} - 1 + X_k^{(A)} \leq 2 \cdot C_k^{(A)} + X_k^{(A)}.
\end{aligned}$$

Estimation

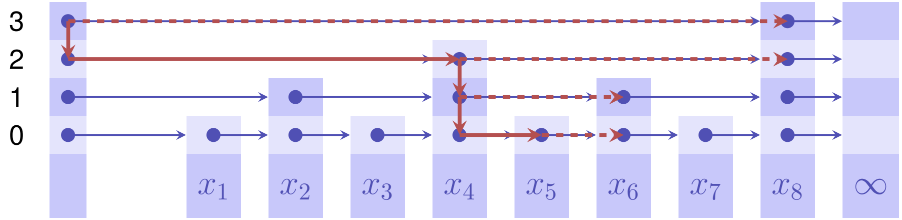
Summing up costs

$$\sum_k G_k^{(M)} = \sum_k C_k^{(M)} \le \sum_k a_k^{(M)} \le \sum_k 2 \cdot C_k^{(A)} + X_k^{(A)}$$
$$\le 2 \cdot \sum_k C_k^{(A)} + X_k^{(A)}$$
$$= 2 \cdot \sum_k G_k^{(A)}$$

In the worst case MTF requires at most twice as many operations as the optimal strategy.

Cool idea: skip lists

Perfect skip list



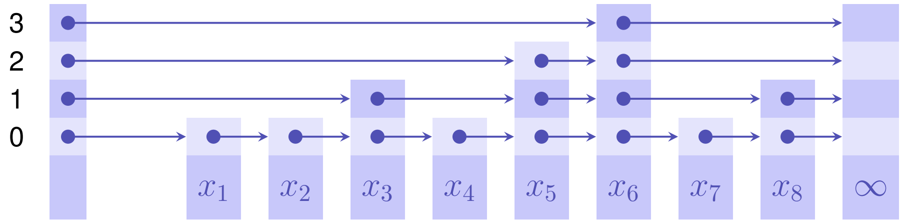
$x_1 \le x_2 \le x_3 \le \cdots \le x_9$.
Example: search for a key x with $x_5 < x < x_6$.

Analysis perfect skip list (worst cases)

Search in $\mathcal{O}(\log n)$. Insert in $\mathcal{O}(n)$.

Randomized Skip List

Idea: insert a key with random height H with $\mathbb{P}(H = i) = \frac{1}{2^{i+1}}$.



Analysis Randomized Skip List

Theorem

The expected number of fundamental operations for Search, Insert and Delete of an element in a randomized skip list is $\mathcal{O}(\log n)$.

The lengthy proof that will not be presented in this courseobserves the length of a path from a searched node back to the starting point in the highest level.

13. C++ advanced (III): Functors and Lambda

Functors: Motivation

A simple output filter

```
template <typename T, typename Function>
void filter(const T& collection, Function f){
    for (const auto& x: collection)
        if (f(x)) std::cout << x << " ";
    std::cout << "\n";
}
```


Functors: Motivation

```
template <typename T, typename Function>
void filter(const T& collection, Function f);

template <typename T>
bool even(T x){
%
    return x % 2 == 0;
}

std::vector<int> a {1,2,3,4,5,6,7,9,11,16,19};
filter(a,even<int>); // output: 2,4,6,16
```

Functor: object with overloaded operator ()

```
class GreaterThan{
  int value; // state
  public:
  GreaterThan(int x):value{x}{}

  bool operator() (int par) const {
    return par > value;
  }
};
```

Functor is a callable object. Can be understood as a stateful function.

```
std::vector<int> a {1,2,3,4,5,6,7,9,11,16,19};
int value=8;
filter(a,GreaterThan(value)); // 9,11,16,19
```


Functor: object with overloaded operator ()

```
template <typename T>
class GreaterThan{
    T value;
public:
    GreaterThan(T x):value{x}{}

    bool operator() (T par) const{
        return par > value;
    }
};
```

also works with a template, of course

```
std::vector<int> a {1,2,3,4,5,6,7,9,11,16,19};
int value=8;
filter(a,GreaterThan<int>(value)); // 9,11,16,19
```


The same with a Lambda-Expression

```
std::vector<int> a {1,2,3,4,5,6,7,9,11,16,19};
int value=8;

filter(a, [value](int x) {return x > value;} );
```


Sum of Elements – Old School

```
std::vector<int> a {1,2,3,4,5,6,7,9,11,16,19};
int sum = 0;
for (auto x: a)
  sum += x;
std::cout << sum << "\n"; // 83
```

Sum of Elements – with Functor

```
template <typename T>
struct Sum{
    T value = 0;

    void operator() (T par){ value += par; }
};

std::vector<int> a {1,2,3,4,5,6,7,9,11,16,19};
Sum<int> sum;
// for_each copies sum: we need to copy the result back
sum = std::for_each(a.begin(), a.end(), sum);
std::cout << sum.value << std::endl; // 83
```


Sum of Elements – with References[16]

```
template <typename T>
struct SumR{
    T& value;
    SumR (T& v):value{v} {}

    void operator() (T par){ value += par; }
};

std::vector<int> a {1,2,3,4,5,6,7,9,11,16,19};
int s=0;
SumR<int> sum{s};
// cannot (and do not need to) assign to sum here
std::for_each(a.begin(), a.end(), sum);
std::cout << s << std::endl; // 83
```

[16] Of course this works, very similarly, using pointers

Sum of Elements – with Λ

```
std::vector<int> a {1,2,3,4,5,6,7,9,11,16,19};
int s=0;

std::for_each(a.begin(), a.end(), [&s] (int x) {s += x;} );

std::cout << s << "\n";
```


Sorting, different

```
// pre: i >= 0
// post: returns sum of digits of i
int q(int i){
    int res =0;
    for(;i>0;i/=10)
        res += i % 10;
    return res;
}

std::vector<int> v {10,12,9,7,28,22,14};
std::sort (v.begin(), v.end(),
  [] (int i, int j) { return q(i) < q(j);}
);
```

Now $v = 10, 12, 22, 14, 7, 9, 28$ (sorted by sum of digits)

Lambda-Expressions in Detail

$$\underbrace{\texttt{[value]}}_{\text{capture}}\ \underbrace{\texttt{(int x)}}_{\text{parameters}}\ \underbrace{\texttt{->bool}}_{\text{return type}}\ \underbrace{\texttt{\{return x > value;\}}}_{\text{statement}}$$

Closure

```
[value] (int x) ->bool {return x > value;}
```

- Lambda expressions evaluate to a temporary object – a closure
- The closure retains the execution context of the function, the captured objects.
- Lambda expressions can be implemented as functors.

Simple Lambda Expression

```
[]()->void {std::cout << "Hello World";}
```

call:

```
[]()->void {std::cout << "Hello World";}();
```


Minimal Lambda Expression

```
[]{}
```

- Return type can be inferred if ≤ 1 return statement.[17]

  ```
  []() {std::cout << "Hello World";}
  ```

- If no parameters and no explcit return type, then () can be omitted.

  ```
  []{std::cout << "Hello World";}
  ```

- `[...]` can never be omitted.

[17] Since C++14 also several returns provided that the same return type is deduced

Examples

```
[](int x, int y) {std::cout << x * y;} (4,5);
```

Output: 20

Examples

```
int k = 8;
[](int& v) {v += v;} (k);
std::cout << k;
```

Output: 16

Examples

```
int k = 8;
[](int v) {v += v;} (k);
std::cout << k;
```

Output: 8

Capture – Lambdas

For Lambda-expressions the capture list determines the context accessible

Syntax:

- `[x]`: Access a copy of x (read-only)
- `[&x]`: Capture x by reference
- `[&x,y]`: Capture x by reference and y by value
- `[&]`: Default capture all objects by reference in the scope of the lambda expression
- `[=]`: Default capture all objects by value in the context of the Lambda-Expression

Capture – Lambdas

```
int elements=0;
int sum=0;
std::for_each(v.begin(), v.end(),
  [&] (int k) {sum += k; elements++;} // capture all by reference
)
```


Capture – Lambdas

```
template <typename T>
void sequence(vector<int> & v, T done){
  int i=0;
  while (!done()) v.push_back(i++);
}

vector<int> s;
sequence(s, [&] {return s.size() >= 5;} )
```

now v = 0 1 2 3 4

The capture list refers to the context of the lambda expression.

Capture – Lambdas

When is the value captured?

```
int v = 42;
auto func = [=] {std::cout << v << "\n"};
v = 7;
func();
```

Output: 42

Values are assigned when the lambda-expression is created.

Capture – Lambdas

(Why) does this work?

```
class Limited{
  int limit = 10;
 public:
  // count entries smaller than limit
  int count(const std::vector<int>& a){
    int c = 0;
    std::for_each(a.begin(), a.end(),
        [=,&c] (int x) {if (x < limit) c++;}
    );
    return c;
  }
};
```

The `this` pointer is implicitly copied by value

Capture – Lambdas

```
struct mutant{
  int i = 0;
  void do(){ [=] {i=42;}();}
};

mutant m;
m.do();
std::cout << m.i;
```

Output: 42

The `this` *pointer* is implicitly copied by value

Lambda Expressions are Functors

```
[x, &y] () {y = x;}
```

can be implemented as

```
unnamed {x,y};
```

with

```
class unnamed {
  int x; int& y;
  unnamed (int x_, int& y_) : x (x_), y (y_) {}
  void operator () () {y = x;}
};
```


Lambda Expressions are Functors

```
[=] () {return x + y;}
```

can be implemented as

```
unnamed {x,y};
```

with

```
class unnamed {
  int x; int y;
  unnamed (int x_, int y_) : x (x_), y (y_) {}
  int operator () () const {return x + y;}
};
```


Polymorphic Function Wrapper `std::function`

```
#include <functional>

int k= 8;
std::function<int(int)> f;
f = [k](int i){ return i+k; };
std::cout << f(8); // 16
```

Kann verwendet werden, um Lambda-Expressions zu speichern.

Other Examples
`std::function<int(int,int)>`;
`std::function<void(double)>` ...

http://en.cppreference.com/w/cpp/utility/functional/function

14. Hashing

Hash Tables, Birthday Paradoxon, Hash functions, Perfect and Universal Hashing, Resolving Collisions with Chaining, Open Addressing, Probing [Ottman/Widmayer, Kap. 4.1-4.3.2, 4.3.4, Cormen et al, Kap. 11-11.4]

Motivation

Gloal: Table of all n students of this course

Requirement: fast access by name

Naive Ideas

Mapping Name $s = s_1 s_2 \ldots s_{l_s}$ to key

$$k(s) = \sum_{i=1}^{l_s} s_i \cdot b^i$$

b large enough such that different names map to different keys.

Store each data set at its index in a huge array.

Example with $b = 100$. Ascii-Values s_i.

Anna $\mapsto 71111065$
Jacqueline $\mapsto 102110609021813999774$

Unrealistic: requires too large arrays.

Better idea?

Allocation of an array of size m ($m > n$).

Mapping Name s to

$$k_m(s) = \left(\sum_{i=1}^{l_s} s_i \cdot b^i \right) \bmod m.$$

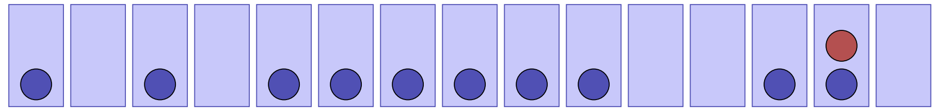
Different names can map to the same key (“Collision”). And then?

Estimation

Maybe collision do not really exist? We make an estimation ...

Abschätzung

Assumption: m urns, n balls (wlog $n \leq m$).
n balls are put uniformly distributed into the urns



What is the collision probability?

Very similar question: with how many people (n) the probability that two of them share the same birthday ($m = 365$) is larger than 50%?

Estimation

$\mathbb{P}(\text{no collision}) = \frac{m}{m} \cdot \frac{m-1}{m} \cdot \dots \cdot \frac{m-n+1}{m} = \frac{m!}{(m-n)! \cdot m^m}$.

Let $a \ll m$. With $e^x = 1 + x + \frac{x^2}{2!} + \dots$ approximate $1 - \frac{a}{m} \approx e^{-\frac{a}{m}}$.
This yields:

$$1 \cdot \left(1 - \frac{1}{m}\right) \cdot \left(1 - \frac{2}{m}\right) \cdot \ldots \cdot \left(1 - \frac{n-1}{m}\right) \approx e^{-\frac{1+\dots+n-1}{m}} = e^{-\frac{n(n-1)}{2m}}.$$

Thus

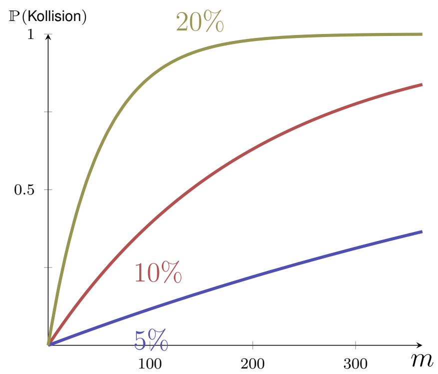
$$\mathbb{P}(\text{Kollision}) = 1 - e^{-\frac{n(n-1)}{2m}}.$$

Puzzle answer: with 23 people the probability for a birthday collision is 50.7%. Derived from the slightly more accurate Stirling formula.

With filling degree:

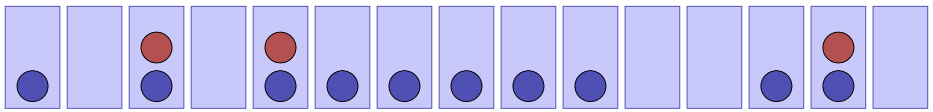
With filling degree $\alpha := n/m$ it holds that (simplified further)

$$\mathbb{P}(\text{collision}) \approx 1 - e^{-\alpha^2 \cdot \frac{m}{2}}.$$



Different Question

Assumption: m urns, n balls (wlog $n \leq m$).
n balls are put uniformly distributed into the urns



What is the expected number of collisions?

Expected Number Collisions

- $\mathbb{P}($Kugel B trifft Kugel $A_i) = 1/m$.
- $\mathbb{P}($Kugel B trifft Kugel A_i nicht$) = 1 - 1/m$.
- $\mathbb{P}(n-1$ Kugeln treffen A_i nicht$) = (1 - 1/m)^{n-1}$.
- $\mathbb{P}(A_i$ getroffen$) = 1 - (1 - 1/m)^{n-1}$.
- Sei X_i Zufallsvariable mit $X_i = \mathbb{1}_{A_i \text{getroffen}}$
- $\mathbb{E}(\sum X_i) = \sum \mathbb{E}(X_i)$
- $\mathbb{E}($Anzahl getroffene Kugeln$) = n(1 - (1 - 1/m)^n) \approx \frac{n^2}{2m}$.

Nomenclature

Hash funtion h: Mapping from the set of keys $\mathcal{K}$ to the index set $\{0, 1, \ldots, m-1\}$ of an array (*hash table*).

$$h : \mathcal{K} \to \{0, 1, \ldots, m-1\}.$$

Normally $|\mathcal{K}| \gg m$. There are $k_1, k_2 \in \mathcal{K}$ with $h(k_1) = h(k_2)$ (*collision*).

A hash function should map the set of keys as uniformly as possible to the hash table.

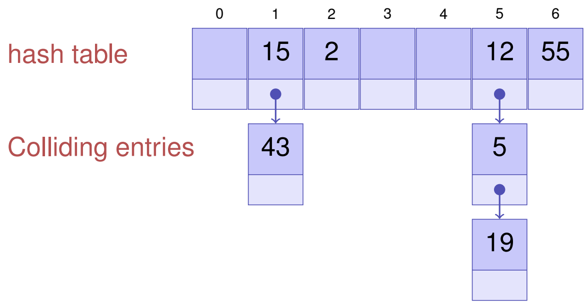
Examples of Good Hash Functions

- $h(k) = k \bmod m$, m prime
- $h(k) = \lfloor m(k \cdot r - \lfloor k \cdot r \rfloor) \rfloor$, r irrational, paritcularly good: $r = \frac{\sqrt{5}-1}{2}$.

Resolving Collisions

Example $m = 7$, $\mathcal{K} = \{0, \ldots, 500\}$, $h(k) = k \bmod m$.
Keys 12 , 55 , 5 , 15 , 2 , 19 , 43

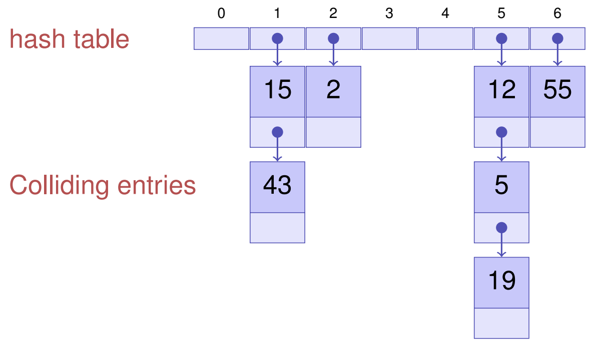
Chaining the Collisions



Resolving Collisions

Example $m = 7$, $\mathcal{K} = \{0, \ldots, 500\}$, $h(k) = k \bmod m$.
Keys 12 , 55 , 5 , 15 , 2 , 19 , 43

Direct Chaining of the Colliding entries



Algorithm for Hashing with Chaining

- `contains`(k) Search in list from position $h(k)$ for k. Return true if found, otherwise false.
- `put`(k) Check if k is in list at position $h(k)$. If no, then append k to the end of the list. Otherwise error message.
- `get`(k) Check if k is in list at position $h(k)$. If yes, return the data associated to key k, otherwise error message.
- `remove`(k) Search the list at position $h(k)$ for k. If successful, remove the list element.

Analysis (directly chained list)

1. Unsuccesful search. The average list lenght is $\alpha = \frac{n}{m}$. The list has to be traversed completely.
   $\Rightarrow$ Average number of entries considered
   $$C'_n = \alpha.$$
2. Successful search Consider the insertion history: key j sees an average list length of $(j-1)/m$.
   $\Rightarrow$ Average number of considered entries
   $$C_n = \frac{1}{n}\sum_{j=1}^{n}(1 + (j-1)/m)) = 1 + \frac{1}{n}\frac{n(n-1)}{2m} \approx 1 + \frac{\alpha}{2}.$$

Advantages and Disadvantages

Advantages

- Possible to overcommit: $\alpha > 1$
- Easy to remove keys.

Disadvantages

- Memory consumption of the chains-

Open Addressing

Store the colliding entries directly in the hash table using a *probing function* $s(j,k)$ ($0 \le j < m$, $k \in \mathcal{K}$)

Key table position along a *probing sequence*

$$S(k) := ((h(k) + s(0,k)) \bmod m, \dots, (h(k) + s(m-1,k)) \bmod m)$$

Algorithms for open addressing

- `contains`(k) Traverse table entries according to $S(k)$. If k is found, return true. If the probing sequence is finished or an empty position is reached, return false.
- `put`(k) Search for k in the table according to $S(k)$. If k is not present, insert k at the first free position in the probing sequence. Otherwise error message.
- `get`(k) Traverse table entries according to $S(k)$. If k is found, return data associated to k. Otherwise error message.
- `remove`(k) Search k in the table according to $S(k)$. If k is found, replace it with a special `removed` key.

Linear Probing

$s(j,k) = j \Rightarrow$
$S(k) = (h(k) \bmod m, (h(k)+1) \bmod m, \dots, (h(k)-1) \bmod m)$

Example $m = 7$, $\mathcal{K} = \{0, \dots, 500\}$, $h(k) = k \bmod m$.
Key 12 , 55 , 5 , 15 , 2 , 19

0	1	2	3	4	5	6
5	15	2	19		12	55

Analysis linear probing (without proof)

1. Unsuccessful search. Average number of considered entries

$$C'_n \approx \frac{1}{2}\left(1 + \frac{1}{(1-\alpha)^2}\right)$$

2. Successful search. Average number of considered entries

$$C_n \approx \frac{1}{2}\left(1 + \frac{1}{1-\alpha}\right).$$

Discussion

Example $\alpha = 0.95$

The unsuccessful search consideres 200 table entries on average!

? Disadvantage of the method?

! *Primary clustering:* similar hash addresses have similar probing sequences $\Rightarrow$ long contiguous areas of used entries.

Quadratic Probing

$s(j,k) = \lceil j/2 \rceil^2 (-1)^{j+1}$
$S(k) = (h(k), h(k)+1, h(k)-1, h(k)+4, h(k)-4, \dots) \mod m$

Example $m = 7$, $\mathcal{K} = \{0, \dots, 500\}$, $h(k) = k \mod m$.
Keys 12 , 55 , 5 , 15 , 2 , 19

0	1	2	3	4	5	6
19	15	2		5	12	55

Analysis Quadratic Probing (without Proof)

1. Unsuccessful search. Average number of entries considered

$$C'_n \approx \frac{1}{1-\alpha} - \alpha + \ln\left(\frac{1}{1-\alpha}\right)$$

2. Successful search. Average number of entries considered

$$C_n \approx 1 + \ln\left(\frac{1}{1-\alpha}\right) - \frac{\alpha}{2}.$$

Discussion

Example $\alpha = 0.95$

Unsuccessfuly search considers 22 entries on average

? Problems of this method?

! *Secondary clustering:* Synonyms k and k' (with $h(k) = h(k')$) travers the same probing sequence.

Double Hashing

Two hash functions $h(k)$ and $h'(k)$. $s(j,k) = j \cdot h'(k)$.
$S(k) = (h(k), h(k) + h'(k), h(k) + 2h'(k), \ldots, h(k) + (m-1)h'(k)) \mod m$

Example:
$m = 7$, $\mathcal{K} = \{0, \ldots, 500\}$, $h(k) = k \mod 7$, $h'(k) = 1 + k \mod 5$.
Keys 12 , 55 , 5 , 15 , 2 , 19

0	1	2	3	4	5	6
5	15	2	19		12	55

Double Hashing

- Probing sequence must permute all hash addresses. Thus $h'(k) \neq 0$ and $h'(k)$ may not divide m, for example guaranteed with m prime.
- h' should be independent of h (avoiding secondary clustering)

Independence:

$$\mathbb{P}\left((h(k) = h(k')) \wedge (h'(k) = h'(k'))\right) = \mathbb{P}\left(h(k) = h(k')\right) \cdot \mathbb{P}\left(h'(k) = h'(k')\right).$$

Independence fulfilled by $h(k) = k \mod m$ and $h'(k) = 1 + k \mod (m-2)$ (m prime).

Analysis Double Hashing

Let h and h' be independent, then:

1. Unsuccessful search. Average number of considered entries:

$$C'_n \approx \frac{1}{1-\alpha}$$

2. Successful search. Average number of considered entries:

$$C_n \approx \frac{1}{\alpha} \ln\left(\frac{1}{1-\alpha}\right)$$

Overview

	$\alpha = 0.50$		$\alpha = 0.90$		$\alpha = 0.95$	
	C_n	C'_n	C_n	C'_n	C_n	C'_n
Separate Chaining	1.250	1.110	1.450	1.307	1.475	1.337
Direct Chaining	1.250	0.500	1.450	0.900	1.475	0.950
Linear Probing	1.500	2.500	5.500	50.500	10.500	200.500
Quadratic Probing	1.440	2.190	2.850	11.400	3.520	22.050
Double Hashing	1.39	2.000	2.560	10.000	3.150	20.000

: C_n: Anzahl Schritte erfolgreiche Suche, C'_n: Anzahl Schritte erfolglose Suche, Belegungsgrad α.

Perfect Hashing

If the set of used keys is known up-front the hash function can be chosen perfectly, i.e. such that there are no collisions. The practical construction is non-trivial.

Example: table of key words of a compiler.

Universal Hashing

- $|\mathcal{K}| > m \Rightarrow$ Set of "similar keys" can be chose such that a large number of collisions occur.
- Impossible to select a "best" hash function for all cases.
- Possible, however[18]: randomize!

Universal hash class $\mathcal{H} \subseteq \{h : \mathcal{K} \to \{0, 1, \dots, m-1\}\}$ is a family of hash functions such that

$$\forall k_1 \neq k_2 \in \mathcal{K} : |\{h \in \mathcal{H} | h(k_1) = h(k_2)\}| \leq \frac{1}{m}|\mathcal{H}|.$$

[18]Similar as for quicksort

Universal Hashing

Theorem

A function h randomly chosen from a universal class $\mathcal{H}$ of hash functions randomly distributes an arbitrary sequence of keys from $\mathcal{K}$ as uniformly as possible on the available slots.

Universal Hashing

Initial remark for the proof of the theorem:

Define with $x, y \in \mathcal{K}$, $h \in \mathcal{H}$, $Y \subseteq \mathcal{K}$:

$$\delta(x,y,h) = \begin{cases} 1, & \text{if } h(x) = h(y), x \neq y \\ 0, & \text{otherwise,} \end{cases}$$

$$\delta(x,Y,h) = \sum_{y\in Y} \delta(x,y,h),$$

$$\delta(x,y,\mathcal{H}) = \sum_{h\in\mathcal{H}} \delta(x,y,h).$$

$\mathcal{H}$ is universal if for all $x, y \in \mathcal{K}$, $x \neq y : \delta(x,y,\mathcal{H}) \leq |\mathcal{H}|/m$.

Universal Hashing

Proof of the theorem

$S \subseteq \mathcal{K}$: keys stored up to now. x is added now:

$$\begin{aligned}\mathbb{E}_{\mathcal{H}}(\delta(x,S,h)) &= \sum_{h\in\mathcal{H}} \delta(x,S,h)/|\mathcal{H}| \\ &= \frac{1}{|\mathcal{H}|} \sum_{h\in\mathcal{H}} \sum_{y\in S} \delta(x,y,h) = \frac{1}{|\mathcal{H}|} \sum_{y\in S} \sum_{h\in\mathcal{H}} \delta(x,y,h) \\ &= \frac{1}{|\mathcal{H}|} \sum_{y\in S} \delta(x,y,\mathcal{H}) \\ &\leq \frac{1}{|\mathcal{H}|} \sum_{y\in S} |\mathcal{H}|/m = \frac{|S|}{m}.\end{aligned}$$

■

Universal Hashing is Relevant!

Let p be prime and $\mathcal{K} = \{0, \ldots, p-1\}$. With $a \in \mathcal{K} \setminus \{0\}$, $b \in \mathcal{K}$ define

$$h_{ab} : \mathcal{K} \to \{0, \ldots, m-1\}, h_{ab}(x) = ((ax+b) \bmod p) \bmod m.$$

Then the following theorem holds:

Theorem

The class $\mathcal{H} = \{h_{ab} | a, b \in \mathcal{K}, a \neq 0\}$ is a universal class of hash functions.

15. C++ advanced (IV): Exceptions

Some operations that can fail

- Opening files for reading and writing

```
std::ifstream input("myfile.txt");
```

- Parsing

```
int value = std::stoi("12−8");
```

- Memory allocation

```
std::vector<double> data(ManyMillions);
```

- Invalid data

```
int a = b/x; // what if x is zero?
```


Possibilities of Error Handling

- None (inacceptable)
- Global error variable (flags)
- Functions returning Error Codes
- Objects that keep error status
- Exceptions

Global error variables

- Common in older C-Code
- Concurrency is a problem.
- Error handling at good will. Requires extreme discipline, documentation and litters the code with seemingly unrelated checks.

Functions Returning Error Codes

- Every call to a function yields a result.
- Typical for large APIs (e.g. OS level). Often combined with global error code.[19]
- Caller can check the return value of a function in order to check the correct execution.

[19] Global error code thread-safety provided via thread-local storage.

Functions Returning Error Codes

Example

```
#include <errno.h>
...

pf = fopen ("notexisting.txt", "r+");
if (pf == NULL) {
  fprintf(stderr, "Error opening file: %s\n", strerror( errno ));
}
else { // ...
  fclose (pf);
}
```


Error state Stored in Object

- Error state of an object stored internally in the object.

Example

```
int i;
std::cin >> i;
if (std::cin.good()){// success, continue
        ...
}
```


Exceptions

- Exceptions break the normal control flow
- Exceptions can be thrown (throw) and catched (catch)
- Exceptions can become effective accross function boundaries.

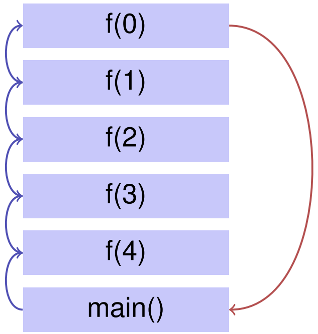
Example: throw exception

```
class MyException{};

void f(int i){
  if (i==0) throw MyException();
  f(i-1);
}

int main()
{
  f(4);
  return 0;
}
```

terminate called after throwing an instance of 'MyException'
Aborted

Example: catch exception

```cpp
class MyException{};

void f(int i){
  if (i==0) throw MyException();
  f(i-1);
}

int main(){
  try{
    f(4);
  }
  catch (MyException e){
     std::cout << "exception caught\n";
  }
}
```




```
exception caught
```


Resources get closed

```cpp
class MyException{};
struct SomeResource{
   ~SomeResource(){std::cout << "closed resource\n";}
};
void f(int i){
  if (i==0) throw MyException();
  SomeResource x;
  f(i-1);
}
int main(){
  try{f(5);}
  catch (MyException e){
     std::cout << "exception caught\n";
  }
}
```

```
closed resource
closed resource
closed resource
closed resource
closed resource
exception caught
```


When Exceptions?

Exceptions are used for *error handling* exclusively.

- Use `throw` only in order to identify an error that violates the post-condition of a function or that makes the continued execution of the code impossible in an other way.
- Use `catch` only when it is clear how to handle the error (potentially re-throwing the exception)
- Do *not* use `throw` in order to show a programming error or a violation of invariants, use `assert` instead.
- Do *not* use exceptions in order to change the control flow. Throw is *not* a better return.

Why Exceptions?

This:

```cpp
int ret = f();
if (ret == 0) {
  // ...
} else {
  // ...code that handles the error...
}
```

may look better than this on a first sight:

```cpp
try {
  f();
  // ...
} catch (std::exception& e) {
  // ...code that handles the error...
}
```

Why exceptions?

Truth is that toy examples do not necessarily hit the point.

Using return-codes for error handling either pollutes the code with checks or the error handling is not done right in the first place.

That's why

Example 1: Expression evaluation (expression parser from Introduction to programming), cf.
`http://codeboard.io/projects/46131`

Input: `1 + (3 * 6 / (/ 7 ))`
Error is deap in the recursion hierarchy. How to produce a meaningful error message (and continue execution)? Would have to pass error code over recursion steps.

Second Example

Value type with guarantee: values in range provided.

```
template <typename T, T min, T max>
class Range{
public:
  Range(){}
  Range (const T& v) : value (v) {
    if (value < min) throw Underflow ();
    if (value > max) throw Overflow ();
  }
  operator const T& () const {return value;}
private:
  T value;
};
```

Error handling in the constructor.

Types of Exceptions, Hierarchical

```
class RangeException {};
class Overflow : public RangeException {};
class Underflow : public RangeException {};
class DivisionByZero: public RangeException {};
class FormatError: public RangeException {};
```

Operators

```
template <typename T, T min, T max>
Range<T, min, max> operator/ (const Range<T, min, max>& a,
                              const Range<T, min, max>& b){
  if (b == 0) throw DivisionByZero();
  return T (a) * T(b);
}

template <typename T, T min, T max>
std::istream& operator >> (std::istream& is, Range<T, min, max>& a){
  T value;
  if (!(is >> value)) throw FormatError();
  a = value;
  return is;
}
```

Error handling in the operator.

Error handling (central)

```
Range<int,-10,10> a,b,c;
try{
  std::cin >> a;
  std::cin >> b;
  std::cin >> c;
  a = a / b + 4 * (b - c);
  std::cout << a;
}
catch(FormatError& e){ std::cout << "Format error\n"; }
catch(Underflow& e){ std::cout << "Underflow\n"; }
catch(Overflow& e){ std::cout << "Overflow\n"; }
catch(DivisionByZero& e){ std::cout << "Divison By Zero\n"; }
```


16. Binary Search Trees

[Ottman/Widmayer, Kap. 5.1, Cormen et al, Kap. 12.1 - 12.3]

Dictionary implementation

Hashing: implementation of dictionaries with expected very fast access times.

Disadvantages of hashing: linear access time in worst case. Some operations not supported at all:

- enumerate keys in increasing order
- next smallest key to given key

Trees

Trees are

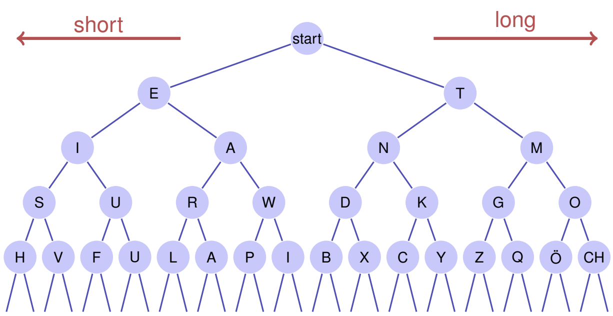
- Generalized lists: nodes can have more than one successor
- Special graphs: graphs consist of nodes and edges. A tree is a fully connected, directed, acyclic graph.

Trees

Use

- Decision trees: hierarchic representation of decision rules
- syntax trees: parsing and traversing of expressions, e.g. in a compiler
- Code tress: representation of a code, e.g. morse alphabet, huffman code
- Search trees: allow efficient searching for an element by value

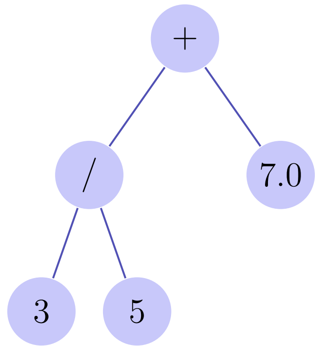
Examples



Morsealphabet

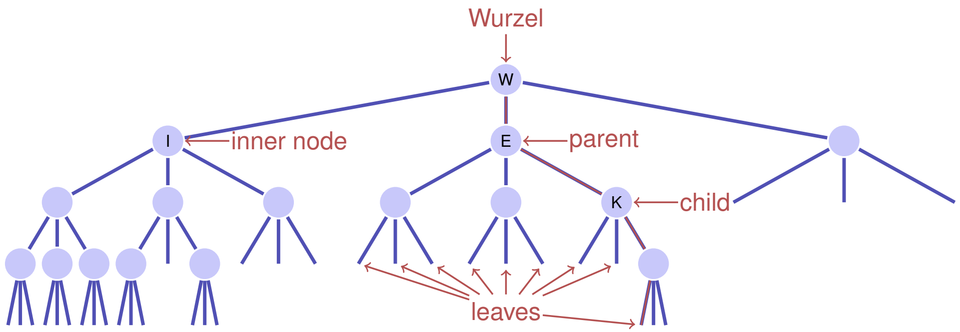
Examples

$3/5 + 7.0$



Expression tree

Nomenclature



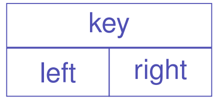
- Order of the tree: maximum number of child nodes, here: 3
- Height of the tree: maximum path length root – leaf (here: 4)

Binary Trees

A binary tree is either

- a leaf, i.e. an empty tree, or
- an inner leaf with two trees T_l (left subtree) and T_r (right subtree) as left and right successor.

In each node v we store

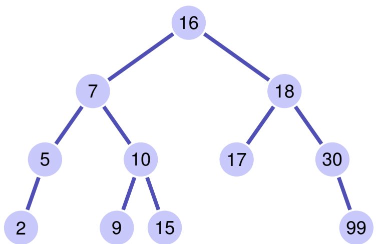


- a key $v.\mathrm{key}$ and
- two nodes $v.\mathrm{left}$ and $v.\mathrm{right}$ to the roots of the left and right subtree.
- a leaf is represented by the **null**-pointer

Binary search tree

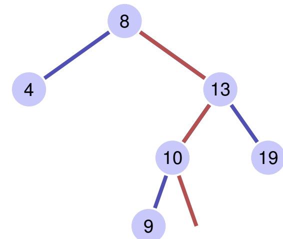
A binary search tree is a binary tree that fulfils the search tree property:

- Every node v stores a key
- Keys in the left subtree $v.\mathrm{left}$ of v are smaller than $v.\mathrm{key}$
- Key in the right subtree $v.\mathrm{right}$ of v are larger than $v.\mathrm{key}$



Searching

Input : Binary search tree with root r, key k
Output : Node v with $v.\mathrm{key} = k$ or **null**
$v \leftarrow r$
while $v \neq$ **null do**
    if $k = v.\mathrm{key}$ **then**
        return v
    else if $k < v.\mathrm{key}$ **then**
        $v \leftarrow v.\mathrm{left}$
    else
        $v \leftarrow v.\mathrm{right}$
return null



Search (12) $\rightarrow$ **null**

Height of a tree

The height $h(T)$ of a tree T with root r is given by

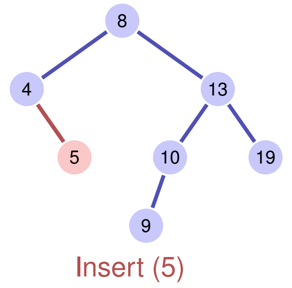
$$h(r) = \begin{cases} 0 & \text{if } r = \textbf{null} \\ 1 + \max\{h(r.\text{left}), h(r.\text{right})\} & \text{otherwise.} \end{cases}$$

The worst case run time of the search is thus $\mathcal{O}(h(T))$

Insertion of a key

Insertion of the key k

- Search for k
- If successful search: output error
- Of no success: insert the key at the leaf reached

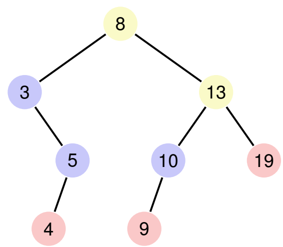


Remove node

Three cases possible:

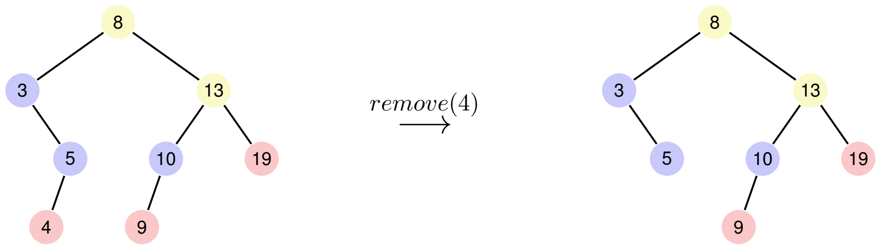
- Node has no children
- Node has one child
- Node has two children

[Leaves do not count here]



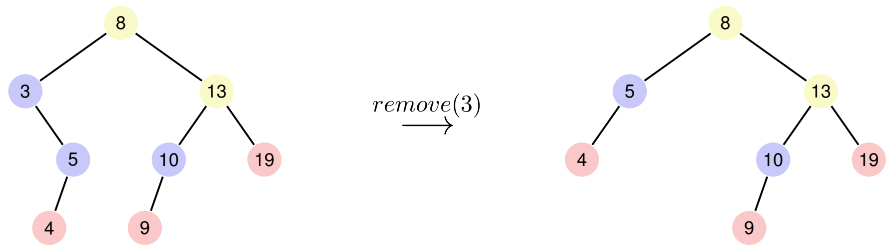
Remove node

Node has no children
Simple case: replace node by leaf.



Remove node

Node has one child
Also simple: replace node by single child.



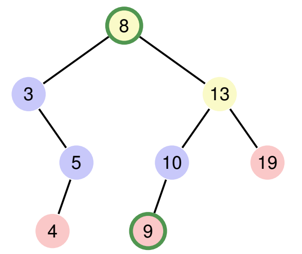
Remove node

Node has two children

The following observation helps: the smallest key in the right subtree $v.\text{right}$ (the *symmetric successor* of v)

- is smaller than all keys in $v.\text{right}$
- is greater than all keys in $v.\text{left}$
- and cannot have a left child.

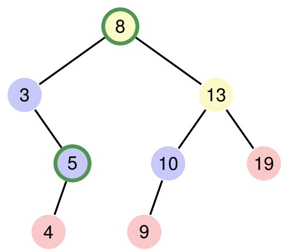
Solution: replace v by its symmetric successor.



By symmetry...

Node has two children

Also possible: replace v by its symmetric predecessor.



Algorithm SymmetricSuccessor(v)

Input : Node v of a binary search tree.
Output : Symmetric successor of v
$w \leftarrow v.\text{right}$
$x \leftarrow w.\text{left}$
while $x \neq$ **null do**
  $w \leftarrow x$
  $x \leftarrow x.\text{left}$
return w

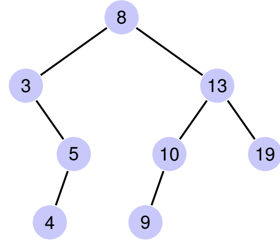
Analysis

Deletion of an element v from a tree T requires $\mathcal{O}(h(T))$ fundamental steps:

- Finding v has costs $\mathcal{O}(h(T))$
- If v has maximal one child unequal to **null**then removal takes $\mathcal{O}(1)$ steps
- Finding the symmetric successor n of v takes $\mathcal{O}(h(T))$ steps. Removal and insertion of n takes $\mathcal{O}(1)$ steps.

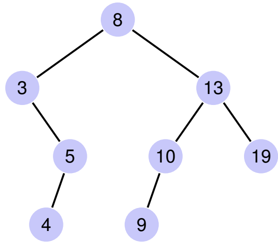
Traversal possibilities

- preorder: v, then $T_{\text{left}}(v)$, then $T_{\text{right}}(v)$.
  8, 3, 5, 4, 13, 10, 9, 19
- postorder: $T_{\text{left}}(v)$, then $T_{\text{right}}(v)$, then v.
  4, 5, 3, 9, 10, 19, 13, 8
- inorder: $T_{\text{left}}(v)$, then v, then $T_{\text{right}}(v)$.
  3, 4, 5, 8, 9, 10, 13, 19

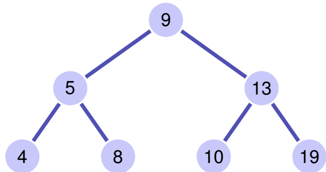


Further supported operations

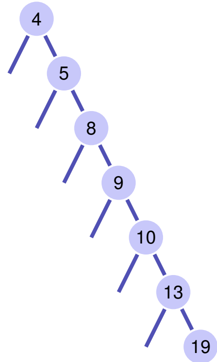
- Min(T): Read-out minimal value in $\mathcal{O}(h)$
- ExtractMin(T): Read-out and remove minimal value in $\mathcal{O}(h)$
- List(T): Output the sorted list of elements
- Join(T_1, T_2): Merge two trees with $\max(T_1) < \min(T_2)$ in $\mathcal{O}(n)$.



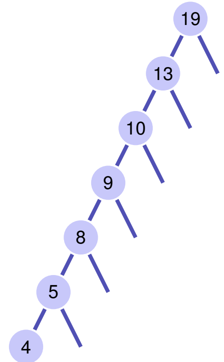
Degenerated search trees



Insert 9,5,13,4,8,10,19
ideally balanced



Insert 4,5,8,9,10,13,19
linear list



Insert 19,13,10,9,8,5,4
linear list

Probabilistically

A search tree constructed from a random sequence of numbers provides an an expected path length of $\mathcal{O}(\log n)$.

Attention: this only holds for insertions. If the tree is constructed by random insertions and deletions, the expected path length is $\mathcal{O}(\sqrt{n})$.

Balanced trees make sure (e.g. with *rotations*) during insertion or deletion that the tree stays balanced and provide a $\mathcal{O}(\log n)$ Worst-case guarantee.

17. AVL Trees

Balanced Trees [Ottman/Widmayer, Kap. 5.2-5.2.1, Cormen et al, Kap. Problem 13-3]

Objective

Searching, insertion and removal of a key in a tree generated from n keys inserted in random order takes expected number of steps $\mathcal{O}(\log_2 n)$.

But worst case $\Theta(n)$ (degenerated tree).

Goal: avoidance of degeneration. Artificial balancing of the tree for each update-operation of a tree.

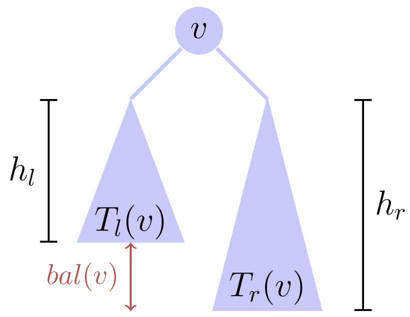
Balancing: guarantee that a tree with n nodes always has a height of $\mathcal{O}(\log n)$.

Adelson-Venskii and Landis (1962): AVL-Trees

Balance of a node

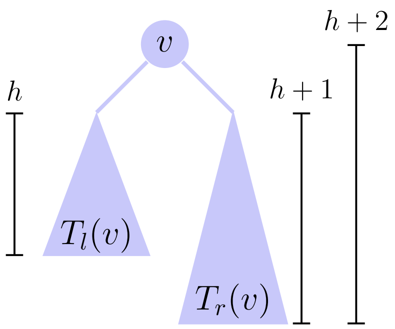
The height *balance* of a node v is defined as the height difference of its sub-trees $T_l(v)$ and $T_r(v)$

$$\mathrm{bal}(v) := h(T_r(v)) - h(T_l(v))$$

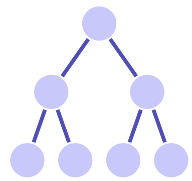


AVL Condition

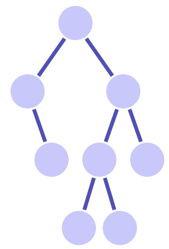
AVL Condition: for eacn node v of a tree $\mathrm{bal}(v) \in \{-1, 0, 1\}$



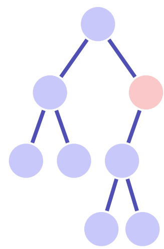
(Counter-)Examples



AVL tree with height 2



AVL tree with height 3

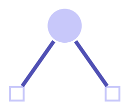


No AVL tree

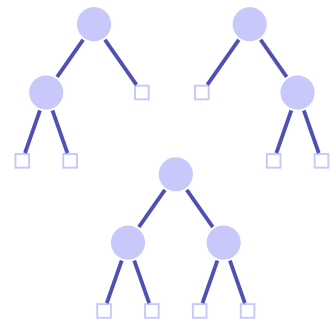
Number of Leaves

- 1. observation: a binary search tree with n keys provides exactly $n+1$ leaves. Simple induction argument.
- 2. observation: a lower bound of the number of leaves in a search tree with given height implies an upper bound of the height of a search tree with given number of keys.

Lower bound of the leaves



AVL tree with height 1 has $M(1) := 2$ leaves.



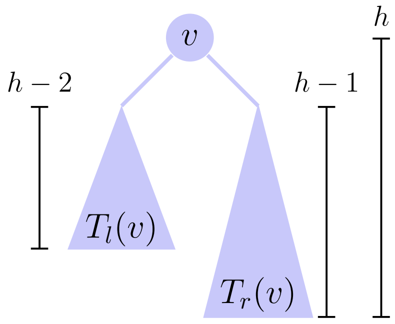
AVL tree with height 2 has at least $M(2) := 3$ leaves.

Lower bound of the leaves for $h > 2$

- Height of one subtree $\geq h - 1$.
- Height of the other subtree $\geq h - 2$.

Minimal number of leaves $M(h)$ is

$$M(h) = M(h-1) + M(h-2)$$



Overal we have $M(h) = F_{h+2}$ with *Fibonacci-numbers* $F_0 := 0$, $F_1 := 1$, $F_n := F_{n-1} + F_{n-2}$ for $n > 1$.

[Fibonacci Numbers: closed form]

Closed form of the Fibonacci numbers: computation via generation functions:

1. Power series approach

$$f(x) := \sum_{i=0}^{\infty} F_i \cdot x^i$$

[Fibonacci Numbers: closed form]

2. For Fibonacci Numbers it holds that $F_0 = 0$, $F_1 = 1$, $F_i = F_{i-1} + F_{i-2}\ \forall i > 1$. Therefore:

$$\begin{aligned}
f(x) &= x + \sum_{i=2}^{\infty} F_i \cdot x^i = x + \sum_{i=2}^{\infty} F_{i-1} \cdot x^i + \sum_{i=2}^{\infty} F_{i-2} \cdot x^i \\
&= x + x \sum_{i=2}^{\infty} F_{i-1} \cdot x^{i-1} + x^2 \sum_{i=2}^{\infty} F_{i-2} \cdot x^{i-2} \\
&= x + x \sum_{i=0}^{\infty} F_i \cdot x^i + x^2 \sum_{i=0}^{\infty} F_i \cdot x^i \\
&= x + x \cdot f(x) + x^2 \cdot f(x).
\end{aligned}$$

[Fibonacci Numbers: closed form]

3. Thus:

$$\begin{aligned}
& f(x) \cdot (1 - x - x^2) = x. \\
\Leftrightarrow\quad & f(x) = \frac{x}{1 - x - x^2} = -\frac{x}{x^2 + x - 1}
\end{aligned}$$

with the roots $-\phi$ and $-\hat{\phi}$ of $x^2 + x - 1$,

$$\phi = \frac{1 + \sqrt{5}}{2} \approx 1.6, \qquad \hat{\phi} = \frac{1 - \sqrt{5}}{2} \approx -0.6.$$

it holds that $\phi \cdot \hat{\phi} = -1$ and thus

$$f(x) = -\frac{x}{(x + \phi) \cdot (x + \hat{\phi})} = \frac{x}{(1 - \phi x) \cdot (1 - \hat{\phi} x)}$$

[Fibonacci Numbers: closed form]

4 It holds that:

$$(1-\hat{\phi}x)-(1-\phi x)=\sqrt{5}\cdot x.$$

Damit:

$$\begin{aligned}f(x)&=\frac{1}{\sqrt{5}}\frac{(1-\hat{\phi}x)-(1-\phi x)}{(1-\phi x)\cdot(1-\hat{\phi}x)}\\&=\frac{1}{\sqrt{5}}\left(\frac{1}{1-\phi x}-\frac{1}{1-\hat{\phi}x}\right)\end{aligned}$$

[Fibonacci Numbers: closed form]

5 Power series of $g_a(x)=\frac{1}{1-a\cdot x}$ $(a\in\mathbb{R})$:

$$\frac{1}{1-a\cdot x}=\sum_{i=0}^{\infty}a^i\cdot x^i.$$

E.g. Taylor series of $g_a(x)$ at $x=0$ or like this: Let $\sum_{i=0}^{\infty}G_i\cdot x^i$ a power series of g. By the identity $g_a(x)(1-a\cdot x)=1$ it holds that for all x (within the radius of convergence)

$$1=\sum_{i=0}^{\infty}G_i\cdot x^i-a\cdot\sum_{i=0}^{\infty}G_i\cdot x^{i+1}=G_0+\sum_{i=1}^{\infty}(G_i-a\cdot G_{i-1})\cdot x^i$$

For $x=0$ it follows $G_0=1$ and for $x\neq 0$ it follows then that $G_i=a\cdot G_{i-1}\Rightarrow G_i=a^i$.

[Fibonacci Numbers: closed form]

6 Fill in the power series:

$$\begin{aligned}f(x)&=\frac{1}{\sqrt{5}}\left(\frac{1}{1-\phi x}-\frac{1}{1-\hat{\phi}x}\right)=\frac{1}{\sqrt{5}}\left(\sum_{i=0}^{\infty}\phi^ix^i-\sum_{i=0}^{\infty}\hat{\phi}^ix^i\right)\\&=\sum_{i=0}^{\infty}\frac{1}{\sqrt{5}}(\phi^i-\hat{\phi}^i)x^i\end{aligned}$$

Comparison of the coefficients with $f(x)=\sum_{i=0}^{\infty}F_i\cdot x^i$ yields

$$F_i=\frac{1}{\sqrt{5}}(\phi^i-\hat{\phi}^i).$$

Fibonacci Numbers, Inductive Proof

It holds that $F_i=\frac{1}{\sqrt{5}}(\phi^i-\hat{\phi}^i)$ with roots ϕ, $\hat{\phi}$ of the equation $x^2=x+1$ (golden ratio), thus $\phi=\frac{1+\sqrt{5}}{2}$, $\hat{\phi}=\frac{1-\sqrt{5}}{2}$.

Proof (induction). Immediate for $i=0, i=1$. Let $i>2$:

$$\begin{aligned}F_i&=F_{i-1}+F_{i-2}=\frac{1}{\sqrt{5}}(\phi^{i-1}-\hat{\phi}^{i-1})+\frac{1}{\sqrt{5}}(\phi^{i-2}-\hat{\phi}^{i-2})\\&=\frac{1}{\sqrt{5}}(\phi^{i-1}+\phi^{i-2})-\frac{1}{\sqrt{5}}(\hat{\phi}^{i-1}+\hat{\phi}^{i-2})=\frac{1}{\sqrt{5}}\phi^{i-2}(\phi+1)-\frac{1}{\sqrt{5}}\hat{\phi}^{i-2}(\hat{\phi}+1)\\&=\frac{1}{\sqrt{5}}\phi^{i-2}(\phi^2)-\frac{1}{\sqrt{5}}\hat{\phi}^{i-2}(\hat{\phi}^2)=\frac{1}{\sqrt{5}}(\phi^i-\hat{\phi}^i).\end{aligned}$$

Tree Height

Because $\hat{\phi} < 1$, overal we have

$$M(h) \in \Theta\left(\left(\frac{1+\sqrt{5}}{2}\right)^h\right) \subseteq \Omega(1.618^h)$$

and thus

$$h \leq 1.44 \log_2 n + c.$$

AVL tree is asymptotically not more than 44% higher than a perfectly balanced tree.

Insertion

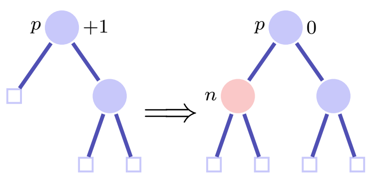
Balance

- Keep the balance stored in each node
- Re-balance the tree in each update-operation

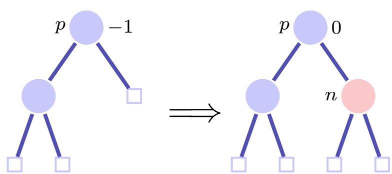
New node n is inserted:

- Insert the node as for a search tree.
- Check the balance condition increasing from n to the root.

Balance at Insertion Point



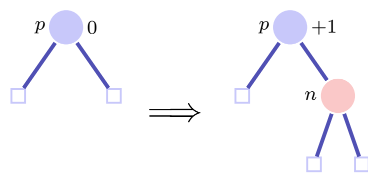
case 1: $\mathrm{bal}(p) = +1$



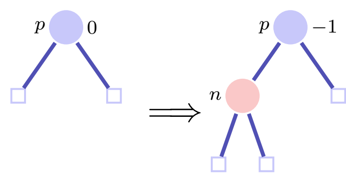
case 2: $\mathrm{bal}(p) = -1$

Finished in both cases because the subtree height did not change

Balance at Insertion Point



case 3.1: $\mathrm{bal}(p) = 0$ right



case 3.2: $\mathrm{bal}(p) = 0$, left

Not finished in both case. Call of `upin(p)`

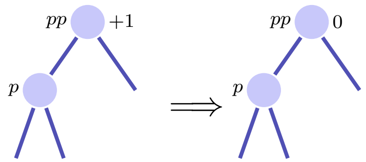
upin(p) - invariant

When `upin(p)` is called it holds that

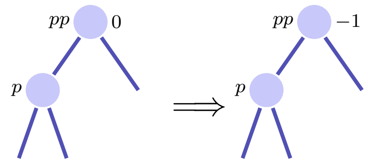
- the subtree from p is grown and
- $\mathrm{bal}(p) \in \{-1, +1\}$

upin(p)

Assumption: p is left son of pp[20]



case 1: $\mathrm{bal}(pp) = +1$, done.



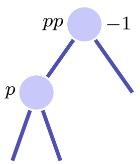
case 2: $\mathrm{bal}(pp) = 0$, `upin(pp)`

In both cases the AVL-Condition holds for the subtree from pp

[20] If p is a right son: symmetric cases with exchange of $+1$ and -1

upin(p)

Assumption: p is left son of pp



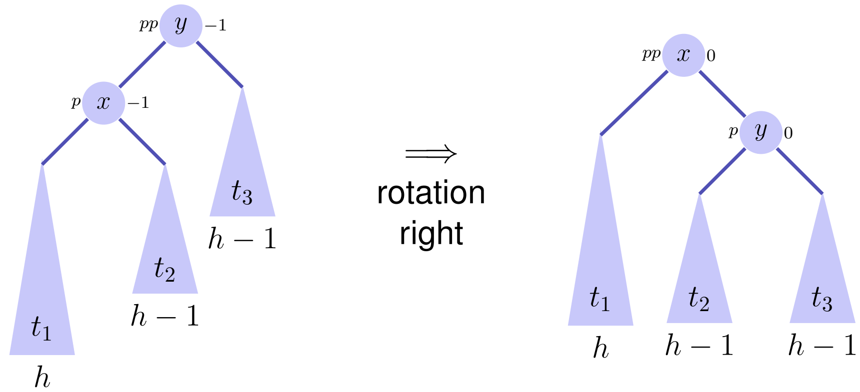
case 3: $\mathrm{bal}(pp) = -1$,

This case is problematic: adding n to the subtree from pp has violated the AVL-condition. Re-balance!

Two cases $\mathrm{bal}(p) = -1$, $\mathrm{bal}(p) = +1$

Rotationen

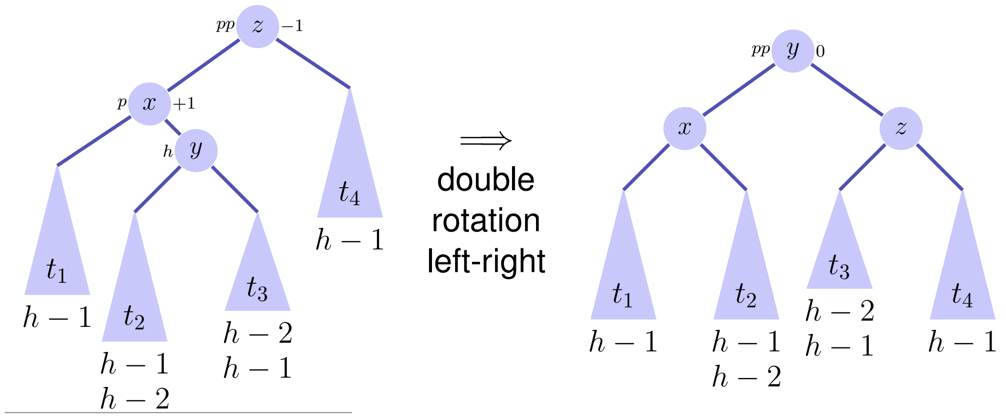
case 1.1 $\mathrm{bal}(p) = -1$. [21]



[21] p right son: $\mathrm{bal}(pp) = \mathrm{bal}(p) = +1$, left rotation

Rotationen

case 1.1 $\mathrm{bal}(p) = -1$. [22]



[22] p right son: $\mathrm{bal}(pp) = +1$, $\mathrm{bal}(p) = -1$, double rotation right left

Analysis

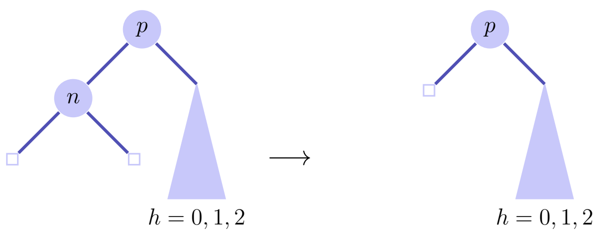
- Tree height: $\mathcal{O}(\log n)$.
- Insertion like in binary search tree.
- Balancing via recursion from node to the root. Maximal path lenght $\mathcal{O}(\log n)$.

Insertion in an AVL-tree provides run time costs of $\mathcal{O}(\log n)$.

Deletion

Case 1: Children of node n are both leaves Let p be parent node of n. $\Rightarrow$ Other subtree has height $h' = 0$, 1 or 2.

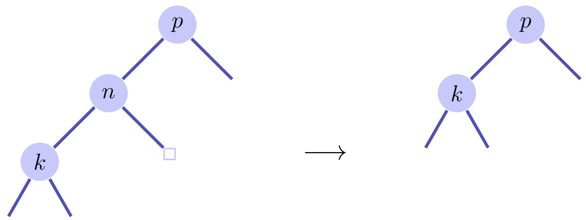
- $h' = 1$: Adapt $\mathrm{bal}(p)$.
- $h' = 0$: Adapt $\mathrm{bal}(p)$. Call `upout(p)`.
- $h' = 2$: Rebalanciere des Teilbaumes. Call `upout(p)`.



Deletion

Case 2: one child k of node n is an inner node

- Replace n by k. `upout(k)`



Deletion

Case 3: both children of node n are inner nodes

- Replace n by symmetric successor. `upout(k)`
- Deletion of the symmetric successor is as in case 1 or 2.

`upout(p)`

Let pp be the parent node of p.

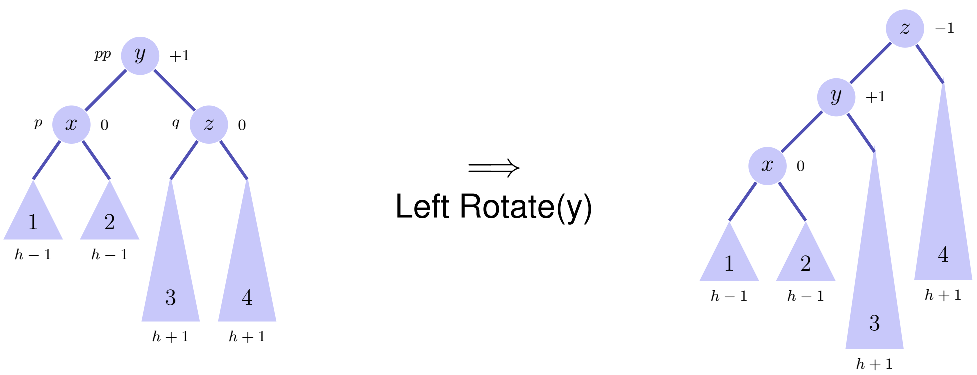
(a) p left child of pp

1. $\text{bal}(pp) = -1 \Rightarrow \text{bal}(pp) \leftarrow 0$. `upout(pp)`
2. $\text{bal}(pp) = 0 \Rightarrow \text{bal}(pp) \leftarrow +1$.
3. $\text{bal}(pp) = +1 \Rightarrow$ next slides.

(b) p right child of pp: Symmetric cases exchanging $+1$ and -1.

`upout(p)`

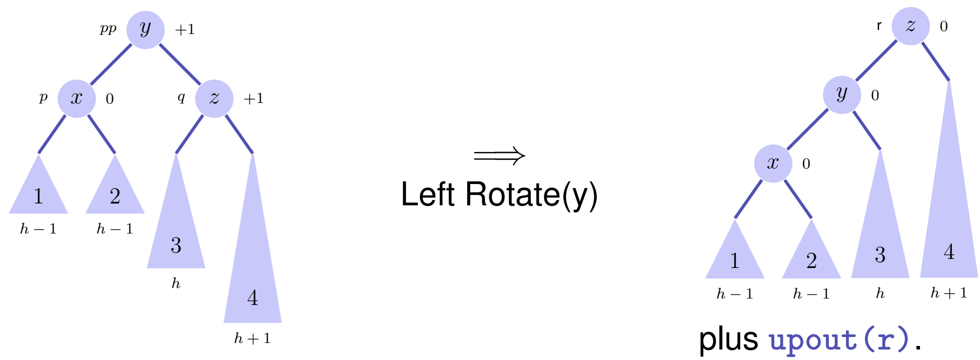
Case (a).3: $\text{bal}(pp) = +1$. Let q be brother of p
(a).3.1: $\text{bal}(q) = 0$.[23]



[23] (b).3.1: $\text{bal}(pp) = -1$, $\text{bal}(q) = -1$, Right rotation

`upout(p)`

Case (a).3: $\text{bal}(pp) = +1$. (a).3.2: $\text{bal}(q) = +1$.[24]

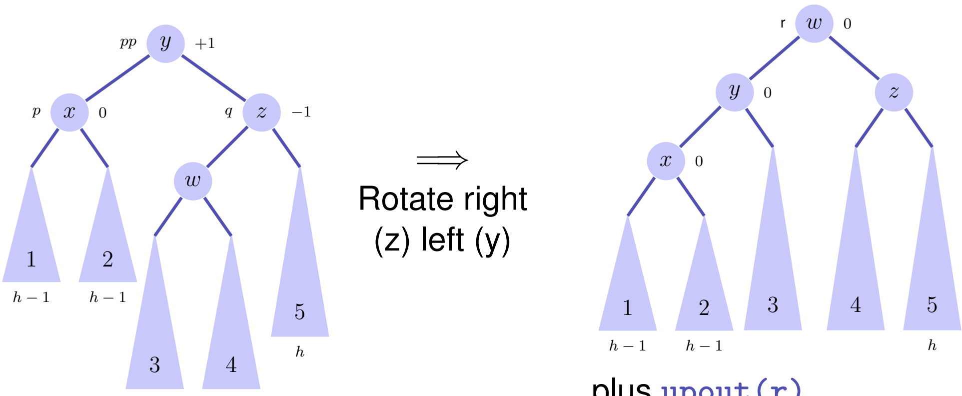


plus `upout(r)`.

[24] (b).3.2: $\text{bal}(pp) = -1$, $\text{bal}(q) = +1$, Right rotation+upout

`upout(p)`

Case (a).3: $\text{bal}(pp) = +1$. (a).3.3: $\text{bal}(q) = -1$.[25]



plus `upout(r)`.

[25] (b).3.3: $\text{bal}(pp) = -1$, $\text{bal}(q) = -1$, left-right rotation + upout

Conclusion

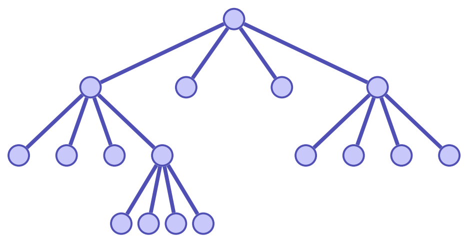
- AVL trees have worst-case asymptotic runtimes of $\mathcal{O}(\log n)$ for searching, insertion and deletion of keys.
- Insertion and deletion is relatively involved and an overkill for really small problems.

18. Quadtrees

Quadtrees, Collision Detection, Image Segmentation

Quadtree

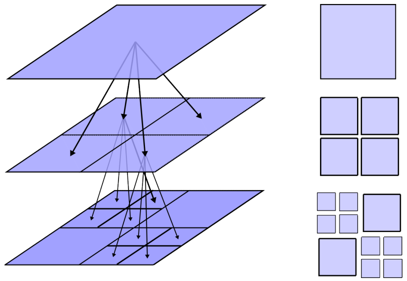
A quad tree is a tree of order 4.



... and as such it is not particularly interesting except when it is used for ...

Quadtree - Interpretation und Nutzen

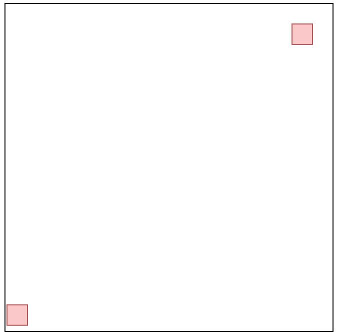
Separation of a two-dimensional range into 4 equally sized parts.



[analogously in three dimensions with an *octtree* (tree of order 8)]

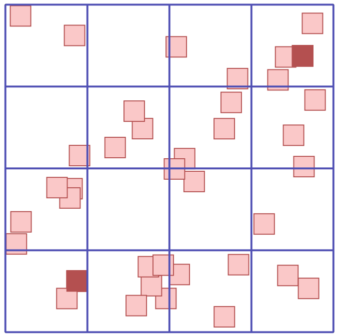
Example 1: Collision Detection

- Objects in the 2D-plane, e.g. particle simulation on the screen.
- Goal: collision detection



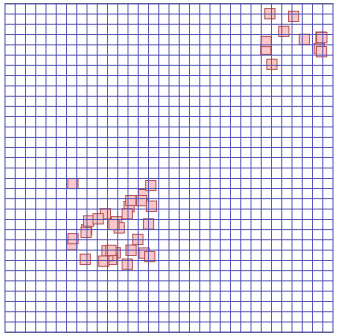
Idea

- Many objects: n^2 detections (naively)
- Improvement?
- Obviously: collision detection not required for objects far away from each other
- What is „far away”?
- Grid $(m \times m)$
- Collision detection per grid cell



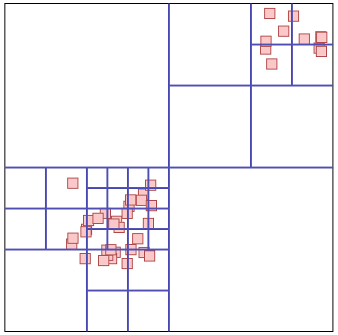
Grids

- A grid often helps, but not always
- Improvement?
- More finegrained grid?
- Too many grid cells!



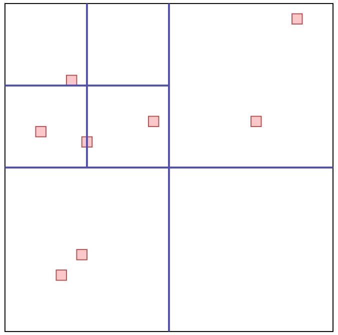
Adaptive Grids

- A grid often helps, but not always
- Improvement?
- *Adaptively* refine grid
- Quadtree!



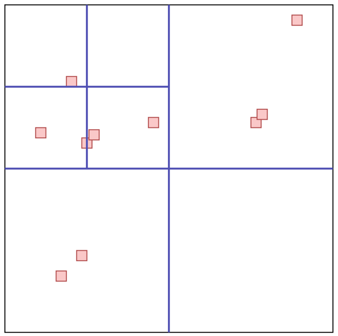
Algorithm: Insertion

- Quadtree starts with a single node
- Objects are added to the node. When a node contains too many objects, the node is split.
- Objects that are on the boundary of the quadtree remain in the higher level node.

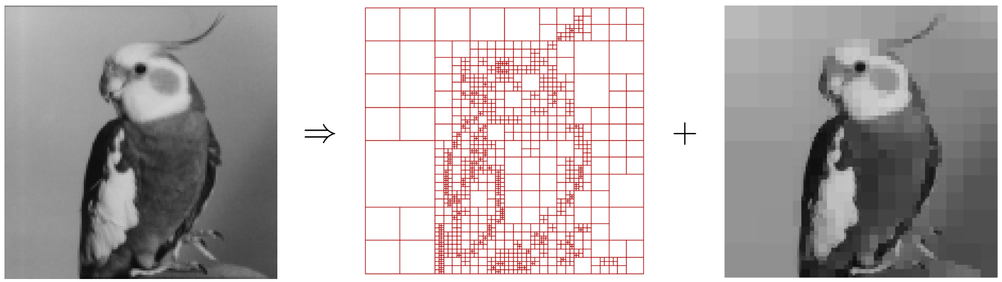


Algorithm: Collision Detection

- Run through the quadtree in a recursive way. For each node test collision with all objects contained in the same or (recursively) contained nodes.

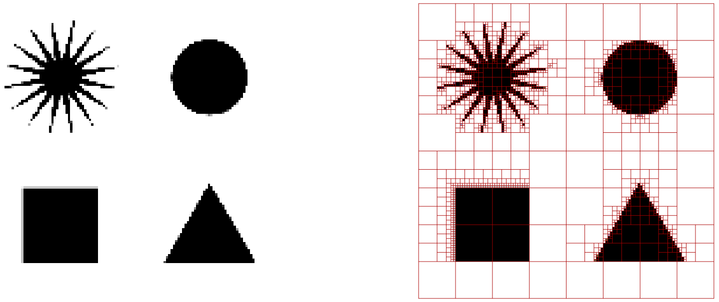


Example 2: Image Segmentation



(Possible applications: compression, denoising, edge detection)

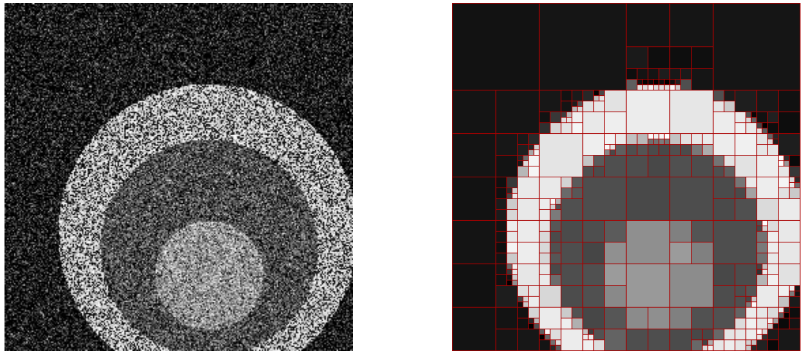
Quadtree on Monochrome Bitmap



Similar procedure to generate the quadtree: split nodes recursively until each node only contains pixels of the same color.

Quadtree with Approximation

When there are more than two color values, the quadtree can get very large. $\Rightarrow$ Compressed representation: *approximate* the image piecewise constant on the rectangles of a quadtree.



Piecewise Constant Approximation

(Grey-value) Image $z \in \mathbb{R}^S$ on pixel indices S. [26]

Rectangle $r \subset S$.

Goal: determine

$$\arg\min_{x \in r} \sum_{s \in r} (z_s - x)^2$$

Solution: the arithmetic mean $\mu_r = \frac{1}{|r|} \sum_{s \in r} z_s$

[26] we assume that S is a square with side length 2^k for some $k \geq 0$

Intermediate Result

The (w.r.t. mean squared error) best approximation

$$\mu_r = \frac{1}{|r|} \sum_{s \in r} z_s$$

and the corresponding error

$$\sum_{s \in r} (z_s - \mu_r)^2 =: \|z_r - \mu_r\|_2^2$$

can be computed quickly after a $\mathcal{O}(|S|)$ tabulation: prefix sums!

Which Quadtree?

Conflict

- *As close as possible to the data* $\Rightarrow$ small rectangles, large quadtree . Extreme case: one node per pixel. Approximation = original
- *Small amount of nodes* $\Rightarrow$ large rectangles, small quadtree Extreme case: a single rectangle. Approximation = a single grey value.

Which Quadtree?

Idea: choose between data fidelity and complexity with a regularisation parameter $\gamma \geq 0$

Choose quadtree T with leaves[27] $L(T)$ such that it minimizes the following function

$$H_\gamma(T, z) := \gamma \cdot \underbrace{|L(T)|}_{\text{Number of Leaves}} + \underbrace{\sum_{r \in L(T)} \|z_r - \mu_r\|_2^2}_{\text{Cummulative approximation error of all leaves}}.$$

[27] here: leaf: node with null-children

Regularisation

Let T be a quadtree over a rectangle S_T and let $T_{ll}, T_{lr}, T_{ul}, T_{ur}$ be the four possible sub-trees and

$$\widehat{H}_\gamma(T, z) := \min_T \gamma \cdot |L(T)| + \sum_{r \in L(T)} \|z_r - \mu_r\|_2^2$$

Extreme cases:
$\gamma = 0 \Rightarrow$ original data;
$\gamma \to \infty \Rightarrow$ a single rectangle

Observation: Recursion

- If the (sub-)quadtree T represents only one pixel, then it cannot be split and it holds that

$$\widehat{H}_\gamma(T, z) = \gamma$$

- Let, otherwise,

$$M_1 := \gamma + \|z_{S_T} - \mu_{S_T}\|_2^2$$
$$M_2 := \widehat{H}_\gamma(T_{ll}, z) + \widehat{H}_\gamma(T_{lr}, z) + \widehat{H}_\gamma(T_{ul}, z) + \widehat{H}_\gamma(T_{ur}, z)$$

then

$$\widehat{H}_\gamma(T, z) = \min\{\underbrace{M_1(T, \gamma, z)}_{\text{no split}}, \underbrace{M_2(T, \gamma, z)}_{\text{split}}\}$$

Algorithmus: Minimize(z,r,γ)

Input : Image data $z \in \mathbb{R}^S$, rectangle $r \subset S$, regularization $\gamma > 0$
Output : $\min_T \gamma |L(T)| + \|z - \mu_{L(T)}\|_2^2$

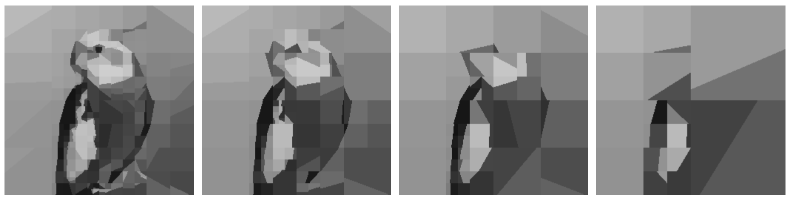
if $|r| = 0$ **then return** 0
$m \leftarrow \gamma + \sum_{s \in r} (z_s - \mu_r)^2$
if $|r| > 1$ **then**
  Split r into r_{ll},r_{lr},r_{ul},r_{ur}
  $m_1 \leftarrow \mathrm{Minimize}(z, r_{ll}, \gamma)$; $m_2 \leftarrow \mathrm{Minimize}(z, r_{lr}, \gamma)$
  $m_3 \leftarrow \mathrm{Minimize}(z, r_{ul}, \gamma)$; $m_4 \leftarrow \mathrm{Minimize}(z, r_{ur}, \gamma)$
  $m' \leftarrow m_1 + m_2 + m_3 + m_4$
else
  $m' \leftarrow \infty$
if $m' < m$ **then** $m \leftarrow m'$
return m

Analysis

The minimization algorithm over dyadic partitions (quadtrees) takes $\mathcal{O}(|S| \log |S|)$ steps.

Application: Denoising (with addditional Wedgelets)

noised — $\gamma = 0.003$ — $\gamma = 0.01$ — $\gamma = 0.03$ — $\gamma = 0.1$

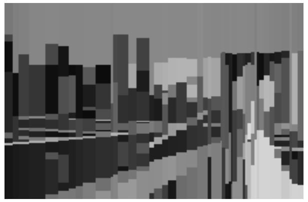
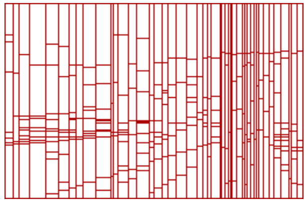


$\gamma = 0.3$ — $\gamma = 1$ — $\gamma = 3$ — $\gamma = 10$

Extensions: Affine Regression + Wedgelets

Other ideas

no quadtree: hierarchical one-dimensional modell (requires dynamic programming)

19. Dynamic Programming I

Fibonacci, Längste aufsteigende Teilfolge, längste gemeinsame Teilfolge, Editierdistanz, Matrixkettenmultiplikation, Matrixmultiplikation nach Strassen [Ottman/Widmayer, Kap. 1.2.3, 7.1, 7.4, Cormen et al, Kap. 15]

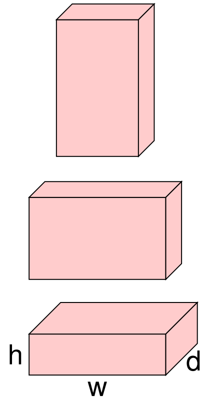
Quiz: Stacking Boxes

- Given: n boxes with sizes $w_i \times d_i \times h_i$
- Wanted: maximal height of a permitted stack
- Permitted stack: the base area of stacked boxes must become strictly smaller in both directions (width and depth)

Boxen Stapeln

We assume that there are enough boxes of a kind such that each box is available in all orientations (right hand side of the figure below).

Solution: later



Box	1	2	3	4	5	6
$[w \times d \times h]$	$[1 \times 2 \times 3]$	$[1 \times 3 \times 2]$	$[2 \times 3 \times 1]$	$[3 \times 4 \times 5]$	$[3 \times 5 \times 4]$	$[4 \times 5 \times 3]$

Simpler: Fibonacci Numbers

 (again)

$$F_n := \begin{cases} n & \text{if } n < 2 \\ F_{n-1} + F_{n-2} & \text{if } n \geq 2. \end{cases}$$

Analysis: why ist the recursive algorithm so slow?

Algorithm FibonacciRecursive(n)

Input : $n \geq 0$
Output : n-th Fibonacci number

if $n < 2$ **then**
  $f \leftarrow n$
else
  $f \leftarrow$ FibonacciRecursive$(n-1)$ + FibonacciRecursive$(n-2)$
return f

Analysis

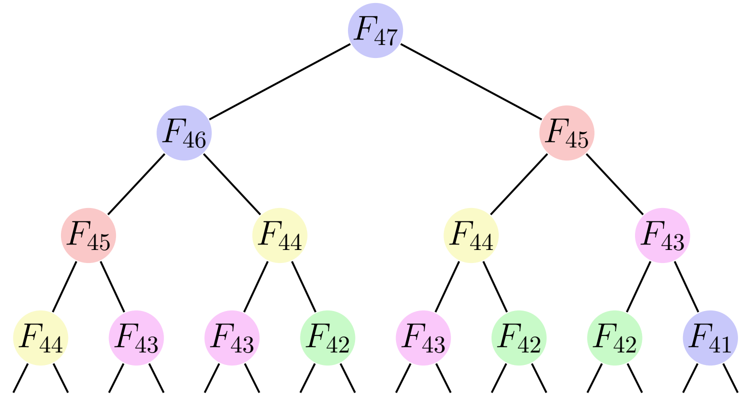
$T(n)$: Number executed operations.

- $n = 0, 1$: $T(n) = \Theta(1)$
- $n \geq 2$: $T(n) = T(n-2) + T(n-1) + c$.

$$T(n) = T(n-2) + T(n-1) + c \geq 2T(n-2) + c \geq 2^{n/2}c' = (\sqrt{2})^n c'$$

Algorithm is *exponential* in n.

Reason (visual)



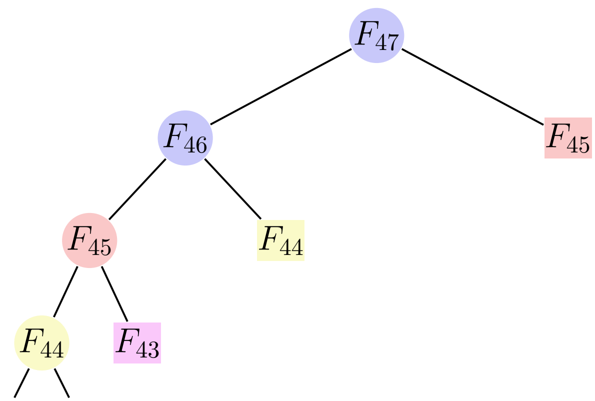
Nodes with same values are evaluated (too) often.

Memoization

Memoization (sic) saving intermediate results.

- Before a subproblem is solved, the existence of the corresponding intermediate result is checked.
- If an intermediate result exists then it is used.
- Otherwise the algorithm is executed and the result is saved accordingly.

Memoization with Fibonacci



Rechteckige Knoten wurden bereits ausgewertet.

Algorithm FibonacciMemoization(n)

Input : $n \geq 0$
Output : n-th Fibonacci number

if $n \leq 2$ **then**
  $f \leftarrow 1$
else if $\exists$memo$[n]$ **then**
  $f \leftarrow$ memo$[n]$
else
  $f \leftarrow$ FibonacciMemoization$(n-1)$ + FibonacciMemoization$(n-2)$
  memo$[n] \leftarrow f$
return f

Analysis

Computational complexity:

$$T(n) = T(n-1) + c = ... = \mathcal{O}(n).$$

Algorithm requires $\Theta(n)$ memory.[28]

[28] But the naive recursive algorithm also requires $\Theta(n)$ memory implicitly.

Looking closer ...

... the algorithm computes the values of F_1, F_2, F_3,... in the *top-down* approach of the recursion.

Can write the algorithm *bottom-up*. Then it is called *dynamic programming*.

Algorithm FibonacciDynamicProgram(n)

Input : $n \geq 0$
Output : n-th Fibonacci number

$F[1] \leftarrow 1$
$F[2] \leftarrow 1$
for $i \leftarrow 3, \ldots, n$ **do**
  $F[i] \leftarrow F[i-1] + F[i-2]$
return $F[n]$

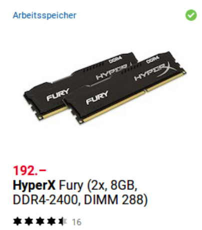
Dynamic Programming: Idea

- Divide a complex problem into a reasonable number of sub-problems
- The solution of the sub-problems will be used to solve the more complex problem
- Identical problems will be computed only once

Dynamic Programming Consequence

Identical problems will be computed only once

$\Rightarrow$ Results are saved



We trade spee against memory consumption

Dynamic Programming = Divide-And-Conquer ?

- In both cases the original problem can be solved (more easily) by utilizing the solutions of sub-problems. The problem provides *optimal substructure*.
- Divide-And-Conquer algorithms (such as Mergesort): sub-problems are independent; their solutions are required only once in the algorithm.
- DP: sub-problems are dependent. The problem is said to have *overlapping sub-problems* that are required multiple-times in the algorithm. In order to avoid redundant computations, results have to be tabulated.

Dynamic Programming: Procedure

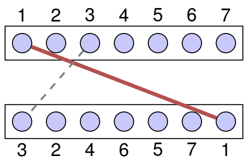
1. Use a *DP-table* with information to the subproblems.
   Dimension of the entries? Semantics of the entries?
2. Computation of the *base cases*
   Which entries do not depend on others?
3. Determine *computation order*.
   In which order can the entries be computed such that dependencies are fulfilled?
4. Read-out the *solution*
   How can the solution be read out from the table?

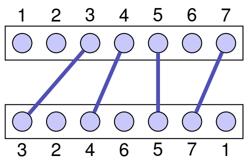
Runtime (typical) = number entries of the table times required operations per entry.

Dynamic Programing: Procedure with the example

1. **Dimension of the table? Semantics of the entries?**
   $n \times 1$ table. nth entry contains nth Fibonacci number.
2. **Which entries do not depend on other entries?**
   Values F_1 and F_2 can be computed easily and independently.
3. **What is the execution order such that required entries are always available?**
   F_i with increasing i.
4. **Wie kann sich Lösung aus der Tabelle konstruieren lassen?**
   F_n ist die n-te Fibonacci-Zahl.

Longest Ascending Sequence (LAS)

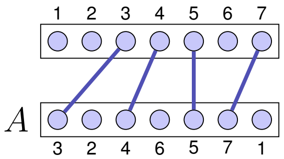




Connect as many as possible fitting ports without lines crossing.

Formally

- Consider Sequence $A = (a_1, \ldots, a_n)$.
- Search for a longest increasing subsequence of A.
- Examples of increasing subsequences: $(3, 4, 5)$, $(2, 4, 5, 7)$, $(3, 4, 5, 7)$, $(3, 7)$.



Generalization: allow any numbers, even with duplicates. But only strictly increasing subsequences are permitted. Example: $(2, 3, 3, 3, 5, 1)$ with increasing subsequence $(2, 3, 5)$.

First idea

Assumption: LAS L_k known for k Now want to compute L_{k+1} for $k + 1$.

If a_{k+1} fits to L_k, then $L_{k+1} = L_k \oplus a_{k+1}$

Counterexample $A_5 = (1, 2, 5, 3, 4)$. Let $A_3 = (1, 2, 5)$ with $L_3 = A$. Determine L_4 from L_3?

It does not work this way, we cannot infer L_{k+1} from L_k.

Second idea.

Assumption: a LAS L_j is known for each $j \leq k$. Now compute LAS L_{k+1} for $k + 1$.

Look at all fitting $L_{k+1} = L_j \oplus a_{k+1}$ ($j \leq k$) and choose a longest sequence.

Counterexample: $A_5 = (1, 2, 5, 3, 4)$. Let $A_4 = (1, 2, 5, 3)$ with $L_1 = (1)$, $L_2 = (1, 2)$, $L_3 = (1, 2, 5)$, $L_4 = (1, 2, 5)$. Determine L_5 from $L_1, \ldots, L_4$?

That does not work either: cannot infer L_{k+1} from only *an arbitrary solution* L_j. We need to consider all LAS. Too many.

Third approach

Assumption: the LAS L_j, *that ends with smallest element* is known for each of the lengths $1 \leq j \leq k$.

Consider all fitting $L_j \oplus a_{k+1}$ ($j \leq k$) and update the table of the LAS,that end with smallest possible element.

Example: $A = (1, 1000, 1001, 2, 3, 4, ..., 999)$

A	LAT
(1)	(1)
$(1, 1000)$	$(1), (1, 1000)$
$(1, 1000, 1001)$	$(1), (1, 1000), (1, 1000, 1001)$
$(1, 1000, 1001, 2)$	$(1), (1, 2), (1, 1000, 1001)$
$(1, 1000, 1001, 2, 3)$	$(1), (1, 2), (1, 2, 3)$

DP Table

- Idea: save the last element of the increasing sequence L_j at slot j.
- Example: 3 2 5 1 6 4
- Problem: Table does not contain the subsequence, only the last value.
- Solution: second table with the predecessors.

Index	1	2	3	4	5	6
Wert	3	2	5	1	6	4
Predecessor	$-\infty$	$-\infty$	2	$-\infty$	5	1

Index	0	1	2	3	4	...
$(L_j)_j$	-∞	1	4	6	∞	

Dynamic Programming Algorithm LAS

1 Table dimension? Semantics?

Two tables $T[0,\dots,n]$ and $V[1,\dots,n]$. Start with $T[0] \leftarrow -\infty$, $T[i] \leftarrow \infty \; \forall i > 1$

2 Computation of an entry

Entries in T sorted in ascending order. For each new entry a_{k+1} binary search for l, such that $T[l] < a_k < T[l+1]$. Set $T[l+1] \leftarrow a_{k+1}$. Set $V[k] = T[l]$.

Dynamic Programming algorithm LAS

3 Computation order

Traverse the list anc compute $T[k]$ and $V[k]$ with ascending k

4 How can the solution be determined from the table?

Search the largest l with $T[l] < \infty$. l is the last index of the LAS. Starting at l search for the index $i < l$ such that $V[l] = A[i]$, i is the predecessor of l. Repeat with $l \leftarrow i$ until $T[l] = -\infty$

Analysis

- Computation of the table:
  - Initialization: $\Theta(n)$ Operations
  - Computation of the kth entry: binary search on positions $\{1,\dots,k\}$ plus constant number of assignments.

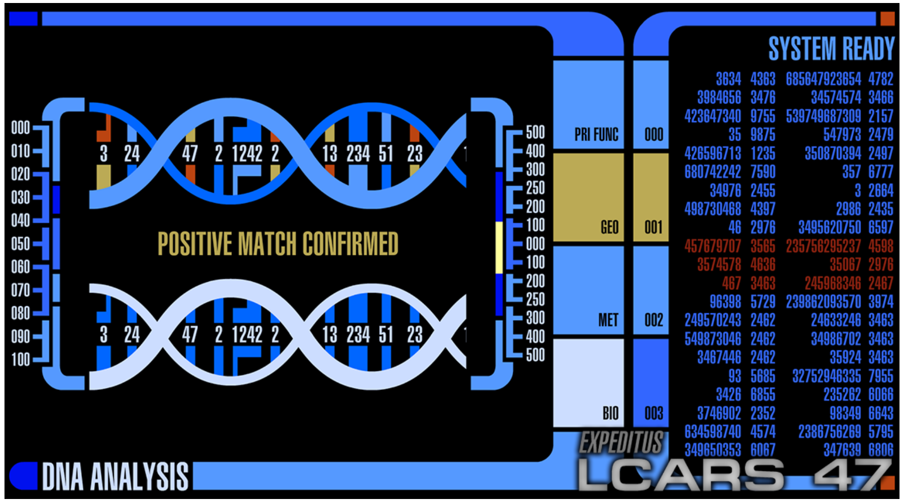
$$\sum_{k=1}^{n}(\log k + \mathcal{O}(1)) = \mathcal{O}(n) + \sum_{k=1}^{n}\log(k) = \Theta(n \log n).$$

- Reconstruction: traverse A from right to left: $\mathcal{O}(n)$.

Overal runtime:

$$\Theta(n \log n).$$

DNA - Comparison (Star Trek)



DNA - Comparison

- DNA consists of sequences of four different nucleotides **A**denine **G**uanine **T**hymine **C**ytosine
- DNA sequences (genes) thus can be described with strings of A, G, T and C.
- Possible comparison of two genes: determine the **longest common subsequence**

Longest common subsequence

Subsequences of a string:

Subsequences(KUH): (), (K), (U), (H), (KU), (KH), (UH), (KUH)

Problem:

- Input: two strings $A = (a_1, \ldots, a_m)$, $B = (b_1, \ldots, b_n)$ with lengths $m > 0$ and $n > 0$.
- Wanted: Longest common subsequecnes (LCS) of A and B.

Longest Common Subsequence

Examples:

LGT(IGEL,KATZE)=E, LGT(TIGER,ZIEGE)=IGE

Ideas to solve?

T $\boxed{\text{I}}$ $\boxed{\text{G}}$ $\boxed{\text{E}}$ R
Z $\boxed{\text{I}}$ E $\boxed{\text{G}}$ $\boxed{\text{E}}$

Recursive Procedure

Assumption: solutions $L(i,j)$ known for $A[1,\dots,i]$ and $B[1,\dots,j]$ for all $1 \le i \le m$ and $1 \le j \le n$, but not for $i = m$ and $j = n$.

T	$\boxed{\text{I}}$		$\boxed{\text{G}}$	$\boxed{\text{E}}$	R
Z	$\boxed{\text{I}}$	E	$\boxed{\text{G}}$	$\boxed{\text{E}}$	

Consider characters a_m, b_n. Three possibilities:

1. A is enlarged by one whitespace. $L(m,n) = L(m, n-1)$
2. B is enlarged by one whitespace. $L(m,n) = L(m-1, n)$
3. $L(m,n) = L(m-1, n-1) + \delta_{mn}$ with $\delta_{mn} = 1$ if $a_m = b_n$ and $\delta_{mn} = 0$ otherwise

Recursion

$$L(m,n) \leftarrow \max\left\{L(m-1,n-1) + \delta_{mn}, L(m,n-1), L(m-1,n)\right\}$$

for $m, n > 0$ and base cases $L(\cdot, 0) = 0$, $L(0, \cdot) = 0$.

	∅	Z	I	E	G	E
∅	0	0	0	0	0	0
T	0	0	0	0	0	0
I	0	0	1	1	1	1
G	0	0	1	1	2	2
E	0	0	1	2	2	3
R	0	0	1	2	2	3

Dynamic Programming algorithm LCS

1. **Dimension of the table? Semantics?**
   Table $L[0,\dots,m][0,\dots,n]$. $L[i,j]$: length of a LCS of the strings $(a_1,\dots,a_i)$ and $(b_1,\dots,b_j)$
2. **Computation of an entry**
   $L[0,i] \leftarrow 0 \; \forall 0 \le i \le m$, $L[j,0] \leftarrow 0 \; \forall 0 \le j \le n$. Computation of $L[i,j]$ otherwise via $L[i,j] = \max(L[i-1,j-1] + \delta_{ij}, L[i,j-1], L[i-1,j])$.

Dynamic Programming algorithm LCS

3. **Computation order**
   Rows increasing and within columns increasing (or the other way round).
4. **Reconstruct solution?**
   Start with $j = m$, $i = n$. If $a_i = b_j$ then output a_i and continue with $(j,i) \leftarrow (j-1, i-1)$; otherwise, if $L[i,j] = L[i,j-1]$ continue with $j \leftarrow j-1$ otherwise, if $L[i,j] = L[i-1,j]$ continue with $i \leftarrow i-1$. Terminate for $i = 0$ or $j = 0$.

Analysis LCS

- Number table entries: $(m + 1) \cdot (n + 1)$.
- Constant number of assignments and comparisons each. Number steps: $\mathcal{O}(mn)$
- Determination of solition: decrease i or j. Maximally $\mathcal{O}(n + m)$ steps.

Runtime overal:

$$\mathcal{O}(mn).$$

Editing Distance

Editing distance of two sequences $A = (a_1, \ldots, a_m)$, $B = (b_1, \ldots, b_m)$.

Editing operations:

- Insertion of a character
- Deletion of a character
- Replacement of a character

Question: how many editing operations at least required in order to transform string A into string B.

TIGER ZIGER ZIEGER ZIEGE

Editing Distance = Levenshtein Distance

Procedure?

- Two dimensional table $E[0, \ldots, m][0, \ldots, n]$ with editing distances $E[i, j]$ of strings $A_i = (a_1, \ldots, a_i)$ and $B_j = (b_1, \ldots, b_j)$.
- Consider the last characters of A_i and B_j. Three possible cases:
  1. Delete last character of A_i: [29] $E[i - 1, j] + 1$.
  2. Append character to A_i:[30] $E[i, j - 1] + 1$.
  3. Replace A_i by B_j: $E[i - 1, j - 1] + 1 - \delta_{ij}$.

$$E[i, j] \leftarrow \min \left\{E[i-1, j]+1, E[i, j-1]+1, E[i-1, j-1]+1-\delta_{ij}\right\}$$

[29] or append character to B_j

[30] or delete last character of B_j

DP Table

$$E[i, j] \leftarrow \min \left\{E[i-1, j]+1, E[i, j-1]+1, E[i-1, j-1]+1-\delta_{ij}\right\}$$

	∅	Z	I	E	G	E
∅	0	1	2	3	4	5
T	1	1	2	3	4	5
I	2	2	1	2	3	4
G	3	3	2	2	2	3
E	4	4	3	2	3	2
R	5	5	4	3	3	3

Algorithm: exercise

Matrix-Chain-Multiplication

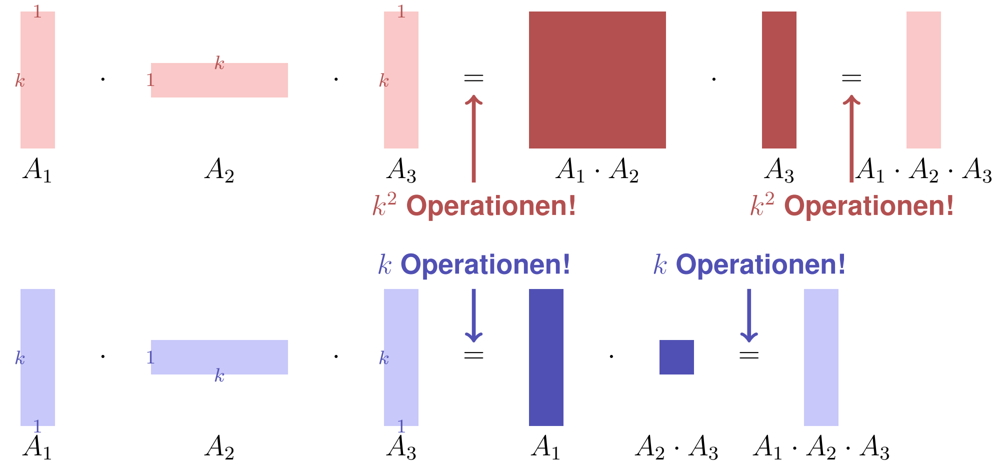
Task: Computation of the product $A_1 \cdot A_2 \cdot ... \cdot A_n$ of matrices $A_1, \dots, A_n$.

Matrix multiplication is associative, i.e. the order of evalution can be chosen arbitrarily

Goal: efficient computation of the product.

Assumption: multiplicaiton of an $(r \times s)$-matrix with an $(s \times u)$-matrix provides costs $r \cdot s \cdot u$.

Does it matter?



Recursion

- Assume that the best possible computation of $(A_1 \cdot A_2 \cdots A_i)$ and $(A_{i+1} \cdot A_{i+2} \cdots A_n)$ is known for each i.
- Compute best i, done.

$n \times n$-table M. entry $M[p, q]$ provides costs of the best possible bracketing $(A_p \cdot A_{p+1} \cdots A_q)$.

$$M[p, q] \leftarrow \min_{p \leq i < q} \left(M[p, i] + M[i+1, q] + \text{costs of the last multiplication}\right)$$

Computation of the DP-table

- Base cases $M[p, p] \leftarrow 0$ for all $1 \leq p \leq n$.
- Computation of $M[p, q]$ depends on $M[i, j]$ with $p \leq i \leq j \leq q$, $(i, j) \neq (p, q)$.
  In particular $M[p, q]$ depends at most from entries $M[i, j]$ with $i - j < q - p$.
  Consequence: fill the table from the diagonal.

Analysis

DP-table has n^2 entries. Computation of an entry requires considering up to $n-1$ other entries.

Overal runtime $\mathcal{O}(n^3)$.

Readout the order from M: exercise!

Digression: matrix multiplication

Consider the mutliplicaiton of two $n \times n$ matrices.

Let

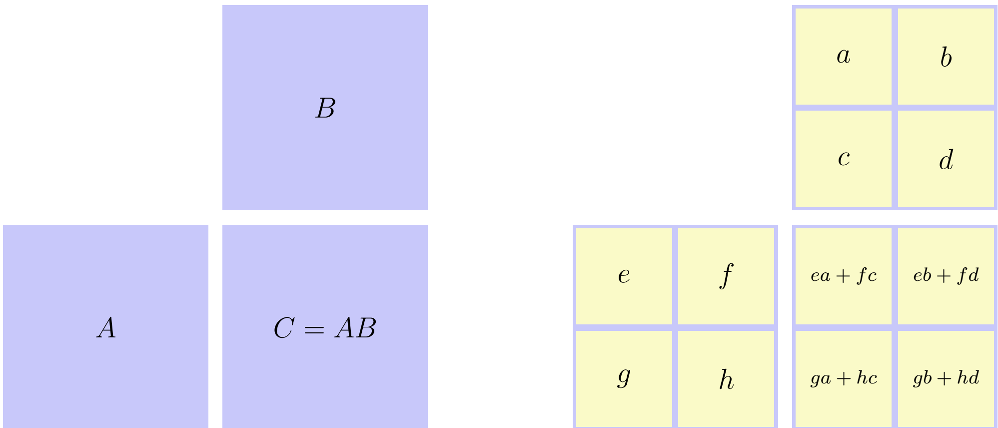
$$A = (a_{ij})_{1\le i,j\le n}, B = (b_{ij})_{1\le i,j\le n}, C = (c_{ij})_{1\le i,j\le n},$$
$$C = A \cdot B$$

then

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

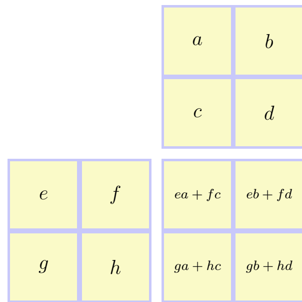
Naive algorithm requires $\Theta(n^3)$ elementary multiplications.

Divide and Conquer



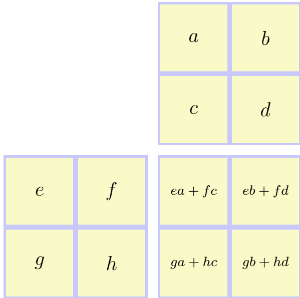
Divide and Conquer

- Assumption $n = 2^k$.
- Number of elementary multiplications: $M(n) = 8M(n/2)$, $M(1) = 1$.
- yields $M(n) = 8^{\log_2 n} = n^{\log_2 8} = n^3$. No advantage ☹



Strassen's Matrix Multiplication

- Nontrivial observation by Strassen (1969):
  It suffices to compute the seven products
  $A = (e+h)\cdot(a+d)$, $B = (g+h)\cdot a$,
  $C = e\cdot(b-d)$, $D = h\cdot(c-a)$, $E = (e+f)\cdot d$,
  $F = (g-e)\cdot(a+b)$, $G = (f-h)\cdot(c+d)$. Denn:
  $ea + fc = A + D - E + G$, $eb + fd = C + E$,
  $ga + hc = B + D$, $gb + hd = A - B + C + F$.
- This yields $M'(n) = 7M(n/2)$, $M'(1) = 1$.
  Thus $M'(n) = 7^{\log_2 n} = n^{\log_2 7} \approx n^{2.807}$.
- Fastest currently known algorithm:
  $\mathcal{O}(n^{2.37})$



20. Dynamic Programming II

Subset sum problem, knapsack problem, greedy algorithm vs dynamic programming [Ottman/Widmayer, Kap. 7.2, 7.3, 5.7, Cormen et al, Kap. 15,35.5]

Quiz Solution

- $n \times n$ Table
- Entry at row i and column j: height of highest possible stack formed from maximally i boxes and basement box j.

$[w \times d]$	$[1 \times 2]$	$[1 \times 3]$	$[2 \times 3]$	$[3 \times 4]$	$[3 \times 5]$	$[4 \times 5]$
h	3	2	1	5	4	3
1	$\underline{3}$	2	1	5	4	3
2	3	2	$\underline{4}$	8	8	8
3	3	2	4	$\underline{9}$	8	11
4	3	2	4	9	8	$\underline{12}$

Determination of the table: $\Theta(n^3)$, for each entry all entries in the row above must be considered. Computation of the optimal solution by traversing back, worst case $\Theta(n^2)$

Quiz Alternative Solution

- $1 \times n$ Table, topologically sorted[31] according to half-order stackability
- Entry at index j: height of highest possible stack with basement box j.

$[w \times d]$	$[1 \times 2]$	$[1 \times 3]$	$[2 \times 3]$	$[3 \times 4]$	$[3 \times 5]$	$[4 \times 5]$
h	3	2	1	5	4	3
	3	2	4	9	8	12

Topological sort in $\Theta(n^2)$. Traverse from left to right in $\Theta(n)$, overal $\Theta(n^2)$. Traversing back also $\Theta(n^2)$

[31] explanation soon

Task

Partition the set of the "item" above into two set such that both sets have the same value.

A solution:

Subset Sum Problem

Consider $n \in \mathbb{N}$ numbers $a_1, \ldots, a_n \in \mathbb{N}$.

Goal: decide if a selection $I \subseteq \{1, \ldots, n\}$ exists such that

$$\sum_{i \in I} a_i = \sum_{i \in \{1,\ldots,n\} \setminus I} a_i.$$

Naive Algorithm

Check for each bit vector $b = (b_1, \ldots, b_n) \in \{0,1\}^n$, if

$$\sum_{i=1}^{n} b_i a_i \stackrel{?}{=} \sum_{i=1}^{n} (1 - b_i) a_i$$

Worst case: n steps for each of the 2^n bit vectors b. Number of steps: $\mathcal{O}(n \cdot 2^n)$.

Algorithm with Partition

- Partition the input into two equally sized parts $a_1, \ldots, a_{n/2}$ and $a_{n/2+1}, \ldots, a_n$.
- Iterate over all subsets of the two parts and compute partial sum $S_1^k, \ldots, S_{2^{n/2}}^k$ $(k = 1, 2)$.
- Sort the partial sums: $S_1^k \le S_2^k \le \cdots \le S_{2^{n/2}}^k$.
- Check if there are partial sums such that $S_i^1 + S_j^2 = \frac{1}{2} \sum_{i=1}^{n} a_i =: h$
    - Start with $i = 1, j = 2^{n/2}$.
    - If $S_i^1 + S_j^2 = h$ then finished
    - If $S_i^1 + S_j^2 > h$ then $j \leftarrow j - 1$
    - If $S_i^1 + S_j^2 < h$ then $i \leftarrow i + 1$

Example

Set $\{1,6,2,3,4\}$ with value sum 16 has 32 subsets.

Partitioning into $\{1,6\}$, $\{2,3,4\}$ yields the following 12 subsets with value sums:

$\{1,6\}$				$\{2,3,4\}$							
$\{\}$	$\{1\}$	$\{6\}$	$\{1,6\}$	$\{\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{2,3\}$	$\{2,4\}$	$\{3,4\}$	$\{2,3,4\}$
0	$\boxed{1}$	6	7	0	2	3	4	5	6	$\boxed{7}$	9

$\Leftrightarrow$ One possible solution: $\{1,3,4\}$

Analysis

- Generate partial sums for each part: $\mathcal{O}(2^{n/2}\cdot n)$.
- Each sorting: $\mathcal{O}(2^{n/2}\log(2^{n/2})) = \mathcal{O}(n2^{n/2})$.
- Merge: $\mathcal{O}(2^{n/2})$

Overal running time

$$\mathcal{O}\left(n\cdot 2^{n/2}\right) = \mathcal{O}\left(n\left(\sqrt{2}\right)^n\right).$$

Substantial improvement over the naive method – but still exponential!

Dynamic programming

Task: let $z = \frac{1}{2}\sum_{i=1}^{n} a_i$. Find a selection $I \subset \{1,\dots,n\}$, such that $\sum_{i\in I} a_i = z$.

DP-table: $[0,\dots,n]\times[0,\dots,z]$-table T with boolean entries. $T[k,s]$ specifies if there is a selection $I_k \subset \{1,\dots,k\}$ such that $\sum_{i\in I_k} a_i = s$.

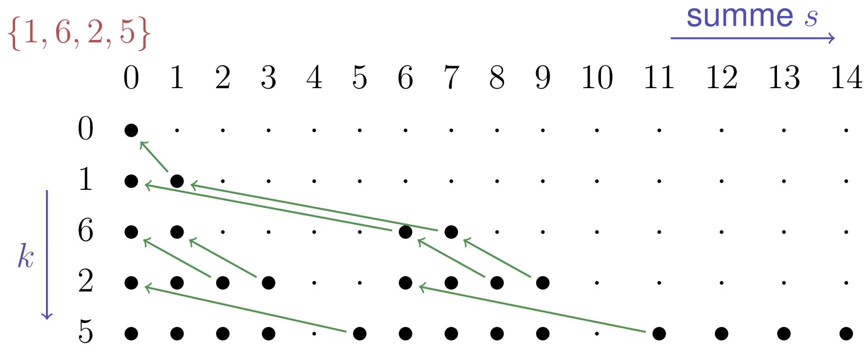
Initialization: $T[0,0]$ = true. $T[0,s]$ = false for $s > 1$.

Computation:

$$T[k,s] \leftarrow \begin{cases} T[k-1,s] & \text{if } s < a_k \\ T[k-1,s] \vee T[k-1,s-a_k] & \text{if } s \geq a_k \end{cases}$$

for increasing k and then within k increasing s.

Example



Determination of the solution: if $T[k,s] = T[k-1,s]$ then a_k unused and continue with $T[k-1,s]$, otherwise a_k used and continue with $T[k-1,s-a_k]$.

That is mysterious

The algorithm requires a number of $\mathcal{O}(n \cdot z)$ fundamental operations.

What is going on now? Does the algorithm suddenly have polynomial running time?

Explained

The algorithm does not necessarily provide a polynomial run time. z is an *number* and not a *quantity*!

Input length of the algorithm $\cong$ number bits to *reasonably* represent the data. With the number z this would be $\zeta = \log z$.

Consequently the algorithm requires $\mathcal{O}(n \cdot 2^\zeta)$ fundamental operations and has a run time exponential in ζ.

If, however, z is polynomial in n then the algorithm has polynomial run time in n. This is called *pseudo-polynomial*.

NP

It is known that the subset-sum algorithm belongs to the class of *NP*-complete problems (and is thus *NP-hard*).

P: Set of all problems that can be solved in polynomial time.

NP: Set of all problems that can be solved Nondeterministically in Polynomial time.

Implications:

- NP contains P.
- Problems can be *verified* in polynomial time.
- Under the not (yet?) proven assumption[32] that NP $\neq$ P, there is *no algorithm with polynomial run time* for the problem considered above.

[32] The most important unsolved question of theoretical computer science.

The knapsack problem

We pack our suitcase with ...

- toothbrush
- dumbell set
- coffee machine
- uh oh – too heavy.

- Toothbrush
- Air balloon
- Pocket knife
- identity card
- dumbell set
- Uh oh – too heavy.

- toothbrush
- coffe machine
- pocket knife
- identity card
- Uh oh – too heavy.

Aim to take as much as possible with us. But some things are more valuable than others!

Knapsack problem

Given:

- set of $n \in \mathbb{N}$ items $\{1, \dots, n\}$.
- Each item i has value $v_i \in \mathbb{N}$ and weight $w_i \in \mathbb{N}$.
- Maximum weight $W \in \mathbb{N}$.
- Input is denoted as $E = (v_i, w_i)_{i=1,\dots,n}$.

Wanted:

a selection $I \subseteq \{1, \dots, n\}$ that maximises $\sum_{i \in I} v_i$ under $\sum_{i \in I} w_i \leq W$.

Greedy heuristics

Sort the items decreasingly by value per weight v_i/w_i: Permutation p with $v_{p_i}/w_{p_i} \geq v_{p_{i+1}}/w_{p_{i+1}}$

Add items in this order ($I \leftarrow I \cup \{p_i\}$), if the maximum weight is not exceeded.

That is fast: $\Theta(n \log n)$ for sorting and $\Theta(n)$ for the selection. But is it good?

Counterexample

$$
\begin{array}{lll}
v_1 = 1 & w_1 = 1 & v_1/w_1 = 1 \\
v_2 = W - 1 & w_2 = W & v_2/w_2 = \frac{W-1}{W}
\end{array}
$$

Greed algorithm chooses $\{v_1\}$ with value 1.
Best selection: $\{v_2\}$ with value $W - 1$ and weight W.

Greedy heuristics can be arbitrarily bad.

Dynamic Programming

Partition the maximum weight.

Three dimensional table $m[i, w, v]$ (“doable”) of boolean values.

$m[i, w, v] =$ true if and only if

- A selection of the first i parts exists ($0 \leq i \leq n$)
- with overal weight w ($0 \leq w \leq W$) and
- a value of at least v ($0 \leq v \leq \sum_{i=1}^{n} v_i$) .

Computation of the DP table

Initially

- $m[i, w, 0] \leftarrow$ true für alle $i \geq 0$ und alle $w \geq 0$.
- $m[0, w, v] \leftarrow$ false für alle $w \geq 0$ und alle $v > 0$.

Computation

$$m[i, w, v] \leftarrow \begin{cases} m[i-1, w, v] \vee m[i-1, w-w_i, v-v_i] & \text{if } w \geq w_i \text{ und } v \geq v_i \\ m[i-1, w, v] & \text{otherwise.} \end{cases}$$

increasing in i and for each i increasing in w and for fixed i and w increasing by v.

Solution: largest v, such that $m[i, w, v] =$ true for some i and w.

Observation

The definition of the problem obviously implies that

- for $m[i, w, v] =$ true it holds:
  $m[i', w, v] =$ true $\forall i' \geq i$,
  $m[i, w', v] =$ true $\forall w' \geq w$,
  $m[i, w, v'] =$ true $\forall v' \leq v$.
- fpr $m[i, w, v] =$ false it holds:
  $m[i', w, v] =$ false $\forall i' \leq i$,
  $m[i, w', v] =$ false $\forall w' \leq w$,
  $m[i, w, v'] =$ false $\forall v' \geq v$.

This strongly suggests that we do not need a 3d table!

2d DP table

Table entry $t[i, w]$ contains, instead of boolean values, the largest v, that can be achieved[33] with

- items $1, \dots, i$ ($0 \leq i \leq n$)
- at maximum weight w ($0 \leq w \leq W$).

[33]We could have followed a similar idea in order to reduce the size of the sparse table.

Computation

Initially

- $t[0, w] \leftarrow 0$ for all $w \geq 0$.

We compute

$$t[i, w] \leftarrow \begin{cases} t[i-1, w] & \text{if } w < w_i \\ \max\{t[i-1, w], t[i-1, w-w_i] + v_i\} & \text{otherwise.} \end{cases}$$

increasing by i and for fixed i increasing by w.

Solution is located in $t[n, w]$

Example

$E = \{(2,3),(4,5),(1,1)\}$

$i \downarrow$ / $w \rightarrow$	0	1	2	3	4	5	6	7
$\emptyset$	0	0	0	0	0	0	0	0
$(2,3)$	0	0	3	3	3	3	3	3
$(4,5)$	0	0	3	3	5	5	8	8
$(1,1)$	0	1	3	4	5	6	8	9

Reading out the solution: if $t[i,w] = t[i-1,w]$ then item i unused and continue with $t[i-1,w]$ otherwise used and continue with $t[i-1, s-w_i]$.

Analysis

The two algorithms for the knapsack problem provide a run time in $\Theta(n \cdot W \cdot \sum_{i=1}^{n} v_i)$ (3d-table) and $\Theta(n \cdot W)$ (2d-table) and are thus both pseudo-polynomial, but they deliver the best possible result.

The greedy algorithm is very fast butmight deliver an arbitrarily bad result.

Now we consider a solution between the two extremes.

21. Dynamic Programming III

FPTAS [Ottman/Widmayer, Kap. 7.2, 7.3, Cormen et al, Kap. 15,35.5]

Approximation

Let $\varepsilon \in (0,1)$ given. Let I_{opt} an optimal selection.

No try to find a valid selection I with

$$\sum_{i \in I} v_i \geq (1-\varepsilon) \sum_{i \in I_{\text{opt}}} v_i.$$

Sum of weights may not violate the weight limit.

Different formulation of the algorithm

Before: weight limit $w \to$ maximal value v

Reversed: value $v \to$ minimal weight w

$\Rightarrow$ **alternative table** $g[i, v]$ provides the minimum weight with

- a selection of the first i items ($0 \leq i \leq n$) that
- provide a value of exactly v ($0 \leq v \leq \sum_{i=1}^{n} v_i$).

Computation

Initially

- $g[0, 0] \leftarrow 0$
- $g[0, v] \leftarrow \infty$ (Value v cannot be achieved with 0 items.).

Computation

$$g[i, v] \leftarrow \begin{cases} g[i-1, v] & \text{falls } v < v_i \\ \min\{g[i-1, v], g[i-1, v-v_i] + w_i\} & \text{sonst.} \end{cases}$$

incrementally in i and for fixed i increasing in v.

Solution can be found at largest index v with $g[n, v] \leq w$.

Example

$E = \{(2, 3), (4, 5), (1, 1)\}$

$v \rightarrow$, $i \downarrow$

	0	1	2	3	4	5	6	7	8	9
$\emptyset$	0	∞	∞	∞	∞	∞	∞	∞	∞	∞
$(2, 3)$	0	∞	∞	2	∞	∞	∞	∞	∞	∞
$(4, 5)$	0	∞	∞	2	∞	4	∞	∞	6	∞
$(1, 1)$	0	1	∞	2	3	4	5	∞	6	7

Read out the solution: if $g[i, v] = g[i-1, v]$ then item i unused and continue with $g[i-1, v]$ otherwise used and continue with $g[i-1, b-v_i]$.

The approximation trick

Pseduopolynomial run time gets polynmial if the number of occuring values can be bounded by a polynom of the input length.

Let $K > 0$ be chosen *appropriately*. Replace values v_i by "rounded values" $\tilde{v}_i = \lfloor v_i / K \rfloor$ delivering a new input $E' = (w_i, \tilde{v}_i)_{i=1\dots n}$.

Apply the algorithm on the input E' with the same weight limit W.

Idea

Example $K = 5$

Values

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \dots, 98, 99, 100$$
$$\to$$
$$0, 0, 0, 0, 1, 1, 1, 1, 1, 2, \dots, 19, 19, 20$$

Obviously less different values

Properties of the new algorithm

- Selection of items in E' is also admissible in E. Weight remains unchanged!
- Run time of the algorithm is bounded by $\mathcal{O}(n^2 \cdot v_{\max}/K)$ $(v_{\max} := \max\{v_i | 1 \leq i \leq n\})$

How good is the approximation?

It holds that

$$v_i - K \leq K \cdot \left\lfloor \frac{v_i}{K} \right\rfloor = K \cdot \tilde{v}_i \leq v_i$$

Let I'_{opt} be an optimal solution of E'. Then

$$\left(\sum_{i \in I_{\text{opt}}} v_i \right) - n \cdot K \overset{|I_{\text{opt}}| \leq n}{\leq} \sum_{i \in I_{\text{opt}}} (v_i - K) \leq \sum_{i \in I_{\text{opt}}} (K \cdot \tilde{v}_i) = K \sum_{i \in I_{\text{opt}}} \tilde{v}_i$$
$$\underset{I'_{\text{opt}}\text{optimal}}{\leq} K \sum_{i \in I'_{\text{opt}}} \tilde{v}_i = \sum_{i \in I'_{\text{opt}}} K \cdot \tilde{v}_i \leq \sum_{i \in I'_{\text{opt}}} v_i.$$

Choice of K

Requirement:

$$\sum_{i \in I'} v_i \geq (1 - \varepsilon) \sum_{i \in I_{\text{opt}}} v_i.$$

Inequality from above:

$$\sum_{i \in I'_{\text{opt}}} v_i \geq \left(\sum_{i \in I_{\text{opt}}} v_i \right) - n \cdot K$$

thus: $K = \varepsilon \frac{\sum_{i \in I_{\text{opt}}} v_i}{n}$.

Choice of K

Choose $K = \varepsilon \frac{\sum_{i \in I_{\text{opt}}} v_i}{n}$. The optimal sum is unknown. Therefore we choose $K' = \varepsilon \frac{v_{\max}}{n}$.[34]

It holds that $v_{\max} \le \sum_{i \in I_{\text{opt}}} v_i$ and thus $K' \le K$ and the approximation is even slightly better.

The run time of the algorithm is bounded by

$$\mathcal{O}(n^2 \cdot v_{\max}/K') = \mathcal{O}(n^2 \cdot v_{\max}/(\varepsilon \cdot v_{\max}/n)) = \mathcal{O}(n^3/\varepsilon).$$

[34] We can assume that items i with $w_i > W$ have been removed in the first place.

FPTAS

Such a family of algorithms is called an *approximation scheme*: the choice of ε controls both running time and approximation quality.

The runtime $\mathcal{O}(n^3/\varepsilon)$ is a polynom in n and in $\frac{1}{\varepsilon}$. The scheme is therefore also called a *FPTAS - Fully Polynomial Time Approximation Scheme*

22. Greedy Algorithms

Fractional Knapsack Problem, Huffman Coding [Cormen et al, Kap. 16.1, 16.3]

The Fractional Knapsack Problem

set of $n \in \mathbb{N}$ items $\{1, \ldots, n\}$ Each item i has value $v_i \in \mathbb{N}$ and weight $w_i \in \mathbb{N}$. The maximum weight is given as $W \in \mathbb{N}$. Input is denoted as $E = (v_i, w_i)_{i=1,\ldots,n}$.

Wanted: Fractions $0 \le q_i \le 1$ $(1 \le i \le n)$ that maximise the sum $\sum_{i=1}^n q_i \cdot v_i$ under $\sum_{i=1}^n q_i \cdot w_i \le W$.

Greedy heuristics

Sort the items decreasingly by value per weight v_i/w_i.

Assumption $v_i/w_i \geq v_{i+1}/w_{i+1}$

Let $j = \max\{0 \leq k \leq n : \sum_{i=1}^{k} w_i \leq W\}$. Set

- $q_i = 1$ for all $1 \leq i \leq j$.
- $q_{j+1} = \frac{W - \sum_{i=1}^{j} w_i}{w_{j+1}}$.
- $q_i = 0$ for all $i > j + 1$.

That is fast: $\Theta(n \log n)$ for sorting and $\Theta(n)$ for the computation of the q_i.

Correctness

Assumption: optimal solution (r_i) ($1 \leq i \leq n$).

The knapsack is full: $\sum_i r_i \cdot w_i = \sum_i q_i \cdot w_i = W$.

Consider k: smallest i with $r_i \neq q_i$ Definition of greedy: $q_k > r_k$. Let $x = q_k - r_k > 0$.

Construct a new solution (r'_i): $r'_i = r_i \forall i < k$. $r'_k = q_k$. Remove weight $\sum_{i=k+1}^{n} \delta_i = x \cdot w_k$ from items $k+1$ to n. This works because $\sum_{i=k}^{n} r_i \cdot w_i = \sum_{i=k}^{n} q_i \cdot w_i$.

Correctness

$$
\begin{aligned}
\sum_{i=k}^{n} r'_i v_i &= r_k v_k + x w_k \frac{v_k}{w_k} + \sum_{i=k+1}^{n} (r_i w_i - \delta_i) \frac{v_i}{w_i} \\
&\geq r_k v_k + x w_k \frac{v_k}{w_k} + \sum_{i=k+1}^{n} r_i w_i \frac{v_i}{w_i} - \delta_i \frac{v_k}{w_k} \\
&= r_k v_k + x w_k \frac{v_k}{w_k} - x w_k \frac{v_k}{w_k} + \sum_{i=k+1}^{n} r_i w_i \frac{v_i}{w_i} = \sum_{i=k}^{n} r_i v_i.
\end{aligned}
$$

Thus (r'_i) is also optimal. Iterative application of this idea generates the solution (q_i).

Huffman-Codes

Goal: memory-efficient saving of a sequence of characters using a binary code with code words..

Example

File consisting of 100.000 characters from the alphabet $\{a, \ldots, f\}$.

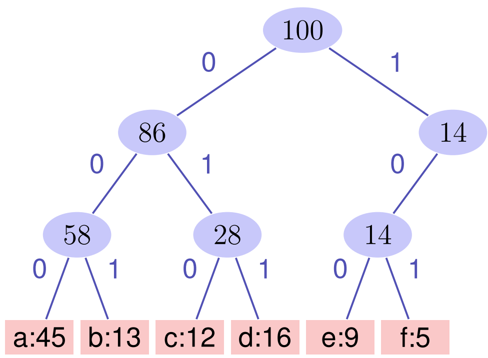
	a	b	c	d	e	f
Frequency (Thousands)	45	13	12	16	9	5
Code word with fix length	000	001	010	011	100	101
Code word variable length	0	101	100	111	1101	1100

File size (code with fix length): 300.000 bits.
File size (code with variable length): 224.000 bits.

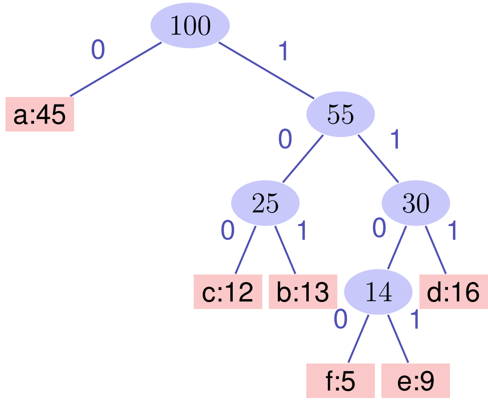
Huffman-Codes

- Consider prefix-codes: no code word can start with a different codeword.
- Prefix codes can, compared with other codes, achieve the optimal *data compression* (without proof here).
- Encoding: concatenation of the code words without stop character (difference to morsing).
  $affe \to 0 \cdot 1100 \cdot 1100 \cdot 1101 \to 0110011001101$
- Decoding simple because prefixcode
  $0110011001101 \to 0 \cdot 1100 \cdot 1100 \cdot 1101 \to affe$

Code trees



Code words with fixed length



Code words with variable length

Properties of the Code Trees

- An optimal coding of a file is alway represented by a complete binary tree: every inner node has two children.
- Let C be the set of all code words, $f(c)$ the frequency of a codeword c and $d_T(c)$ the depth of a code word in tree T. Define the cost of a tree as

$$B(T) = \sum_{c \in C} f(c) \cdot d_T(c).$$

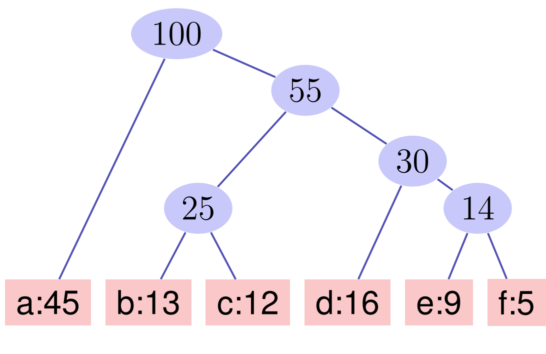
(cost = number bits of the encoded file)

In the following a code tree is called optimal when it minimizes the costs.

Algorithm Idea

Tree construction bottom up

- Start with the set C of code words
- Replace iteriatively the two nodes with smallest frequency by a new parent node.



Algorithm Huffman(C)

Input : code words $c \in C$
Output : Root of an optimal code tree

$n \leftarrow |C|$
$Q \leftarrow C$
for $i = 1$ **to** $n - 1$ **do**
  allocate a new node z
  z.left $\leftarrow$ ExtractMin(Q) // extract word with minimal frequency.
  z.right $\leftarrow$ ExtractMin(Q)
  z.freq $\leftarrow$ z.left.freq $+$ z.right.freq
  Insert(Q, z)
return ExtractMin(Q)

Analyse

Use a heap: build Heap in $\mathcal{O}(n)$. Extract-Min in $O(\log n)$ for n Elements. Yields a runtime of $O(n \log n)$.

The greedy approach is correct

Theorem

Let x, y be two symbols with smallest frequencies in C and let $T'(C')$ be an optimal code tree to the alphabet $C' = C - \{x, y\} + \{z\}$ with a new symbol z with $f(z) = f(x) + f(y)$. Then the tree $T(C)$ that is constructed from $T'(C')$ by replacing the node z by an inner node with children x and y is an optimal code tree for the alphabet C.

Proof

It holds that $f(x) \cdot d_T(x) + f(y) \cdot d_T(y) = (f(x) + f(y)) \cdot (d_{T'}(z) + 1) = f(z) \cdot d_{T'}(x) + f(x) + f(y)$. Thus $B(T') = B(T) - f(x) - f(y)$.

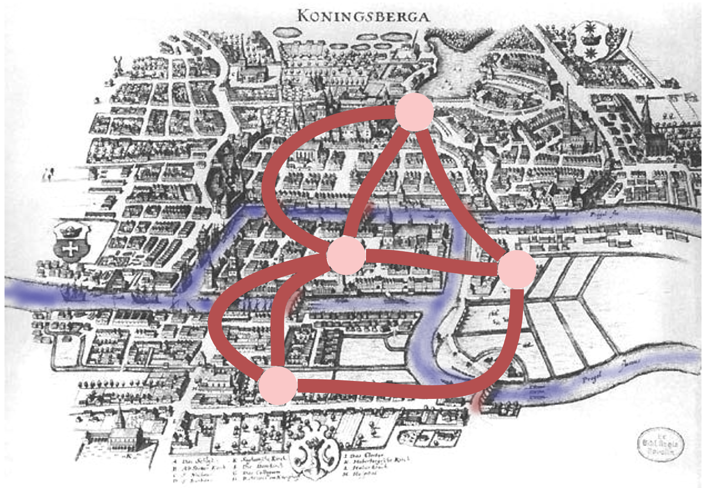
Assumption: T is not optimal. Then there is an optimal tree T'' with $B(T'') < B(T)$. We assume that x and y are brothers in T''. Let T''' be the tree where the inner node with children x and y is replaced by z. Then it holds that $B(T''') = B(T'') - f(x) - f(y) < B(T) - f(x) - f(y) = B(T')$. Contradiction to the optimality of T'.

The assumption that x and y are brothers in T'' can be justified because a swap of elements with smallest frequency to the lowest level of the tree can at most decrease the value of B.

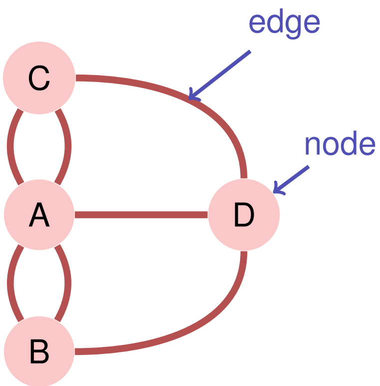
23. Graphs

Notation, Representation, Reflexive transitive closure, Graph Traversal (DFS, BFS), Connected components, Topological Sorting
Ottman/Widmayer, Kap. 9.1 - 9.4,Cormen et al, Kap. 22

Königsberg 1736



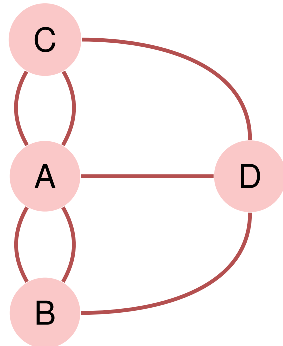
[Multi]Graph



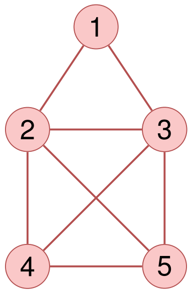
Cycles

- Is there a cycle through the town (the graph) that uses each bridge (each edge) exactly once?
- Euler (1736): no.
- Such a *cycle* is called *Eulerian path*.
- Eulerian path ⇔ each node provides an even number of edges (each node is of an *even degree*).

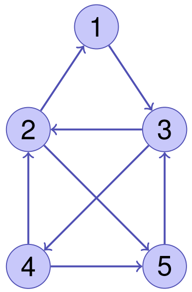
'⇒" ist straightforward, "⇐" ist a bit more difficult



Notation



undirected



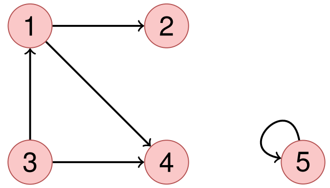
directed

$$V = \{1,2,3,4,5\}$$
$$E = \{\{1,2\},\{1,3\},\{2,3\},\{2,4\},\{2,5\},\{3,4\},\{3,5\},\{4,5\}\}$$

$$V = \{1,2,3,4,5\}$$
$$E = \{(1,3),(2,1),(2,5),(3,2),(3,4),(4,2),(4,5),(5,3)\}$$

Notation

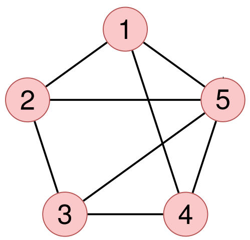
A *directed graph* consists of a set $V = \{v_1, \dots, v_n\}$ of nodes (*Vertices) and a set $E \subseteq V \times V$ of* Edges. The same edges may not be contained more than once.



loop

Notation

An *undirected graph* consists of a set $V = \{v_1, \dots, v_n\}$ of nodes a and a set $E \subseteq \{\{u,v\} | u, v \in V\}$ of edges. Edges may bot be contained more than once.[35]

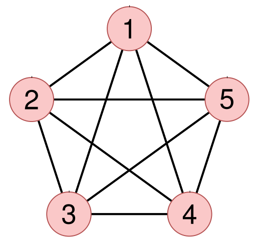


undirected graph

[35] As opposed to the introductory example – it is then called multi-graph.

Notation

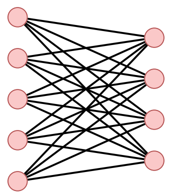
An undirected graph $G = (V, E)$ without loops where E comprises all edges between pairwise different nodes is called *complete*.



a complete undirected graph

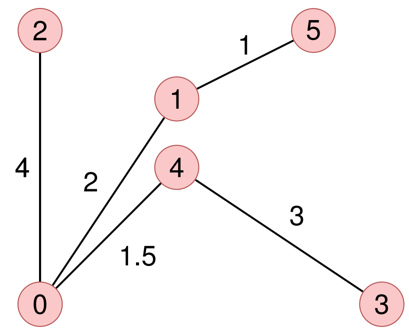
Notation

A graph where V can be partitioned into disjoint sets U and W such that each $e \in E$ provides a node in U and a node in W is called *bipartite*.



Notation

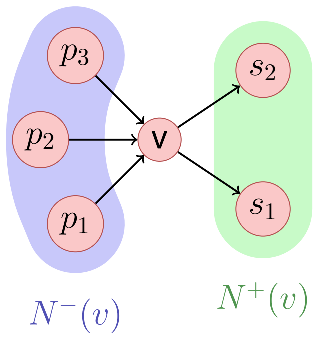
A *weighted graph* $G = (V, E, c)$ is a graph $G = (V, E)$ with an *edge weight function* $c : E \to \mathbb{R}$. $c(e)$ is called *weight* of the edge e.



Notation

For directed graphs $G = (V, E)$

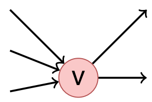
- $w \in V$ is called adjacent to $v \in V$, if $(v, w) \in E$
- *Predecessors* of $v \in V$: $N^-(v) := \{u \in V | (u, v) \in E\}$.
  Successors: $N^+(v) := \{u \in V | (v, u) \in E\}$

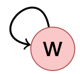


Notation

For directed graphs $G = (V, E)$

- *In-Degree*: $\deg^-(v) = |N^-(v)|$,
  Out-Degree: $\deg^+(v) = |N^+(v)|$



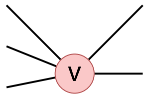


$\deg^-(v) = 3, \deg^+(v) = 2$ $\quad$ $\deg^-(w) = 1, \deg^+(w) = 1$

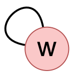
Notation

For undirected graphs $G = (V, E)$:

- $w \in V$ is called *adjacent* to $v \in V$, if $\{v, w\} \in E$
- *Neighbourhood* of $v \in V$: $N(v) = \{w \in V | \{v, w\} \in E\}$
- *Degree* of v: $\deg(v) = |N(v)|$ with a special case for the loops: increase the degree by 2.



$\deg(v) = 5$



$\deg(w) = 2$

Relationship between node degrees and number of edges

For each graph $G = (V, E)$ it holds

1. $\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|$, for G directed
2. $\sum_{v \in V} \deg(v) = 2|E|$, for G undirected.

Paths

- *Path*: a sequence of nodes $\langle v_1, \ldots, v_{k+1} \rangle$ such that for each $i \in \{1 \ldots k\}$ there is an edge from v_i to v_{i+1} .
- *Length* of a path: number of contained edges k.
- *Weight* of a path (in weighted graphs): $\sum_{i=1}^{k} c((v_i, v_{i+1}))$ (bzw. $\sum_{i=1}^{k} c(\{v_i, v_{i+1}\})$)
- *Simple path*: path without repeating vertices

Connectedness

- An undirected graph is called *connected*, if for eacheach pair $v, w \in V$ there is a connecting path.
- A directed graph is called *strongly connected*, if for each pair $v, w \in V$ there is a connecting path.
- A directed graph is called *weakly connected*, if the corresponding undirected graph is connected.

Simple Observations

- generally: $0 \leq |E| \in \mathcal{O}(|V|^2)$
- connected graph: $|E| \in \Omega(|V|)$
- complete graph: $|E| = \frac{|V|\cdot(|V|-1)}{2}$ (undirected)
- Maximally $|E| = |V|^2$ (directed),$|E| = \frac{|V|\cdot(|V|+1)}{2}$ (undirected)

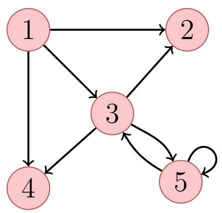
Cycles

- *Cycle*: path $\langle v_1, \ldots, v_{k+1} \rangle$ with $v_1 = v_{k+1}$
- *Simple cycle*: Cycle with pairwise different $v_1, \ldots, v_k$, that does not use an edge more than once.
- *Acyclic*: graph without any cycles.

Conclusion: undirected graphs cannot contain cycles with length 2 (loops have length 1)

Representation using a Matrix

Graph $G = (V, E)$ with nodes $v_1 \ldots, v_n$ stored as *adjacency matrix* $A_G = (a_{ij})_{1 \leq i,j \leq n}$ with entries from $\{0, 1\}$. $a_{ij} = 1$ if and only if edge from v_i to v_j.

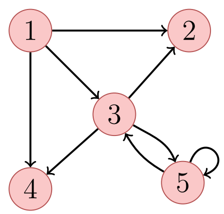


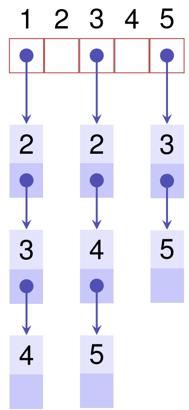
$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Memory consumption $\Theta(|V|^2)$. A_G is symmetric, if G undirected.

Representation with a List

Many graphs $G = (V, E)$ with nodes $v_1, \ldots, v_n$ provide much less than n^2 edges. Representation with *adjacency list*: Array $A[1], \ldots, A[n]$, A_i comprises a linked list of nodes in $N^+(v_i)$.



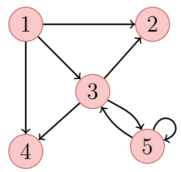


Memory Consumption $\Theta(|V| + |E|)$.

Runtimes of simple Operations

Operation	Matrix	List
Find neighbours/successors of $v \in V$	$\Theta(n)$	$\Theta(\deg^+ v)$
find $v \in V$ without neighbour/successor	$\Theta(n^2)$	$\Theta(n)$
$(u, v) \in E$?	$\Theta(1)$	$\Theta(\deg^+ v)$
Insert edge	$\Theta(1)$	$\Theta(1)$
Delete edge	$\Theta(1)$	$\Theta(\deg^+ v)$

Adjacency Matrix Product



$$B := A_G^2 = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 2 \end{pmatrix}$$

Interpretation

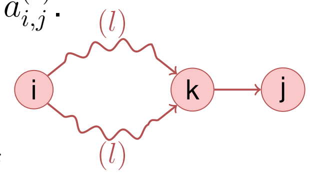
Theorem

Let $G = (V, E)$ be a graph and $k \in \mathbb{N}$. Then the element $a_{i,j}^{(k)}$ of the matrix $(a_{i,j}^{(k)})_{1 \le i,j \le n} = (A_G)^k$ provides the number of paths with length k from v_i to v_j .

Proof

By Induction.

Base case: straightforward for $k = 1$. $a_{i,j} = a_{i,j}^{(1)}$.
Hypothesis: claim is true for all $k \le l$
Step ($l \to l + 1$):



$$a_{i,j}^{(l+1)} = \sum_{k=1}^{n} a_{i,k}^{(l)} \cdot a_{k,j}$$

$a_{k,j} = 1$ iff egde k to j, 0 otherwise. Sum counts the number paths of length l from node v_i to all nodes v_k that provide a direct direction to node v_j, i.e. all paths with length $l + 1$.

Example: Shortest Path

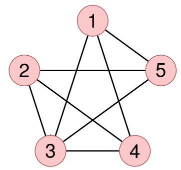
Question: is there a path from i to j? How long is the shortest path?

Answer: exponentiate A_G until for some $k < n$ it holds that $a_{i,j}^{(k)} > 0$. k provides the path length of the shortest path. If $a_{i,j}^{(k)} = 0$ for all $1 \leq k < n$, then there is no path from i to j.

Example: Number triangles

Question: How many triangular path does an undirected graph contain?

Answer: Remove all cycles (diagonal entries). Compute A_G^3. $a_{ii}^{(3)}$ determines the number of paths of length 3 that contain i. There are 6 different permutations of a triangular path. Thus for the number of triangles: $\sum_{i=1}^{n} a_{ii}^{(3)}/6$.



$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}^3 = \begin{pmatrix} 4 & 4 & 8 & 8 & 8 \\ 4 & 4 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 & 8 \\ 8 & 8 & 8 & 4 & 4 \\ 8 & 8 & 8 & 4 & 4 \end{pmatrix}$$ $\Rightarrow 24/6 = 4$ Dreiecke.

Relation

Given a finite set V

(Binary) **Relation** R on V: Subset of the cartesian product $V \times V = \{(a, b) | a \in V, b \in V\}$

Relation $R \subseteq V \times V$ is called

- *reflexive*, if $(v, v) \in R$ for all $v \in V$
- *symmetric*, if $(v, w) \in R \Rightarrow (w, v) \in R$
- *transitive*, if $(v, x) \in R$, $(x, w) \in R \Rightarrow (v, w) \in R$

The (Reflexive) Transitive Closure R^* of R is the smallest extension $R \subseteq R^* \subseteq V \times V$ such that R^* is reflexive and transitive.

Graphs and Relations

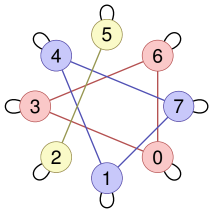
Graph $G = (V, E)$
adjacencies $A_G \mathrel{\widehat{=}}$ Relation $E \subseteq V \times V$ over V

- *reflexive* $\Leftrightarrow a_{i,i} = 1$ for all $i = 1, \ldots, n$. (loops)
- *symmetric* $\Leftrightarrow a_{i,j} = a_{j,i}$ for all $i, j = 1, \ldots, n$ (undirected)
- *transitive* $\Leftrightarrow (u, v) \in E$, $(v, w) \in E \Rightarrow (u, w) \in E$. (reachability)

Example: Equivalence Relation

Equivalence relation $\Leftrightarrow$ symmetric, transitive, reflexive relation $\Leftrightarrow$ collection of complete, undirected graphs where each element has a loop.

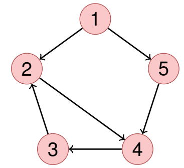
Example: Equivalence classes of the numbers $\{0, ..., 7\}$ modulo 3



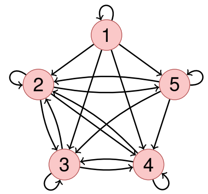
Reflexive Transitive Closure

Reflexive transitive closure of G $\Leftrightarrow$ *Reachability relation* E^*: $(v, w) \in E^*$ iff $\exists$ path from node v to w.

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$



$G = (V, E)$



$G^* = (V, E^*)$

Computation of the Reflexive Transitive Closure

Goal: computation of $B = (b_{ij})_{1 \le i,j \le n}$ with $b_{ij} = 1 \Leftrightarrow (v_i, v_j) \in E^*$

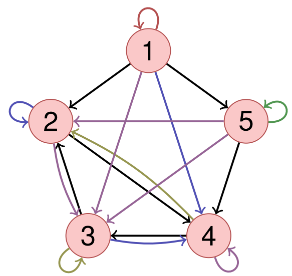
Observation: $a_{ij} = 1$ already implies $(v_i, v_j) \in E^*$.

First idea:

- Start with $B \leftarrow A$ and set $b_{ii} = 1$ for each i (Reflexivity.).
- Iterate over i, j, k and set $b_{ij} = 1$, if $b_{ik} = 1$ and $b_{kj} = 1$. Then all paths with lenght 1 and 2 taken into account.
- Repeated iteration $\Rightarrow$ all paths with length $1 \dots 4$ taken into account.
- $\lceil \log_2 n \rceil$ iterations required. $\Rightarrow$ running time $n^3 \lceil \log_2 n \rceil$

Improvement: Algorithm of Warshall (1962)

Inductive procedure: all paths known over nodes from $\{v_i : i < k\}$. Add node v_k.



$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Algorithm TransitiveClosure(A_G)

Input : Adjacency matrix $A_G = (a_{ij})_{i,j=1\dots n}$
Output : Reflexive transitive closure $B = (b_{ij})_{i,j=1\dots n}$ of G

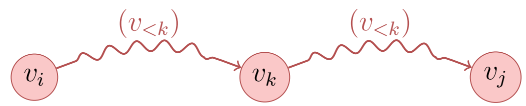
$B \leftarrow A_G$
for $k \leftarrow 1$ **to** n **do**
    $a_{kk} \leftarrow 1$ // Reflexivity
    for $i \leftarrow 1$ **to** n **do**
        for $j \leftarrow 1$ **to** n **do**
            $b_{ij} \leftarrow \max\{b_{ij}, b_{ik} \cdot b_{kj}\}$ // All paths via v_k
return B

Runtime $\Theta(n^3)$.

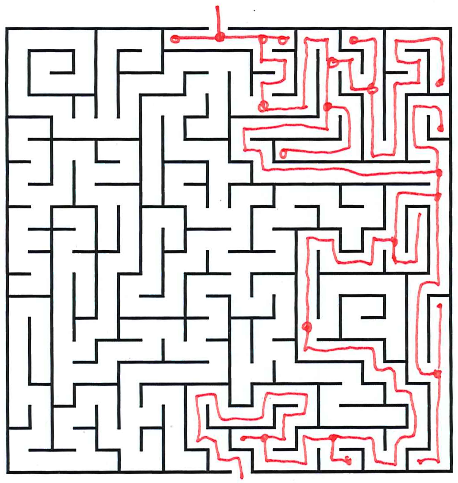
Correctness of the Algorithm (Induction)

Invariant (k): all paths via nodes with maximal index $< k$ considered.

- **Base case ($k = 1$)**: All directed paths (all edges) in A_G considered.
- **Hypothesis**: invariant (k) fulfilled.
- **Step** ($k \to k + 1$): For each path from v_i to v_j via nodes with maximal index k: by the hypothesis $b_{ik} = 1$ and $b_{kj} = 1$. Therefore in the k-th iteration: $b_{ij} \leftarrow 1$.

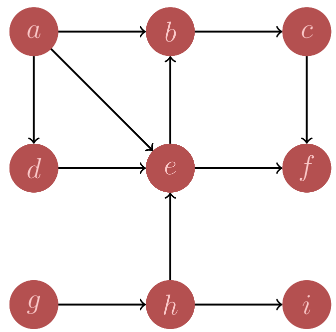


Depth First Search



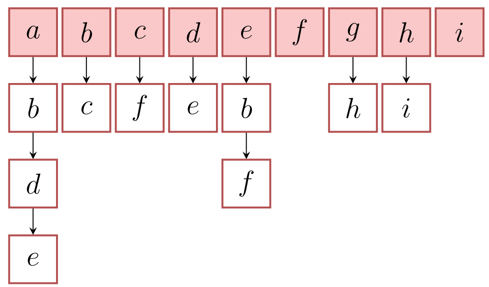
Graph Traversal: Depth First Search

Follow the path into its depth until nothing is left to visit.



Order $a, b, c, f, d, e, g, h, i$

Adjazenzliste



Algorithm Depth First visit DFS-Visit(G, v)

Input : graph $G = (V, E)$, Knoten v.

Mark v visited
foreach $w \in N^+(v)$ **do**
  if $\neg(w$ visited$)$ **then**
    DFS-Visit(G, w)

Depth First Search starting from node v. Running time (without recursion): $\Theta(\deg^+ v)$

Algorithm Depth First visit DFS-Visit(G)

Input : graph $G = (V, E)$

foreach $v \in V$ **do**
  Mark v not visited

foreach $v \in V$ **do**
  if $\neg(v$ visited$)$ **then**
    DFS-Visit(G,v)

Depth First Search for all nodes of a graph. Running time: $\Theta(|V| + \sum_{v \in V}(\deg^+(v) + 1)) = \Theta(|V| + |E|)$.

Iterative DFS-Visit(G, v)

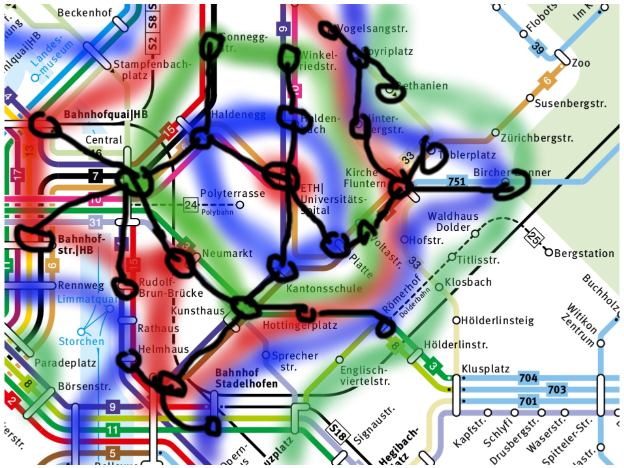
Input : graph $G = (V, E)$

Stack $S \leftarrow \emptyset$; push(S, v)
while $S \neq \emptyset$ **do**
  $w \leftarrow$ pop(S)
  if $\neg(w$ visited$)$ **then**
    mark w visited
    foreach $(w, c) \in E$ **do** // (in reverse order, potentially)
      if $\neg(c$ visited$)$ **then**
        push(S, c)

Stack size up to $|E|$, for each node an extra of $\Theta(\deg^+(w) + 1)$ operations. Overal: $\Theta(|V| + |E|)$

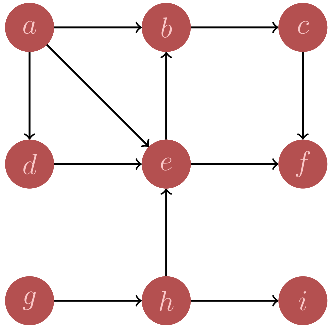
Including all calls from the above main program: $\Theta(|V| + |E|)$

Breadth First Search



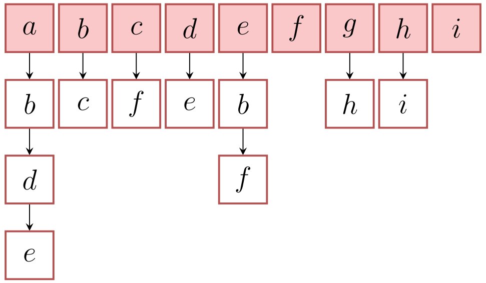
Graph Traversal: Breadth First Search

Follow the path in breadth and only then descend into depth.



Order $a, b, d, e, c, f, g, h, i$

Adjazenzliste



Iterative BFS-Visit(G, v)

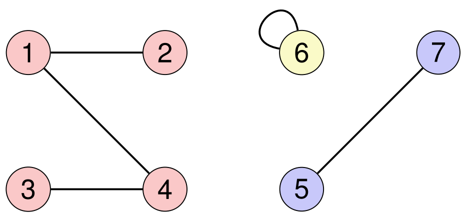
Input : graph $G = (V, E)$

Queue $Q \leftarrow \emptyset$
Mark v as active
enqueue(Q, v)
while $Q \neq \emptyset$ **do**
  $w \leftarrow$ dequeue(Q)
  mark w visited
  foreach $c \in N^+(w)$ **do**
    if $\neg(c$ visited $\vee$ c active$)$ **then**
      Mark c as active
      enqueue(Q, c)

- Algorithm requires extra space of $\mathcal{O}(|V|)$. (Why does that simple approach not work with DFS?)
- Running time including main program: $\Theta(|V| + |E|)$.

Connected Components

Connected components of an undirected graph G: equivalence classes of the reflexive, transitive closure of G. Connected component = subgraph $G' = (V', E')$, $E' = \{\{v, w\} \in E | v, w \in V'\}$ with
$\{\{v, w\} \in E | v \in V' \vee w \in V'\} = E = \{\{v, w\} \in E | v \in V' \wedge w \in V'\}$

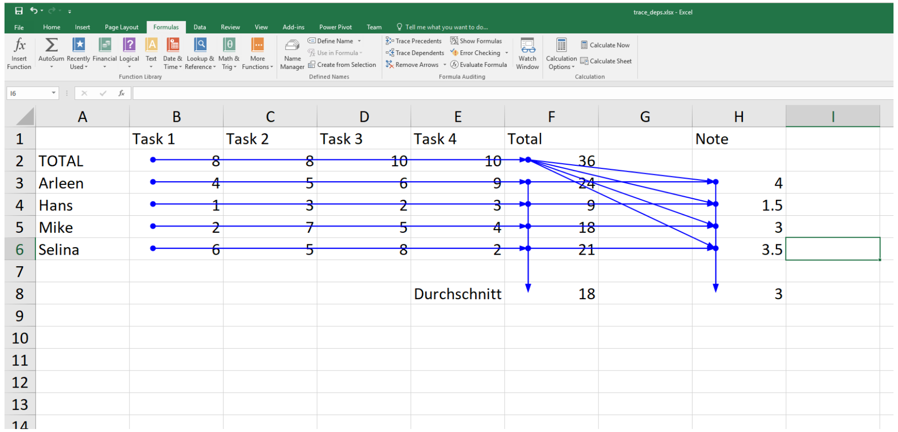


Graph with connected components $\{1, 2, 3, 4\}$, $\{5, 7\}$, $\{6\}$.

Computation of the Connected Components

- Computation of a partitioning of V into pairwise disjoint subsets $V_1, \ldots, V_k$
- such that each V_i contains the nodes of a connected component.
- Algorithm: depth-first search or breadth-first search. Upon each new start of DFSSearch(G, v) or BFSSearch(G, v) a new empty connected component is created and all nodes being traversed are added.

Topological Sorting



	A	B	C	D	E	F	G	H	I
1		Task 1	Task 2	Task 3	Task 4	Total		Note	
2	TOTAL	8	8	10	10	36			
3	Arleen	4	5	6	9	24		4	
4	Hans	1	3	2	3	9		1.5	
5	Mike	2	7	5	4	18		3	
6	Selina	6	5	8	2	21		3.5	
7									
8					Durchschnitt	18		3	
9									
10									
11									
12									
13									
14									

Evaluation Order?

Topological Sorting

Topological Sorting of an acyclic directed graph $G = (V, E)$:

Bijective mapping

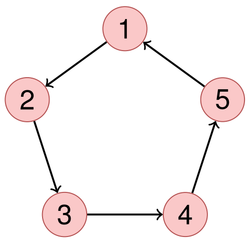
$$\mathrm{ord} : V \to \{1, \ldots, |V|\}$$

such that

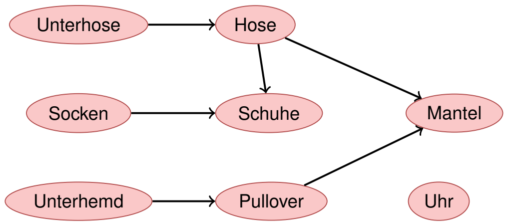
$$\mathrm{ord}(v) < \mathrm{ord}(w)\ \forall\ (v, w) \in E.$$

Identify i with Element $v_i := \mathrm{ord}^1(i)$. Topological sorting $\widehat{=}$ $\langle v_1, \ldots, v_{|V|}\rangle$.

(Counter-)Examples



Cyclic graph: cannot be sorted topologically.



A possible toplogical sorting of the graph:
Unterhemd,Pullover,Unterhose,Uhr,Hose,Mantel,Socken,Schuhe

Observation

Theorem

A directed graph $G = (V, E)$ permits a topological sorting if and only if it is acyclic.

Proof "$\Rightarrow$": If G contains a cycle it cannot permit a topological sorting, because in a cycle $\langle v_{i_1}, \ldots, v_{i_m}\rangle$ it would hold that $v_{i_1} < \cdots < v_{i_m} < v_{i_1}$.

Inductive Proof Opposite Direction

- Base case ($n = 1$): Graph with a single node without loop can be sorted topologically, set$\mathrm{ord}(v_1) = 1$.
- Hypothesis: Graph with n nodes can be sorted topologically
- Step ($n \to n + 1$):
  1. G contains a node v_q with in-degree $\deg^-(v_q) = 0$. Otherwise iteratively follow edges backwards – after at most $n + 1$ iterations a node would be revisited. Contradiction to the cycle-freeness.
  2. Graph without node v_q and without its edges can be topologically sorted by the hypothesis. Now use this sorting and set $\mathrm{ord}(v_i) \leftarrow \mathrm{ord}(v_i) + 1$ for all $i \neq q$ and set $\mathrm{ord}(v_q) \leftarrow 1$.

Preliminary Sketch of an Algorithm

Graph $G = (V, E)$. $d \leftarrow 1$

1. Traverse backwards starting from any node until a node v_q with in-degree 0 is found.
2. If no node with in-degree 0 found after n stepsm, then the graph has a cycle.
3. Set $\mathrm{ord}(v_q) \leftarrow d$.
4. Remove v_q and his edges from G.
5. If $V \neq \emptyset$, then $d \leftarrow d + 1$, go to step 1.

Worst case runtime: $\Theta(|V|^2)$.

Improvement

Idea?

Compute the in-degree of all nodes in advance and traverse the nodes with in-degree 0 while correcting the in-degrees of following nodes.

Algorithm Topological-Sort(G)

Input : graph $G = (V, E)$.
Output : Topological sorting ord

```
Stack S ← ∅
foreach v ∈ V do A[v] ← 0
foreach (v, w) ∈ E do A[w] ← A[w] + 1 // Compute in-degrees
foreach v ∈ V with A[v] = 0 do push(S, v) // Memorize nodes with in-degree 0
i ← 1
while S ≠ ∅ do
    v ← pop(S); ord[v] ← i; i ← i + 1 // Choose node with in-degree 0
    foreach (v, w) ∈ E do // Decrease in-degree of successors
        A[w] ← A[w] − 1
        if A[w] = 0 then push(S, w)
if i = |V| + 1 then return ord else return "Cycle Detected"
```

Algorithm Correctness

Theorem

Let $G = (V, E)$ be a directed acyclic graph. Algorithm TopologicalSort(G) computes a topological sorting ord for G with runtime $\Theta(|V| + |E|)$.

Proof: follows from previous theorem:

1. Decreasing the in-degree corresponds with node removal.
2. In the algorithm it holds for each node v with $A[v] = 0$ that either the node has in-degree 0 or that previously all predecessors have been assigned a value $\mathrm{ord}[u] \leftarrow i$ and thus $\mathrm{ord}[v] > \mathrm{ord}[u]$ for all predecessors u of v. Nodes are put to the stack only once.
3. Runtime: inspection of the algorithm (with some arguments like with graph traversal)

Algorithm Correctness

Theorem

Let $G = (V, E)$ be a directed graph containing a cycle. Algorithm TopologicalSort(G) terminates within $\Theta(|V| + |E|)$ steps and detects a cycle.

Proof: let $\langle v_{i_1}, \dots, v_{i_k} \rangle$ be a cycle in G. In each step of the algorithm remains $A[v_{i_j}] \geq 1$ for all $j = 1, \dots, k$. Thus k nodes are never pushed on the stack und therefore at the end it holds that $i \leq V + 1 - k$.

The runtime of the second part of the algorithm can become shorter. But the computation of the in-degree costs already $\Theta(|V| + |E|)$.

Alternative: Algorithm DFS-Topsort(G, v)

Input : graph $G = (V, E)$, node v, node list L.

```
if v active then
    stop (Cycle)
if v visited then
    return
Mark v active
foreach w ∈ N⁺(v) do
    DFS-Topsort(G, w)
Mark v visited
Add v to head of L
```

Call this algorithm for each node that has not yet been visited.
Asymptotic Running Time $\Theta(|V| + |E|$.

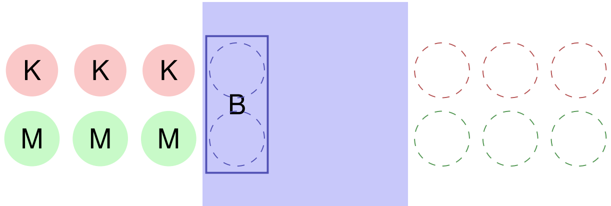
24. Shortest Paths

Motivation, Dijkstra's algorithm on distance graphs, Bellman-Ford Algorithm, Floyd-Warshall Algorithm

[Ottman/Widmayer, Kap. 9.5 Cormen et al, Kap. 24.1-24.3, 25.2-25.3]

River Crossing (Missionaries and Cannibals)

Problem: Three cannibals and three missionaries are standing at a river bank. The available boat can carry two people. At no time may at any place (banks or boat) be more cannibals than missionaries. How can the missionaries and cannibals cross the river as fast as possible? [36]



[36] There are slight variations of this problem. It is equivalent to the jealous husbands problem.

Problem as Graph

Enumerate permitted configurations as nodes and connect them with an edge, when a crossing is allowed. The problem then becomes a shortest path problem.

Example

	links	rechts
Missionare	3	0
Kannibalen	3	0
Boot	x	

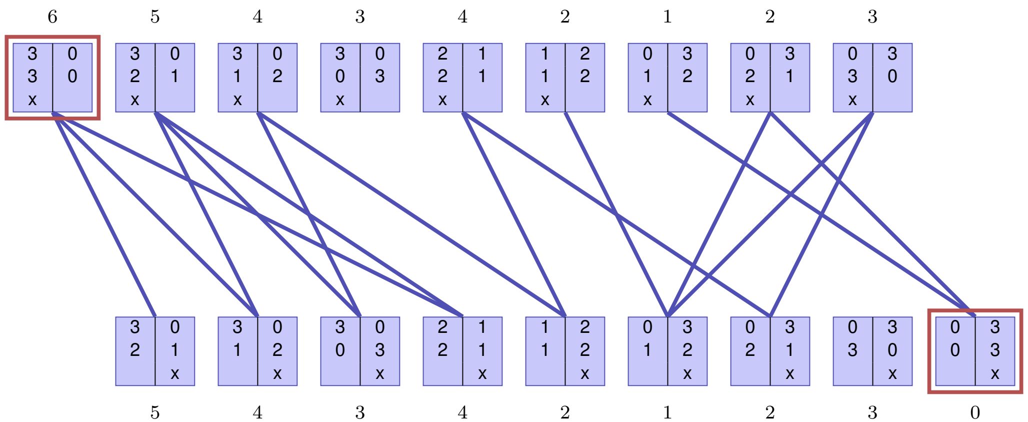
6 Personen am linken Ufer

Überfahrt möglich

	links	rechts
Missionare	2	1
Kannibalen	2	1
Boot		x

4 Personen am linken Ufer

The whole problem as a graph



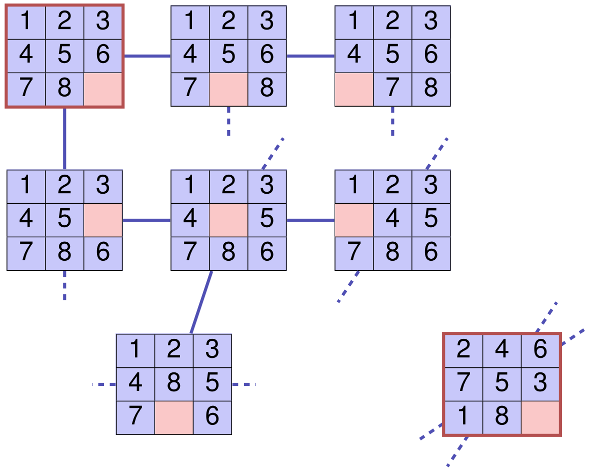
Example Mystic Square

Want to find the fastest solution for

2	4	6
7	5	3
1	8	

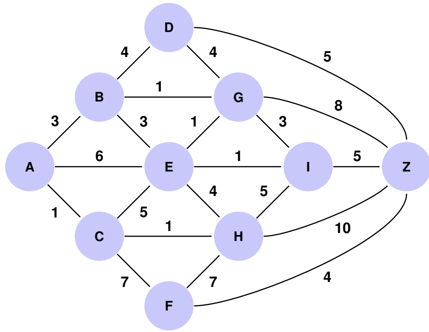
1	2	3
4	5	6
7	8	

Problem as Graph



Route Finding

Provided cities A - Z and Distances between cities.

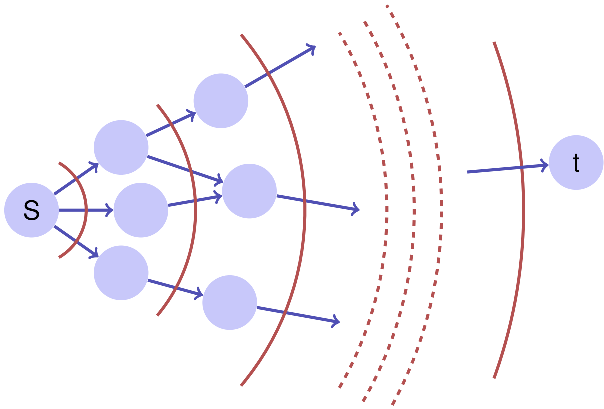


What is the shortest path from A to Z?

Simplest Case

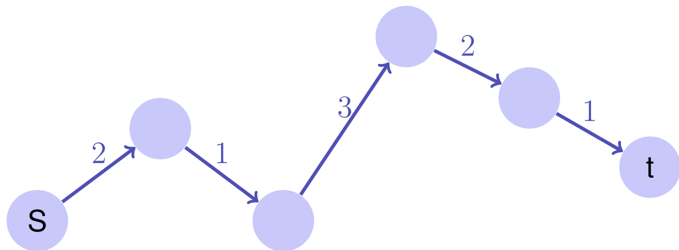
Constant edge weight 1 (wlog)

Solution: Breadth First Search



Graphs with positive weights

Given: $G = (V, E, c)$, $c : E \to \mathbb{R}^+$, $s, t \in V$.
Wanted: Length of a shortest path (weight) from s to t.
Path: $\langle s = v_0, v_1, \dots, v_k = t \rangle$, $(v_i, v_{i+1}) \in E$ $(0 \le i < k)$
Weight: $\sum_{i=0}^{k-1} c((v_i, v_{i+1}))$.



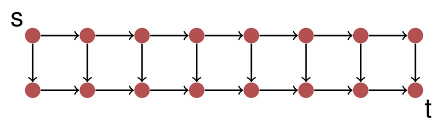
Path with weight 9

Existence of Shortest Path

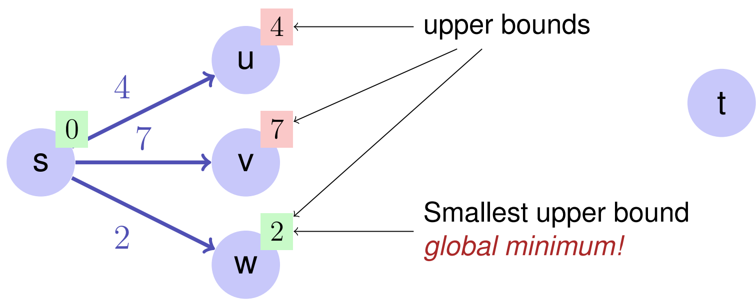
Assumption: There is a path from s to t in G.
Claim: There is a shortest path from s to t in G.

Proof: There can be infinitely many paths from s to t (cycles are possible). However, since c is positive, a shortest path must be acyclic. Thus the maximal length of a shortest path is bounded by some $n \in \mathbb{N}$ and there are only finitely many candidates for a shortest path.

Remark: There can be exponentially many paths. Example



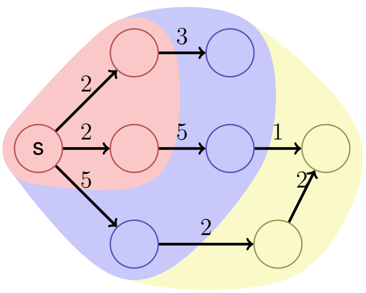
Observation



Basic Idea

Set V of nodes is partitioned into

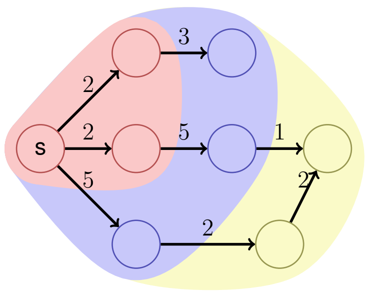
- the set M of nodes for which a shortest path from s is already known,
- the set $R = \bigcup_{v \in M} N^+(v) \setminus M$ of nodes where a shortest path is not yet known but that are accessible directly from M,
- the set $U = V \setminus (M \cup R)$ of nodes that have not yet been considered.



Induction

Induction over $|M|$: choose nodes from R with smallest upper bound. Add r to M and update R and U accordingly.

Correctness: if within the "wavefront" a node with minimal weight has been found then no path with greater weight over different nodes can provide any improvement.



Algorithm Dijkstra(G, s) [formal]

Input : Positively weighted Graph $G = (V, E, c)$, starting point $s \in V$,
Output : Minimal weights d of the shortest paths.

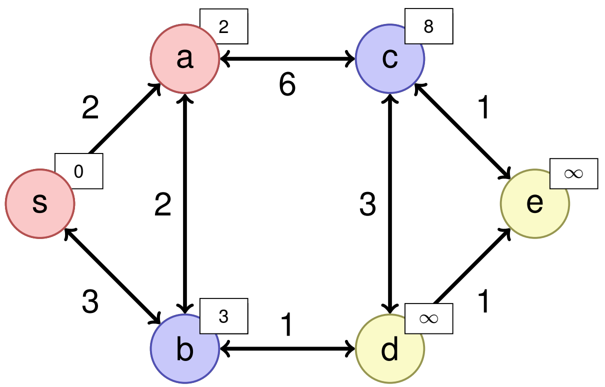
$M = \{s\}$; $R = N^+(s)$, $U = V \setminus R$
$d(s) \leftarrow 0$; $d(u) \leftarrow \infty \ \forall u \neq s$
while $R \neq \emptyset$ **do**
    $r \leftarrow \arg\min_{r \in R} \min_{m \in N^-(r) \cap M} d(m) + c(m, r)$
    $d(r) \leftarrow \min_{m \in N^-(r) \cap M} d(m) + c(m, r)$
    $M \leftarrow M \cup \{r\}$
    $R \leftarrow R - \{r\} \cup N^+(r) \setminus M$
return d

Algorithmus Dijkstra

Initial: $\mathrm{PL}(n) \leftarrow \infty$ für alle Knoten.

- Set $\mathrm{PL}(s) \leftarrow 0$
- Start with $M = \{s\}$. Set $k \leftarrow s$.
- While a new node k is added and this is not the target node
  1. For each neighbour node n of k:
     - compute path length x to n via k
     - If $\mathrm{PL}(n) = \infty$, than add n to R
     - If $x < \mathrm{PL}(n) < \infty$, then set $\mathrm{PL}(n) \leftarrow x$ and adapt R .
  2. Choose as new node k the node with smallest path length in R.

Example



$$M = \{s, a\}$$
$$R = \{b, c\}$$
$$U = \{d, e\}$$

Implementation: Naive Variant

- Find minimum: traverse all edges (u, v) for $u \in M, v \in R$.
- Overal costs: $\mathcal{O}(|V| \cdot |E|)$

Implementation: Better Variant

- Update of all outgoing edges when inserting new w in M:

  foreach $v \in N^+(w)$ **do**
      if $d(w) + c(w,v) < d(v)$ **then**
          $d(v) \leftarrow d(w) + c(w,v)$

- Costs of updates: $\mathcal{O}(|E|)$, Find minima: $\mathcal{O}(|V|^2)$, overal costs $\mathcal{O}(|V|^2)$

Implementation: Data Structure for R?

Required operations:

- Insert (add to R)
- ExtractMin (over R) and DecreaseKey (Update in R)

  foreach $v \in N^+(m)$ **do**
      if $d(m) + c(m,v) < d(v)$ **then**
          $d(v) \leftarrow d(m) + c(m,v)$
          if $v \in R$ **then**
              DecreaseKey(R, v) // Update of a $d(v)$ in the heap of R
          else
              $R \leftarrow R \cup \{v\}$ // Update of $d(v)$ in the heap of R

MinHeap!

DecreaseKey

- DecreaseKey: climbing in MinHeap in $\mathcal{O}(\log |V|)$
- Position in the heap?
  - alternative (a): Store position at the nodes
  - alternative (b): Hashtable of the nodes
  - alterantive (c): re-insert node after update-operation and mark it "deleted" once extracted (Lazy Deletion)

Runtime

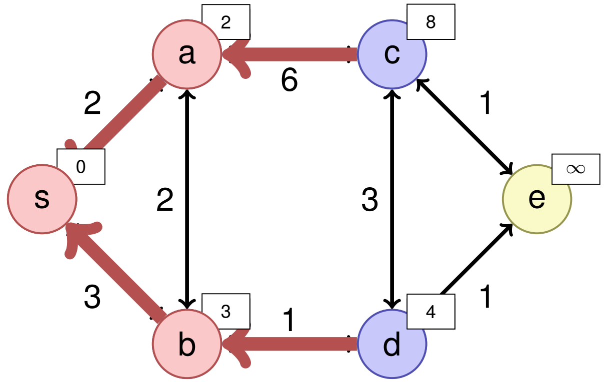
- $|V|\times$ ExtractMin: $\mathcal{O}(|V| \log |V|)$
- $|E|\times$ Insert or DecreaseKey: $\mathcal{O}(|E| \log |V|)$
- $1\times$ Init: $\mathcal{O}(|V|)$
- Overal: $\mathcal{O}(|E| \log |V|)$.

Can be improved when a data structure optimized for ExtractMin and DecreaseKey ist used (Fibonacci Heap), then runtime $\mathcal{O}(|E| + |V| \log |V|)$.

Reconstruct shortest Path

- Memorize best predecessor during the update step in the algorithm above. Store it with the node or in a separate data structure.
- Reconstruct best path by traversing backwards via best predecessor

Example



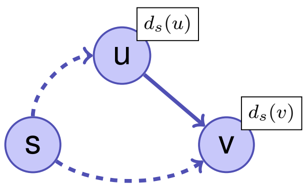
$$M = \{s, a, b\}$$
$$R = \{c, d\}$$
$$U = \{e\}$$

General Weighted Graphs

Relaxing Step as with Dijkstra:

Relax(u, v) ($u, v \in V$, $(u, v) \in E$)

if $d_s(v) > d_s(u) + c(u, v)$ **then**
  $d_s(v) \leftarrow d_s(u) + c(u, v)$
  return true
return false



Problem: cycles with negative weights can shorten the path, a shortest path is not guaranteed to exist.

Observations

- *Observation 1:* Sub-paths of shortest paths are shortest paths. Let $p = \langle v_0, \dots, v_k \rangle$ be a shortest path from v_0 to v_k. Then each of the sub-paths $p_{ij} = \langle v_i, \dots, v_j \rangle$ ($0 \le i < j \le k$) is a shortest path from v_i to v_j.
  Proof: if not, then one of the sub-paths could be shortened which immediately leads to a contradiction.
- *Observation:* If there is a shortest path then it is simple, thus does not provide a node more than once.
  Immediate Consequence of observation 1.

Dynamic Programming Approach (Bellman)

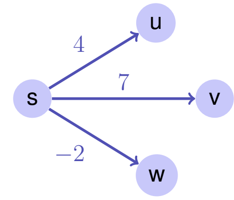
Induction over number of edges $d_s[i,v]$: Shortest path from s to v via maximally i edges.

$$d_s[i,v] = \min\{d_s[i-1,v], \min_{(u,v)\in E}(d_s[i-1,u] + c(u,v))\}$$

$$d_s[0,s] = 0, d_s[0,v] = \infty \ \forall v \neq s.$$

Dynamic Programming Approach (Bellman)

	s	$\cdots$	v	$\cdots$	w
0	0	∞	∞	∞	∞
1	0	∞	7	∞	-2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1$	0	$\cdots$	$\cdots$	$\cdots$	$\cdots$



Algorithm: Iterate over last row until the relaxation steps do not provide any further changes, maximally $n-1$ iterations. If still changes, then there is no shortest path.

Algorithm Bellman-Ford(G, s)

Input : Graph $G = (V, E, c)$, starting point $s \in V$
Output : If return value true, minimal weights d for all shortest paths from s, otherwise no shortest path.

```
d(v) ← ∞ ∀v ∈ V; d(s) ← 0
for i ← 1 to |V| do
    f ← false
    foreach (u, v) ∈ E do
        f ← f ∨ Relax(u, v)
    if f = false then return true
return false;
```

Runtime $\mathcal{O}(|E| \cdot |V|)$.

All shortest Paths

Compute the weight of a shortest path for each pair of nodes.

- $|V|\times$ Application of Dijkstra's Shortest Path algorithm $\mathcal{O}(|V| \cdot |E| \cdot \log |V|)$ (with Fibonacci Heap: $\mathcal{O}(|V|^2 \log |V| + |V| \cdot |E|)$)
- $|V|\times$ Application of Bellman-Ford: $\mathcal{O}(|E| \cdot |V|^2)$
- There are better ways!

Induction via node number[37]

Consider weights of all shortest paths S^k with intermediate nodes in $V^k := \{v_1, \ldots, v_k\}$, provided that weights for all shortest paths S^{k-1} with intermediate nodes in V^{k-1} are given.

- v_k no intermediate node of a shortest path of $v_i \rightsquigarrow v_j$ in V^k: Weight of a shortest path $v_i \rightsquigarrow v_j$ in S^{k-1} is then also weight of shortest path in S^k.
- v_k intermediate node of a shortest path $v_i \rightsquigarrow v_j$ in V^k: Sub-paths $v_i \rightsquigarrow v_k$ and $v_k \rightsquigarrow v_j$ contain intermediate nodes only from S^{k-1}.

[37] like for the algorithm of the reflexive transitive closure of Warshall

DP Induction

$d^k(u, v)$ = Minimal weight of a path $u \rightsquigarrow v$ with intermediate nodes in V^k

Induktion

$$d^k(u, v) = \min\{d^{k-1}(u, v), d^{k-1}(u, k) + d^{k-1}(k, v)\}(k \geq 1)$$
$$d^0(u, v) = c(u, v)$$

DP Algorithm Floyd-Warshall(G)

Input : Acyclic Graph $G = (V, E, c)$
Output : Minimal weights of all paths d

```
d^0 ← c
for k ← 1 to |V| do
    for i ← 1 to |V| do
        for j ← 1 to |V| do
            d^k(v_i, v_j) = min{d^{k-1}(v_i, v_j), d^{k-1}(v_i, v_k) + d^{k-1}(v_k, v_j)}
```

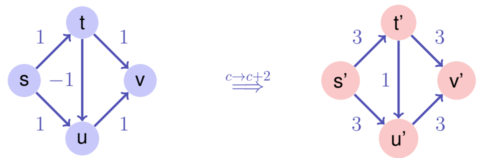
Runtime: $\Theta(|V|^3)$

Remark: Algorithm can be executed with a single matrix d (in place).

Reweighting

Idea: Reweighting the graph in order to apply Dijkstra's algorithm.

The following does *not* work. The graphs are not equivalent in terms of shortest paths.



Reweighting

Other Idea: “Potential” (Height) on the nodes

- $G = (V, E, c)$ a weighted graph.
- Mapping $h : V \to \mathbb{R}$
- New weights

$$\tilde{c}(u, v) = c(u, v) + h(u) - h(v), \ (u, v \in V)$$

Reweighting

Observation: A path p is shortest path in in $G = (V, E, c)$ iff it is shortest path in in $\tilde{G} = (V, E, \tilde{c})$

$$\begin{aligned}\tilde{c}(p) &= \sum_{i=1}^{k} \tilde{c}(v_{i-1}, v_i) = \sum_{i=1}^{k} c(v_{i-1}, v_i) + h(v_{i-1}) - h(v_i) \\ &= h(v_0) - h(v_k) + \sum_{i=1}^{k} c(v_{i-1}, v_i) = c(p) + h(v_0) - h(v_k)\end{aligned}$$

Thus $\tilde{c}(p)$ minimal in all $v_0 \rightsquigarrow v_k \iff c(p)$ minimal in all $v_0 \rightsquigarrow v_k$.

Weights of cycles are invariant: $\tilde{c}(v_0, \dots, v_k = v_0) = c(v_0, \dots, v_k = v_0)$

Johnson’s Algorithm

Add a new node $s \notin V$:

$$\begin{aligned}G' &= (V', E', c') \\ V' &= V \cup \{s\} \\ E' &= E \cup \{(s, v) : v \in V\} \\ c'(u, v) &= c(u, v), \ u \neq s \\ c'(s, v) &= 0 (v \in V)\end{aligned}$$

Johnson’s Algorithm

If no negative cycles, choose as height function the weight of the shortest paths from s,

$$h(v) = d(s, v).$$

For a minimal weight d of a path the following triangular inequality holds:

$$d(s, v) \leq d(s, u) + c(u, v).$$

Substitution yields $h(v) \leq h(u) + c(u, v)$. Therefore

$$\tilde{c}(u, v) = c(u, v) + h(u) - h(v) \geq 0.$$

Algorithm Johnson(G)

Input : Weighted Graph $G = (V, E, c)$
Output : Minimal weights of all paths D.

New node s. Compute $G' = (V', E', c')$
if BellmanFord(G', s) = false **then** return "graph has negative cycles"
foreach $v \in V'$ **do**
  $h(v) \leftarrow d(s, v)$ // d aus BellmanFord Algorithmus
foreach $(u, v) \in E'$ **do**
  $\tilde{c}(u, v) \leftarrow c(u, v) + h(u) - h(v)$
foreach $u \in V$ **do**
  $\tilde{d}(u, \cdot) \leftarrow$ Dijkstra($\tilde{G}', u$)
  foreach $v \in V$ **do**
    $D(u, v) \leftarrow \tilde{d}(u, v) + h(v) - h(u)$

Analysis

Runtimes

- Computation of G': $\mathcal{O}(|V|)$
- Bellman Ford G': $\mathcal{O}(|V| \cdot |E|)$
- $|V| \times$ Dijkstra $\mathcal{O}(|V| \cdot |E| \cdot \log |V|)$
  (with Fibonacci Heap: $\mathcal{O}(|V|^2 \log |V| + |V| \cdot |E|)$)

Overal $\mathcal{O}(|V| \cdot |E| \cdot \log |V|)$
($\mathcal{O}(|V|^2 \log |V| + |V| \cdot |E|)$)

25. Minimum Spanning Trees

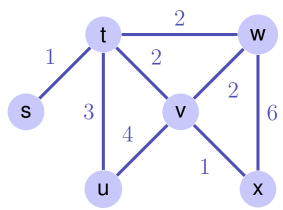
Motivation, Greedy, Algorithm Kruskal, General Rules, ADT Union-Find, Algorithm Jarnik, Prim, Dijkstra, Fibonacci Heaps

[Ottman/Widmayer, Kap. 9.6, 6.2, 6.1, Cormen et al, Kap. 23, 19]

Problem

Given: Undirected, weighted, connected graph $G = (V, E, c)$.

Wanted: Minimum Spanning Tree $T = (V, E')$: connected subgraph $E' \subset E$, such that $\sum_{e \in E'} c(e)$ minimal.



Application: cheapest / shortest cable network

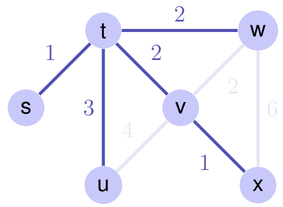
Greedy Procedure

Recall:

- Greedy algorithms compute the solution stepwise choosing locally optimal solutions.
- Most problems cannot be solved with a greedy algorithm.
- The Minimum Spanning Tree problem constitutes one of the exceptions.

Greedy Idea

Construct T by adding the cheapest edge that does not generate a cycle.



(Solution is not unique.)

Algorithm MST-Kruskal(G)

Input : Weighted Graph $G = (V, E, c)$
Output : Minimum spanning tree with edges A.

Sort edges by weight $c(e_1) \leq ... \leq c(e_m)$
$A \leftarrow \emptyset$
for $k = 1$ **to** $|E|$ **do**
    if $(V, A \cup \{e_k\})$ acyclic **then**
        $A \leftarrow A \cup \{e_k\}$
return (V, A, c)

Correctness

At each point in the algorithm (V, A) is a forest, a set of trees.

MST-Kruskal considers each edge e_k exactly once and either chooses or rejects e_k

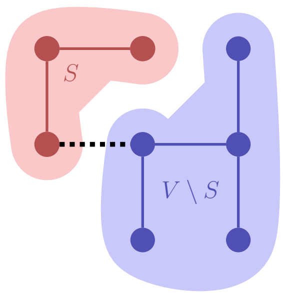
Notation (snapshot of the state in the running algorithm)

- A: Set of selected edges
- R: Set of rejected edges
- U: Set of yet undecided edges

Cut

A cut of G is a partition $S, V - S$ of V. ($S \subseteq V$).

An edge crosses a cut when one of its endpoints is in S and the other is in $V \setminus S$.



Rules

1. Selection rule: choose a cut that is not crossed by a selected edge. Of all undecided edges that cross the cut, select the one with minimal weight.
2. Rejection rule: choose a circle without rejected edges. Of all undecided edges of the circle, reject those with minimal weight.

Rules

Kruskal applies both rules:

1. A selected e_k connects two connection components, otherwise it would generate a circle. e_k is minimal, i.e. a cut can be chosen such that e_k crosses and e_k has minimal weight.
2. A rejected e_k is contained in a circle. Within the circle e_k has minimal weight.

Correctness

Theorem

Every algorithm that applies the rules above in a step-wise manner until $U = \emptyset$ is correct.

Consequence: MST-Kruskal is correct.

Selection invariant

Invariant: At each step there is a minimal spanning tree that contains all selected and none of the rejected edges.

If both rules satisfy the invariant, then the algorithm is correct. Induction:

- At beginning: $U = E$, $R = A = \emptyset$. Invariant obviously holds.
- Invariant is preserved.
- At the end: $U = \emptyset$, $R \cup A = E \Rightarrow (V, A)$ is a spanning tree.

Proof of the theorem: show that both rules preserve the invariant.

Selection rule preserves the invariant

At each step there is a minimal spanning tree T that contains all selected and none of the rejected edges.

Choose a cut that is not crossed by a selected edge. Of all undecided edges that cross the cut, select the egde e with minimal weight.

- Case 1: $e \in T$ (done)
- Case 2: $e \notin T$. Then $T \cup \{e\}$ contains a circle that contains e Circle must have a second edge e' that also crosses the cut.[38] Because $e' \notin R$, $e' \in U$. Thus $c(e) \leq c(e')$ and $T' = T \setminus \{e'\} \cup \{e\}$ is also a minimal spanning tree (and $c(e) = c(e')$).

[38] Such a circle contains at least one node in S and one node in $V \setminus S$ and therefore at lease to edges between S and $V \setminus S$.

Rejection rule preserves the invariant

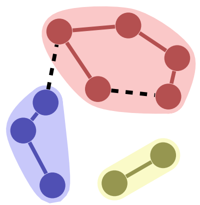
At each step there is a minimal spanning tree T that contains all selected and none of the rejected edges.

Choose a circle without rejected edges. Of all undecided edges of the circle, reject an edge e with minimal weight.

- Case 1: $e \notin T$ (done)
- Case 2: $e \in T$. Remove e from T, This yields a cut. This cut must be crossed by another edge e' of the circle. Because $c(e') \leq c(e)$, $T' = T \setminus \{e\} \cup \{e'\}$ is also minimal (and $c(e) = c(e')$).

Implementation Issues

Consider a set of sets $i \equiv A_i \subset V$. To identify cuts and circles: membership of the both ends of an edge to sets?



Implementation Issues

General problem: partition (set of subsets) .e.g.
$\{\{1,2,3,9\},\{7,6,4\},\{5,8\},\{10\}\}$

Required: ADT (Union-Find-Structure) with the following operations

- Make-Set(i): create a new set represented by i.
- Find(e): name of the set i that contains e .
- Union(i, j): union of the sets with names i and j.

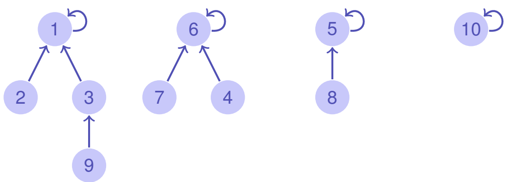
Union-Find Algorithm MST-Kruskal(G)

Input : Weighted Graph $G = (V, E, c)$
Output : Minimum spanning tree with edges A.

Sort edges by weight $c(e_1) \leq ... \leq c(e_m)$
$A \leftarrow \emptyset$
for $k = 1$ **to** $|V|$ **do**
    MakeSet(k)
for $k = 1$ **to** $|E|$ **do**
    $(u, v) \leftarrow e_k$
    if Find(u) $\neq$ Find(v) **then**
        Union(Find(u), Find(v))
        $A \leftarrow A \cup e_k$
return (V, A, c)

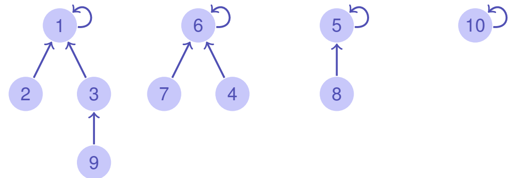
Implementation Union-Find

Idea: tree for each subset in the partition,e.g.
$\{\{1,2,3,9\},\{7,6,4\},\{5,8\},\{10\}\}$



roots = names of the sets,
trees = elements of the sets

Implementation Union-Find



Representation as array:

Index	1	2	3	4	5	6	7	8	9	10
Parent	**1**	1	1	6	**5**	**6**	5	5	3	**10**

Implementation Union-Find

Index	1	2	3	4	5	6	7	8	9	10
Parent	**1**	1	1	6	**5**	**6**	5	5	3	**10**

Operations:

- Make-Set(i): $p[i] \leftarrow i$; **return** i
- Find(i): **while** $(p[i] \neq i)$ **do** $i \leftarrow p[i]$
  return i
- Union(i, j): [39] $p[j] \leftarrow i$; **return** i

[39] i and j need to be names (roots) of the sets. typically: Union(Find(a),Find(b))

Optimisation of the runtime for Find

Tree may degenerate. Example: Union$(1, 2)$, Union$(2, 3)$, Union$(3, 4)$, ...
Idea: always append smaller tree to larger tree. Additionally required: size information g

Operations:

- Make-Set(i): $p[i] \leftarrow i$; $g[i] \leftarrow 1$; **return** i
- Union(i, j):
  if $g[j] > g[i]$ **then** swap(i, j)
  $p[j] \leftarrow i$
  $g[i] \leftarrow g[i] + g[j]$
  return i

Observation

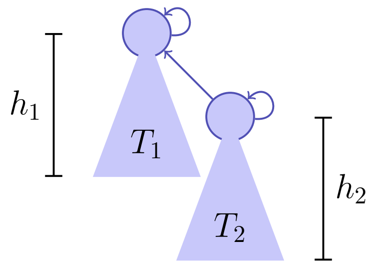
Theorem

The method above (union by size) preserves the following property of the trees: a tree of height h has at least 2^h nodes.

Immediate consequence: runtime Find = $\mathcal{O}(\log n)$.

Proof

Induction: by assumption, sub-trees have at least 2^{h_i} nodes. WLOG: $h_2 \leq h_1$



- $h_2 < h_1$:

$$h(T_1 \oplus T_2) = h_1 \Rightarrow g(T_1 \oplus T_2) \geq 2^h$$

- $h_2 = h_1$:

$$g(T_1) \geq g(T_2) \geq 2^{h_2}$$
$$\Rightarrow g(T_1 \oplus T_2) = g(T_1) + g(T_2) \geq 2 \cdot 2^{h_2} = 2^{h(T_1 \oplus T_2)}$$

Further improvement

Link all nodes to the root when Find is called.

Find(i):

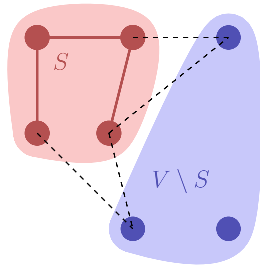
$j \leftarrow i$
while $(p[i] \neq i)$ **do** $i \leftarrow p[i]$
while $(j \neq i)$ **do**
  $t \leftarrow j$
  $j \leftarrow p[j]$
  $p[t] \leftarrow i$
return i

Amortised cost: amortised *nearly* constant (inverse of the Ackermann-function).

MST algorithm of Jarnik, Prim, Dijkstra

Idea: start with some $v \in V$ and grow the spanning tree from here by the acceptance rule.

$S \leftarrow \{v_0\}$
for $i \leftarrow 1$ **to** $|V|$ **do**
  Choose cheapest (u, v) mit $u \in S$, $v \notin S$
  $A \leftarrow A \cup \{(u, v)\}$
  $S \leftarrow S \cup \{v\}$



Running time

Trivially $\mathcal{O}(|V| \cdot |E|)$.

Improvements (like with Dijkstra's ShortestPath)

- Memorize cheapest edge to S: for each $v \in V \setminus S$. $\deg^+(v)$ many updates for each new $v \in S$. Costs: $|V|$ many minima and updates: $\mathcal{O}(|V|^2 + \sum_{v \in V} \deg^+(v)) = \mathcal{O}(|V|^2 + |E|)$
- With Minheap: costs $|V|$ many minima = $\mathcal{O}(|V| \log |V|)$, $|E|$ Updates: $\mathcal{O}(|E| \log |V|)$, Initialization $\mathcal{O}(|V|)$: $\mathcal{O}(|E| \cdot \log |V|.)$
- With a Fibonacci-Heap: $\mathcal{O}(|E| + |V| \cdot \log |V|)$.

Fibonacci Heaps

Data structure for elements with key with operations

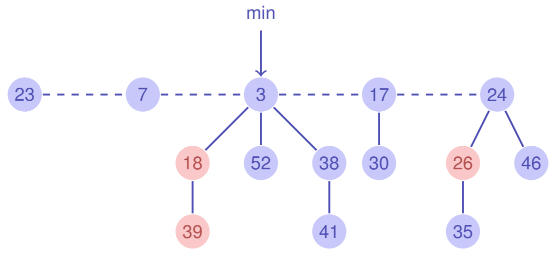
- MakeHeap(): Return new heap without elements
- Insert(H, x): Add x to H
- Minimum(H): return a pointer to element m with minimal key
- ExtractMin(H): return and remove (from H) pointer to the element m
- Union(H_1, H_2): return a heap merged from H_1 and H_2
- DecreaseKey(H, x, k): decrease the key of x in H to k
- Delete (H, x): remove element x from H

Advantage over binary heap?

	Binary Heap (worst-Case)	Fibonacci Heap (amortized)
MakeHeap	$\Theta(1)$	$\Theta(1)$
Insert	$\Theta(\log n)$	$\Theta(1)$
Minimum	$\Theta(1)$	$\Theta(1)$
ExtractMin	$\Theta(\log n)$	$\Theta(\log n)$
Union	$\Theta(n)$	$\Theta(1)$
DecreaseKey	$\Theta(\log n)$	$\Theta(1)$
Delete	$\Theta(\log n)$	$\Theta(\log n)$

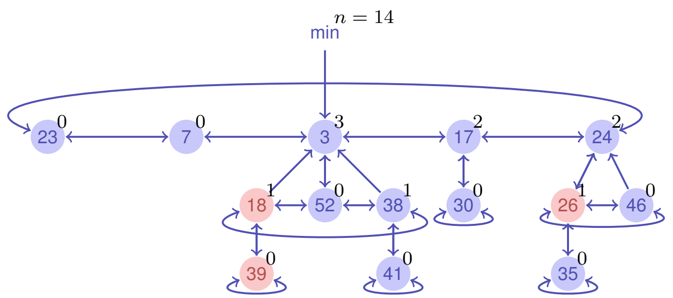
Structure

Set of trees that respect the Min-Heap property. Nodes that can be marked.



Implementation

Doubly linked lists of nodes with a marked-flag and number of children. Pointer to minimal Element and number nodes.



Simple Operations

- MakeHeap (trivial)
- Minimum (trivial)
- Insert(H, e)
  1. Insert new element into root-list
  2. If key is smaller than minimum, reset min-pointer.
- Union (H_1, H_2)
  1. Concatenate root-lists of H_1 and H_2
  2. Reset min-pointer.
- Delete(H, e)
  1. DecreaseKey$(H, e, -\infty)$
  2. ExtractMin(H)

ExtractMin

1. Remove minimal node m from the root list
2. Insert children of m into the root list
3. Merge heap-ordered trees with the same degrees until all trees have a different degree:
   Array of degrees $a[0, \ldots, n]$ of elements, empty at beginning.
   For each element e of the root list:
   a. Let g be the degree of e
   b. If $a[g] = nil$: $a[g] \leftarrow e$.
   c. If $e' := a[g] \neq nil$: Merge e with e' resutling in e'' and set $a[g] \leftarrow nil$. Set e'' unmarked. Re-iterate with $e \leftarrow e''$ having degree $g + 1$.

DecreaseKey (H, e, k)

1. Remove e from its parent node p (if existing) and decrease the degree of p by one.
2. Insert(H, e)
3. Avoid too thin trees:
   a. If $p = nil$ then done.
   b. If p is unmarked: mark p and done.
   c. If p marked: unmark p and cut p from its parent pp. Insert (H, p). Iterate with $p \leftarrow pp$.

Estimation of the degree

Theorem

Let p be a node of a F-Heap H. If child nodes of p are sorted by time of insertion (Union), then it holds that the ith child node has a degree of at least $i - 2$.

Proof: p may have had more children and lost by cutting. When the ith child p_i was linked, p and p_i must at least have had degree $i - 1$. p_i may have lost at least one child (marking!), thus at least degree $i - 2$ remains.

Estimation of the degree

Theorem

Every node p with degree k of a F-Heap is the root of a subtree with at least F_{k+1} nodes. (F: Fibonacci-Folge)

Proof: Let S_k be the minimal number of successors of a node of degree k in a F-Heap plus 1 (the node itself). Clearly $S_0 = 1$, $S_1 = 2$. With the previous theorem $S_k \geq 2 + \sum_{i=0}^{k-2} S_i, k \geq 2$ (p and nodes p_1 each 1). For Fibonacci numbers it holds that (induction) $F_k \geq 2 + \sum_{i=2}^{k} F_i, k \geq 2$ and thus (also induction) $S_k \geq F_{k+2}$.

Fibonacci numbers grow exponentially fast ($\mathcal{O}(\varphi^k)$) Consequence: maximal degree of an arbitrary node in a Fibonacci-Heap with n nodes is $\mathcal{O}(\log n)$.

Amortized worst-case analysis Fibonacci Heap

$t(H)$: number of trees in the root list of H, $m(H)$: number of marked nodes in H not within the root-list, Potential function $\Phi(H) = t(H) + 2 \cdot m(H)$. At the beginnning $\Phi(H) = 0$. Potential always non-negative.

Amortized costs:

- Insert(H, x): $t'(H) = t(H) + 1$, $m'(H) = m(H)$, Increase of the potential: 1, Amortized costs $\Theta(1) + 1 = \Theta(1)$
- Minimum(H): Amortized costs = real costs = $\Theta(1)$
- Union(H_1, H_2): Amortized costs = real costs = $\Theta(1)$

Amortized costs of ExtractMin

- Number trees in the root list $t(H)$.
- Real costs of ExtractMin operation $\mathcal{O}(\log n + t(H))$.
- When merged still $\mathcal{O}(\log n)$ nodes.
- Number of markings can only get smaller when trees are merged
- Thus maximal amortized costs of ExtractMin

$$\mathcal{O}(\log n + t(H)) + \mathcal{O}(\log n) - \mathcal{O}(t(H)) = \mathcal{O}(\log n).$$

Amortized costs of DecreaseKey

- Assumption: DecreaseKey leads to c cuts of a node from its parent node, real costs $\mathcal{O}(c)$
- c nodes are added to the root list
- Delete $(c-1)$ mark flags, addition of at most one mark flag
- Amortized costs of DecreaseKey:

$$\mathcal{O}(c) + (t(H) + c) + 2 \cdot (m(H) - c + 2)) - (t(H) + 2m(H)) = \mathcal{O}(1)$$

26. Flow in Networks

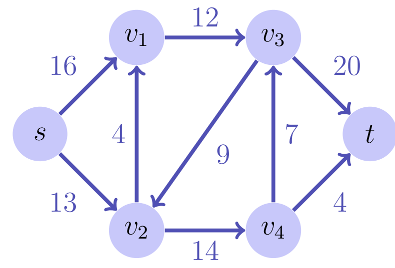
Flow Network, Maximal Flow, Cut, Rest Network, Max-flow Min-cut Theorem, Ford-Fulkerson Method, Edmonds-Karp Algorithm, Maximal Bipartite Matching [Ottman/Widmayer, Kap. 9.7, 9.8.1], [Cormen et al, Kap. 26.1-26.3]

Motivation

- Modelling flow of fluents, components on conveyors, current in electrical networks or information flow in communication networks.
- Connectivity of Communication Networks, Bipartite Matching, Circulation, Scheduling, Image Segmentation, Baseball Eliminination...

Flow Network

- *Flow network* $G = (V, E, c)$: directed graph with *capacities*
- Antiparallel edges forbidden: $(u, v) \in E \;\Rightarrow\; (v, u) \notin E$.
- Model a missing edge (u, v) by $c(u, v) = 0$.
- *Source* s and *sink* t: special nodes. Every node v is on a path between s and t : $s \rightsquigarrow v \rightsquigarrow t$

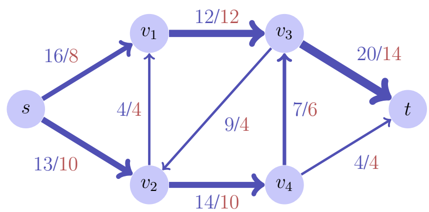


Flow

A *Flow* $f : V \times V \to \mathbb{R}$ fulfills the following conditions:

- *Bounded Capacity*:
  For all $u, v \in V$: $f(u, v) \leq c(u, v)$.
- *Skew Symmetry*:
  For all $u, v \in V$: $f(u, v) = -f(v, u)$.
- *Conservation of flow*:
  For all $u \in V \setminus \{s, t\}$:

$$\sum_{v \in V} f(u, v) = 0.$$



Value of the flow: $|f| = \sum_{v \in V} f(s, v)$.
Here $|f| = 18$.

How large can a flow possibly be?

Limiting factors: cuts

- *cut separating s from t*: Partition of V into S and T with $s \in S$, $t \in T$.
- *Capacity* of a cut: $c(S, T) = \sum_{v \in S, v' \in T} c(v, v')$
- *Minimal cut*: cut with minimal capacity.
- *Flow over the cut*: $f(S, T) = \sum_{v \in S, v' \in T} f(v, v')$

Implicit Summation

Notation: Let $U, U' \subseteq V$

$$f(U,U') := \sum_{\substack{u\in U\\ u'\in U'}} f(u,u'), \qquad f(u,U') := f(\{u\},U')$$

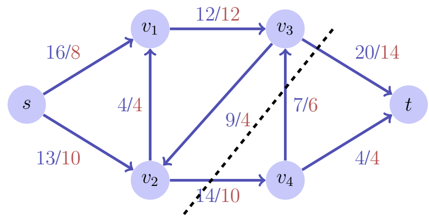
Thus

- $|f| = f(s,V)$
- $f(U,U) = 0$
- $f(U,U') = -f(U',U)$
- $f(X \cup Y, Z) = f(X,Z) + f(Y,Z)$, if $X \cap Y = \emptyset$.
- $f(R,V) = 0$ if $R \cap \{s,t\} = \emptyset$. [flow conversation!]

How large can a flow possibly be?

For each flow and each cut it holds that $f(S,T) = |f|$:

$$\begin{aligned} f(S,T) &= f(S,V) - \underbrace{f(S,S)}_{0} = f(S,V) \\ &= f(s,V) + f(\underbrace{S-\{s\}}_{\not\ni t, \not\ni s}, V) = |f|. \end{aligned}$$

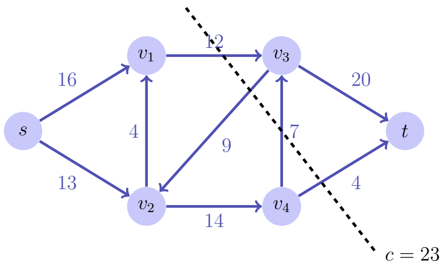


Maximal Flow ?

In particular, for each cut (S,T) of V.

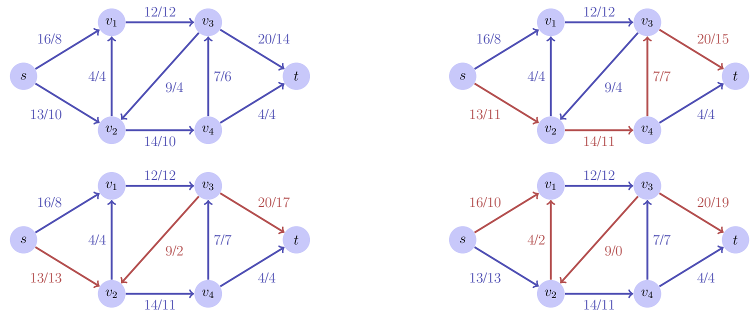
$$|f| \le \sum_{v\in S, v'\in T} c(v,v') = c(S,T)$$

Will discover that equality holds for $\min_{S,T} c(S,T)$.



Maximal Flow ?

Naive Procedure



Conclusion: greedy increase of flow does not solve the problem.

The Method of Ford-Fulkerson

- Start with $f(u,v) = 0$ for all $u, v \in V$
- Determine rest network* G_f and expansion path in G_f
- Increase flow via expansion path*
- Repeat until no expansion path available.

$$
\begin{aligned}
G_f &:= (V, E_f, c_f) \\
c_f(u,v) &:= c(u,v) - f(u,v) \quad \forall u, v \in V \\
E_f &:= \{(u,v) \in V \times V | c_f(u,v) > 0\}
\end{aligned}
$$

*Will now be explained

Increase of flow, negative!

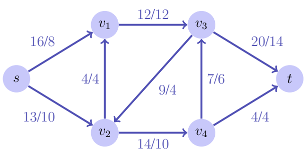
Let some flow f in the network be given.

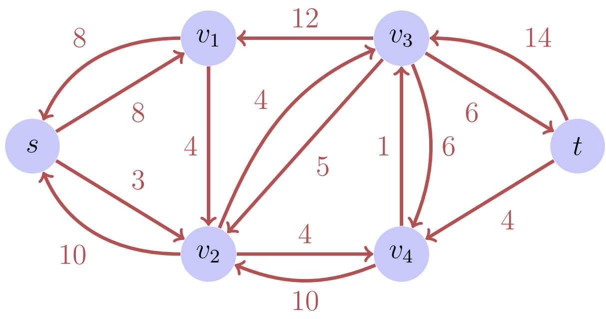
Finding:

- Increase of the flow along some edge possible, when flow can be increased along the edge,i.e. if $f(u,v) < c(u,v)$.
  Rest capacity $c_f(u,v) = c(u,v) - f(u,v) > 0$.
- Increase of flow *against the direction* of the edge possible, if flow can be reduced along the edge, i.e. if $f(u,v) > 0$.
  Rest capacity $c_f(v,u) = f(u,v) > 0$.

Rest Network

Rest network G_f provided by the edges with positive rest capacity:





Rest networks provide the same kind of properties as flow networks with the exception of permitting antiparallel capacity-edges

Observation

Theorem

Let $G = (V, E, c)$ be a flow network with source s and sink t and f a flow in G. Let G_f be the corresponding rest networks and let f' be a flow in G_f. Then $f \oplus f'$ with

$$(f \oplus f')(u,v) = f(u,v) + f'(u,v)$$

defines a flow in G with value $|f| + |f'|$.

Proof

$f \oplus f'$ defines a flow in G:

- capacity limit

$$(f \oplus f')(u,v) = f(u,v) + \underbrace{f'(u,v)}_{\leq c(u,v)-f(u,v)} \leq c(u,v)$$

- skew symmetry

$$(f \oplus f')(u,v) = -f(v,u) + -f'(v,u) = -(f \oplus f')(v,u)$$

- flow conservation $u \in V - \{s,t\}$:

$$\sum_{v\in V}(f \oplus f')(u,v) = \sum_{v\in V} f(u,v) + \sum_{v\in V} f'(u,v) = 0$$

Proof

Value of $f \oplus f'$

$$\begin{aligned} |f \oplus f'| &= (f \oplus f')(s,V) \\ &= \sum_{u\in V} f(s,u) + f'(s,u) \\ &= f(s,V) + f'(s,V) \\ &= |f| + |f'| \end{aligned}$$

$\blacksquare$

Augmenting Paths

expansion path p: simple path from s to t in the rest network G_f.

Rest capacity $c_f(p) = \min\{c_f(u,v) : (u,v) \text{ edge in } p\}$

Flow in G_f

Theorem

The mapping $f_p : V \times V \to \mathbb{R}$,

$$f_p(u,v) = \begin{cases} c_f(p) & \text{if } (u,v) \text{ edge in } p \\ -c_f(p) & \text{if } (v,u) \text{ edge in } p \\ 0 & \text{otherwise} \end{cases}$$

provides a flow in G_f with value $|f_p| = c_f(p) > 0$.

f_p is a flow (easy to show). there is one and only one $u \in V$ with $(s,u) \in p$. Thus $|f_p| = \sum_{v\in V} f_p(s,v) = f_p(s,u) = c_f(p)$.

Consequence

Strategy for an algorithm:

With an expansion path p in G_f the flow $f \oplus f_p$ defines a new flow with value $|f \oplus f_p| = |f| + |f_p| > |f|$.

Max-Flow Min-Cut Theorem

Theorem

Let f be a flow in a flow network $G = (V, E, c)$ with source s and sink t. The following statementsa are equivalent:

1. *f is a maximal flow in G*
2. *The rest network G_f does not provide any expansion paths*
3. *It holds that $|f| = c(S, T)$ for a cut (S, T) of G.*

Proof

- $(3) \Rightarrow (1)$:
  It holds that $|f| \leq c(S, T)$ for all cuts S, T. From $|f| = c(S, T)$ it follows that $|f|$ is maximal.
- $(1) \Rightarrow (2)$:
  f maximal Flow in G. Assumption: G_f has some expansion path $|f \oplus f_p| = |f| + |f_p| > |f|$. Contradiction.

Proof $(2) \Rightarrow (3)$

Assumption: G_f has no expansion path

Define $S = \{v \in V : \text{ there is a path } s \rightsquigarrow v \text{ in } G_f\}$.

$(S, T) := (S, V \setminus S)$ is a cut: $s \in S, t \in T$.

Let $u \in S$ and $v \in T$. Then $c_f(u, v) = 0$, also $c_f(u, v) = c(u, v) - f(u, v) = 0$. Somit $f(u, v) = c(u, v)$.

Thus

$$|f| = f(S, T) = \sum_{u \in S} \sum_{v \in T} f(u, v) = \sum_{u \in S} \sum_{v \in T} c(u, v) = C(S, T).$$

■

Algorithm Ford-Fulkerson(G, s, t)

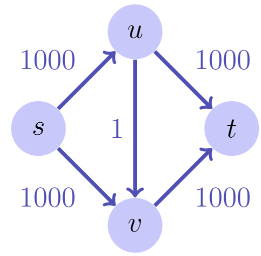
Input : Flow network $G = (V, E, c)$
Output : Maximal flow f.

for $(u, v) \in E$ **do**
    $f(u, v) \leftarrow 0$

while Exists path $p : s \rightsquigarrow t$ in rest network G_f **do**
    $c_f(p) \leftarrow \min\{c_f(u, v) : (u, v) \in p\}$
    foreach $(u, v) \in p$ **do**
        if $(u, v) \in E$ **then**
            $f(u, v) \leftarrow f(u, v) + c_f(p)$
        else
            $f(v, u) \leftarrow f(u, v) - c_f(p)$

Analysis

- The Ford-Fulkerson algorithm does not necessarily have to converge for irrational capacities. For integers or rational numbers it terminates.
- For an integer flow, the algorithms requires maximally $|f_{\max}|$ iterations of the while loop (because the flow increases minimally by 1). Search a single increasing path (e.g. with DFS or BFS) $\mathcal{O}(|E|)$ Therefore $\mathcal{O}(f_{\max}|E|)$.



With an unlucky choice the algorithm may require up to 2000 iterations here.

Edmonds-Karp Algorithm

Choose in the Ford-Fulkerson-Method for finding a path in G_f the expansion path of shortest possible length (e.g. with BFS)

Edmonds-Karp Algorithm

Theorem

When the Edmonds-Karp algorithm is applied to some integer valued flow network $G = (V, E)$ with source s and sink t then the number of flow increases applied by the algorithm is in $\mathcal{O}(|V| \cdot |E|)$.
$\Rightarrow$ Overal asymptotic runtime: $\mathcal{O}(|V| \cdot |E|^2)$

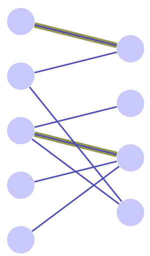
[Without proof]

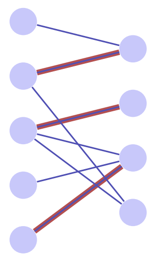
Application: maximal bipartite matching

Given: bipartite undirected graph $G = (V, E)$.

Matching M: $M \subseteq E$ such that $|\{m \in M : v \in m\}| \leq 1$ for all $v \in V$.

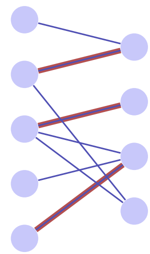
Maximal Matching M: Matching M, such that $|M| \geq |M'|$ for each matching M'.



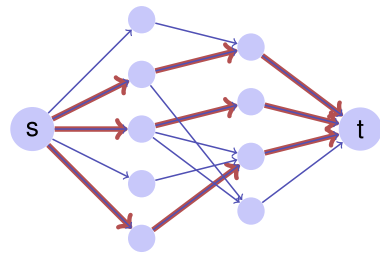


Corresponding flow network

Construct a flow network that corresponds to the partition L, R of a bipartite graph with source s and sink t, with directed edges from s to L, from L to R and from R to t. Each edge has capacity 1.



L R



L R

Integer number theorem

Theorem

If the capacities of a flow network are integers, then the maximal flow generated by the Ford-Fulkerson method provides integer numbers for each $f(u, v)$, $u, v \in V$.

[without proof]

Consequence: Ford-Fulkerson generates for a flow network that corresponds to a bipartite graph a maximal matching $M = \{(u, v) : f(u, v) = 1\}$.

27. Parallel Programming I

Moore's Law and the Free Lunch, Hardware Architectures, Parallel Execution, Flynn's Taxonomy, Scalability: Amdahl and Gustafson, Data-parallelism, Task-parallelism, Scheduling

[Task-Scheduling: Cormen et al, Kap. 27] [Concurrency, Scheduling: Williams, Kap. 1.1 – 1.2]

The Free Lunch

The free lunch is over [40]

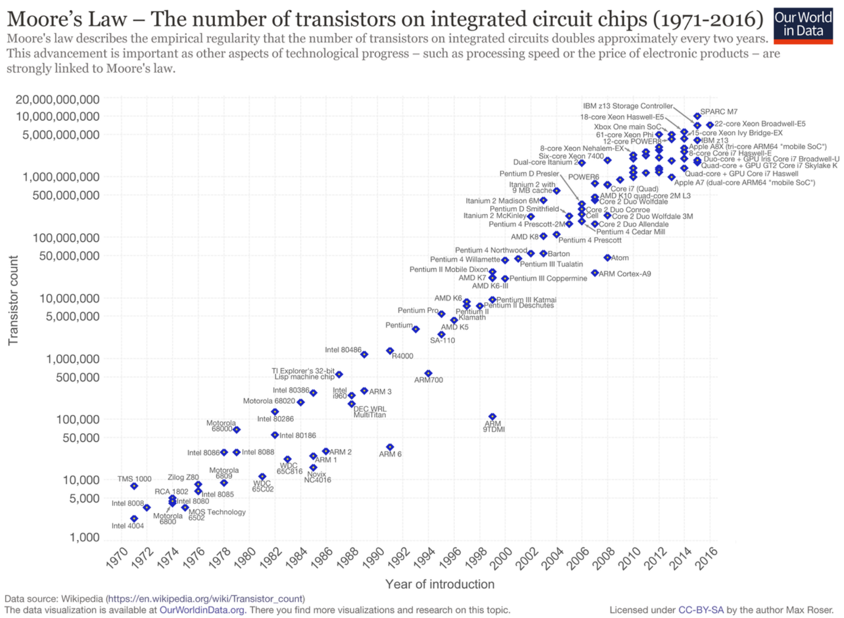
[40] "The Free Lunch is Over", a fundamental turn toward concurrency in software, Herb Sutter, Dr. Dobb's Journal, 2005

Moore's Law


Gordon E. Moore (1929)

Observation by Gordon E. Moore:

The number of transistors on integrated circuits doubles approximately every two years.



ourworldindata.org, https://en.wikipedia.org/wiki/Transistor_count

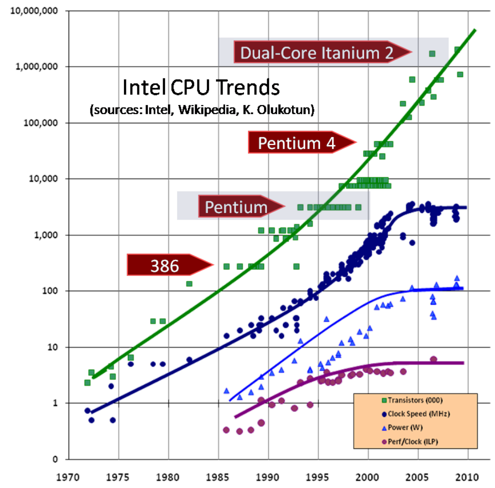
For a long time...

- the sequential execution became faster ("Instruction Level Parallelism", "Pipelining", Higher Frequencies)
- more and smaller transistors = more performance
- programmers simply waited for the next processor generation

Today

- the frequency of processors does not increase significantly and more (heat dissipation problems)
- the instruction level parallelism does not increase significantly any more
- the execution speed is dominated by memory access times (but caches still become larger and faster)

Trends



http://www.gotw.ca/publications/concurrency-ddj.htm

Multicore

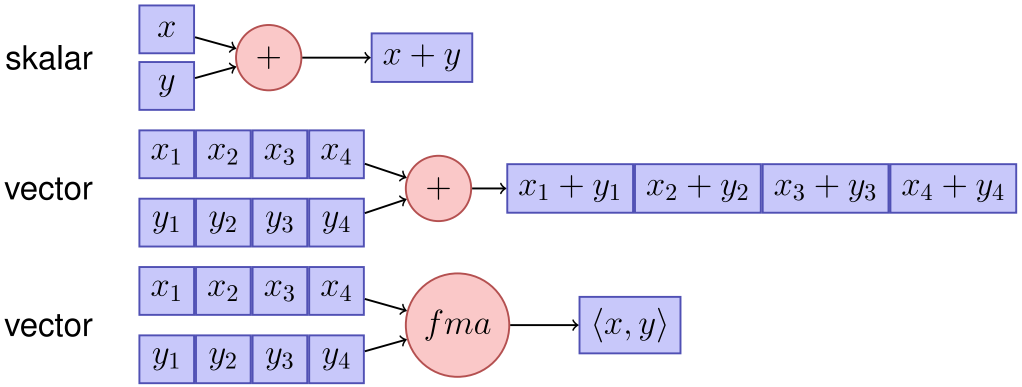
- Use transistors for more compute cores
- Parallelism in the software
- Programmers have to write parallel programs to benefit from new hardware

Forms of Parallel Execution

- Vectorization
- Pipelining
- Instruction Level Parallelism
- Multicore / Multiprocessing
- Distributed Computing

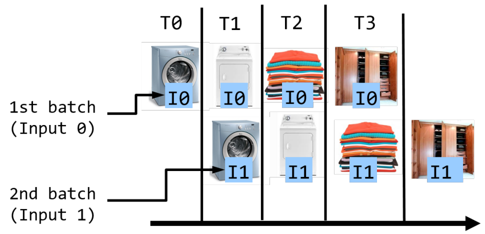
Vectorization

Parallel Execution of the same operations on elements of a vector (register)



Home Work

More efficient



Pipeline

Time

Inputs	T0	T1	T2	T3	T4	T5	T6	T7	T8	T9
I0	S0	S1	S2	S3						
I1		S0	S1	S2	S3					
I2			S0	S1	S2	S3				
I3				S0	S1	S2	S3			
I4					S0	S1	S2	S3		
I5						S0	S1	S2	S3	
I6							S0	S1	S2	S3

Lead In — Full Utilization — Lead out

Balanced / Unbalanced Pipeline

A pipeline is called balanced, if each step takes the same computation time.

Software-Pipelines are often unbalanced.

In the following we assume that each step of the pipeline takes as long as the longest step.

Throughput

- Throughput = Input or output data rate
- Number operations per time unit
- larger througput is better
- $$\text{throughput} = \frac{1}{\max(\text{computationtime}(\text{stages}))}$$
  ignores lead-in and lead-out times

Latency

- Time to perform a computation
- latency $= \#\text{stages} \cdot \max(\text{computationtime}(\text{stages}))$

Homework Example

Washing $T_0 = 1h$, Drying $T_1 = 2h$, Ironing $T_2 = 1h$, Tidy up $T_3 = 0.5h$

- Latency $L = 8h$
- In the long run: 1 batch every $2h$ $(0.5/h)$.

Throughput vs. Latency

- Increasing throughput can increase latency
- Stages of the pipeline need to communicate and synchronize: overhead

Pipelines in CPUs

Fetch | Decode | Execute | Data Fetch | Writeback

Multiple Stages

- Every instruction takes 5 time units (cycles)
- In the best case: 1 instruction per cycle, not always possible ("stalls")

Paralellism (several functional units) leads to *faster execution.*

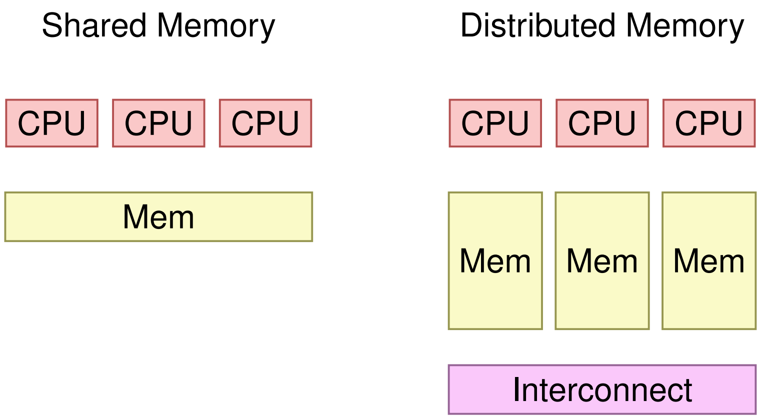
ILP – Instruction Level Parallelism

Modern CPUs provide several hardware units and execute independent instructions in parallel.

- Pipelining
- Superscalar CPUs (multiple instructions per cycle)
- Out-Of-Order Execution (Programmer observes the sequential execution)
- Speculative Execution ()

27.2 Hardware Architectures

Shared vs. Distributed Memory



Shared vs. Distributed Memory Programming

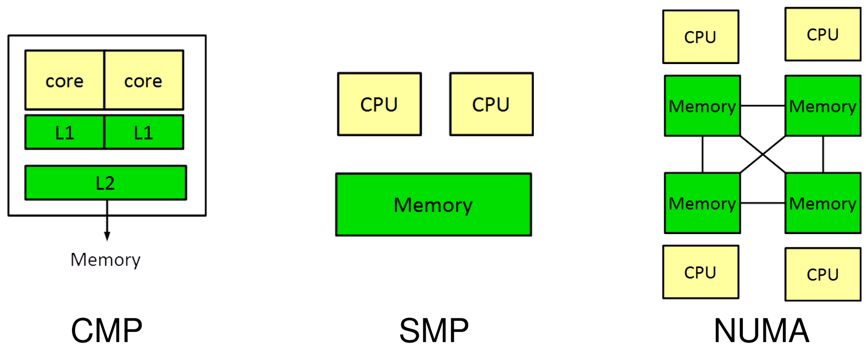
- Categories of programming interfaces
  - Communication via message passing
  - Communication via memory sharing
- It is possible:
  - to program shared memory systems as distributed systems (e.g. with message passing MPI)
  - program systems with distributed memory as shared memory systems (e.g. partitioned global address space PGAS)

Shared Memory Architectures

- Multicore (Chip Multiprocessor - CMP)
- Symmetric Multiprocessor Systems (SMP)
- Simultaneous Multithreading (SMT = Hyperthreading)
  - one physical core, Several Instruction Streams/Threads: several virtual cores
  - Between ILP (several units for a stream) and multicore (several units for several streams). Limited parallel performance.
- Non-Uniform Memory Access (NUMA)

Same programming interface

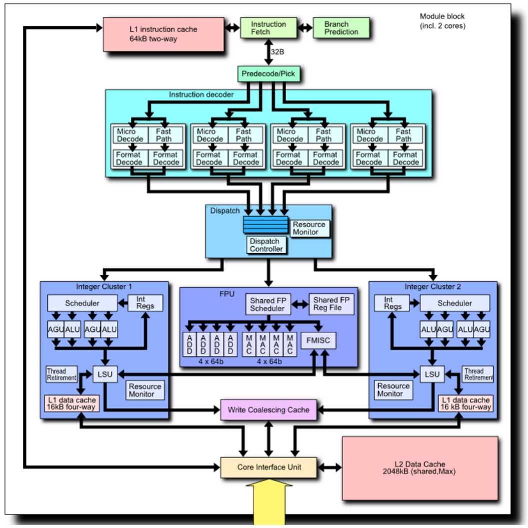
Overview



An Example

AMD Bulldozer: between CMP and SMT

- 2x integer core
- 1x floating point core

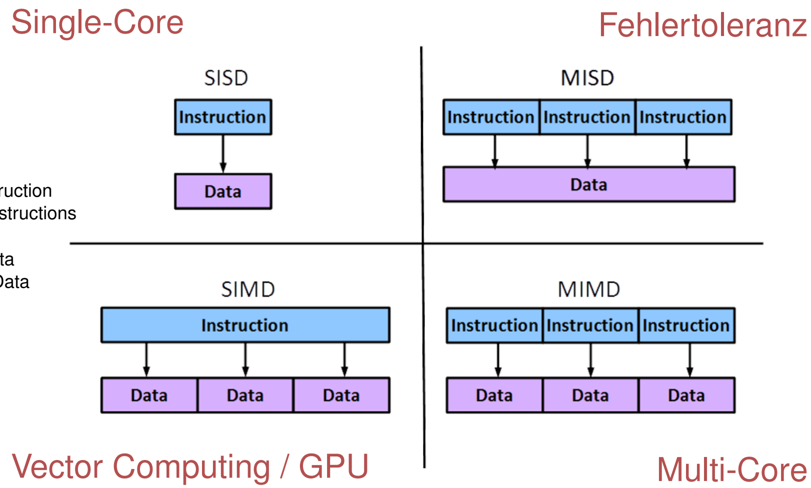


Wikipedia

Flynn's Taxonomy

SI = Single Instruction
MI = Multiple Instructions

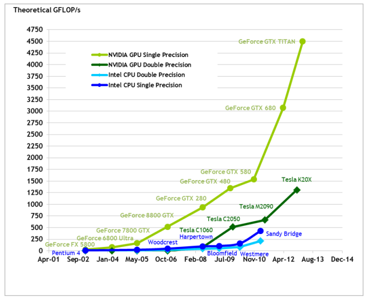
SD = Single Data
MD = Multiple Data



Massively Parallel Hardware

[General Purpose] Graphical Processing Units ([GP]GPUs)

- Revolution in High Performance Computing
  - Calculation 4.5 TFlops vs. 500 GFlops
  - Memory Bandwidth 170 GB/s vs. 40 GB/s
- SIMD
  - High data parallelism
  - Requires own programming model. Z.B. CUDA / OpenCL



27.3 Multi-Threading, Parallelism and Concurrency

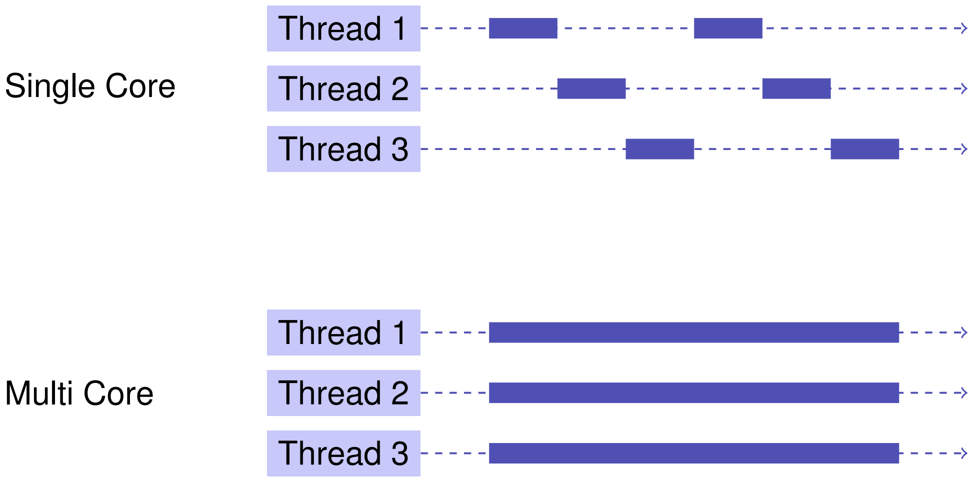
Processes and Threads

- Process: instance of a program
  - each process has a separate context, even a separate address space
  - OS manages processes (resource control, scheduling, synchronisation)
- Threads: threads of execution of a program
  - Threads share the address space
  - fast context switch between threads

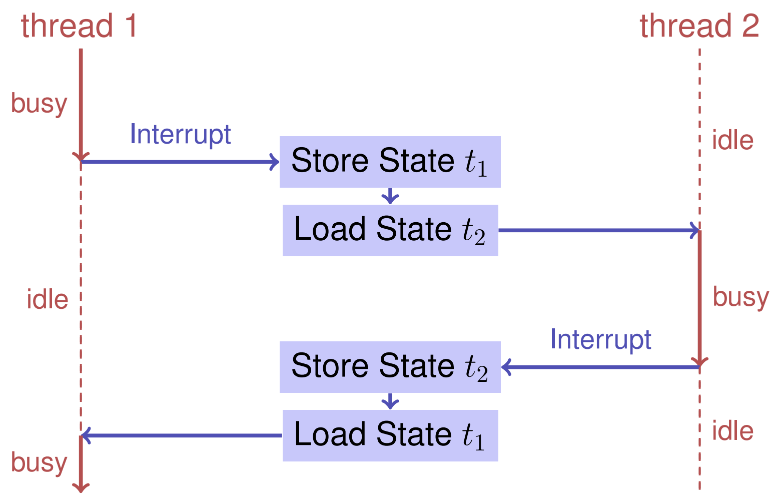
Why Multithreading?

- Avoid “polling” resources (files, network, keyboard)
- Interactivity (e.g. responsivity of GUI programs)
- Several applications / clients in parallel
- Parallelism (performance!)

Multithreading conceptually



Thread switch on one core (Preemption)



Parallelität vs. Concurrency

- *Parallelism:* Use extra resources to solve a problem faster
- *Concurrency:* Correctly and efficiently manage access to shared resources
- Begriffe überlappen offensichtlich. Bei parallelen Berechnungen besteht fast immer Synchronisierungsbedarf.

Parallelism

Work → Resources

Concurrency

Requests → Resources

Thread Safety

Thread Safety means that in a concurrent application of a program this always yields the desired results.

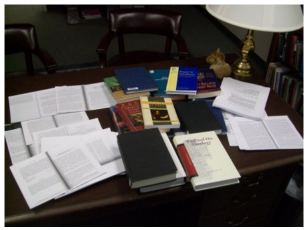
Many optimisations (Hardware, Compiler) target towards the correct execution of a *sequential* program.

Concurrent programs need an annotation that switches off certain optimisations selectively.

Example: Caches

- Access to registers faster than to shared memory.
- Principle of locality.
- Use of Caches (transparent to the programmer)

If and how far a cache coherency is guaranteed depends on the used system.



27.4 Scalability: Amdahl and Gustafson

Scalability

In parallel Programming:

- Speedup when increasing number p of processors
- What happens if $p \to \infty$?
- Program scales linearly: Linear speedup.

Parallel Performance

Given a fixed amount of computing work W (number computing steps)

Sequential execution time T_1

Parallel execution time on p CPUs

- Perfection: $T_p = T_1/p$
- Performance loss: $T_p > T_1/p$ (usual case)
- Sorcery: $T_p < T_1/p$

Parallel Speedup

Parallel speedup S_p on p CPUs:

$$S_p = \frac{W/T_p}{W/T_1} = \frac{T_1}{T_p}.$$

- Perfection: linear speedup $S_p = p$
- Performance loss: sublinear speedup $S_p < p$ (the usual case)
- Sorcery: superlinear speedup $S_p > p$

Efficiency:$E_p = S_p/p$

Reachable Speedup?

Parallel Program

Parallel Part	Seq. Part
80%	20%

$$T_1 = 10$$
$$T_8 = ?$$
$$T_8 = \frac{10 \cdot 0.8}{8} + 10 \cdot 0.2 = 1 + 2 = 3$$
$$S_8 = \frac{T_1}{T_8} = \frac{10}{3} \approx 3.3 < 8 \quad (!)$$

Amdahl's Law: Ingredients

Computational work W falls into two categories

- Paralellisable part W_p
- Not parallelisable, sequential part W_s

Assumption: W can be processed sequentially by *one* processor in W time units ($T_1 = W$):

$$T_1 = W_s + W_p$$
$$T_p \geq W_s + W_p/p$$

Amdahl's Law

$$S_p = \frac{T_1}{T_p} \leq \frac{W_s + W_p}{W_s + \frac{W_p}{p}}$$

Amdahl's Law

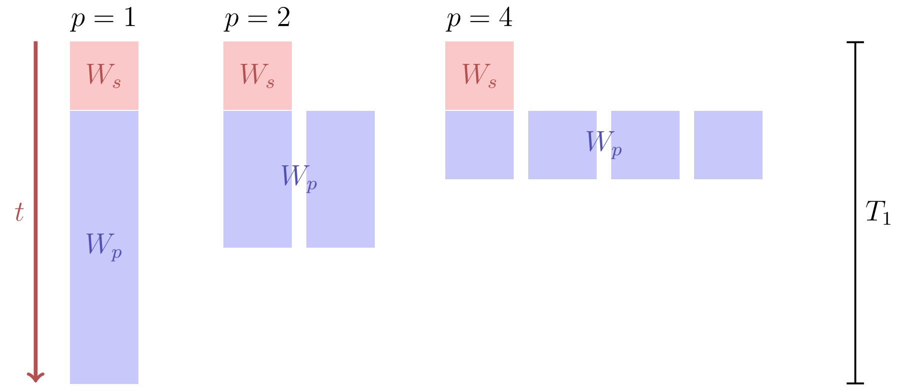
With sequential, not parallelizable fraction λ: $W_s = \lambda W$, $W_p = (1 - \lambda)W$:

$$S_p \leq \frac{1}{\lambda + \frac{1-\lambda}{p}}$$

Thus

$$S_\infty \leq \frac{1}{\lambda}$$

Illustration Amdahl's Law



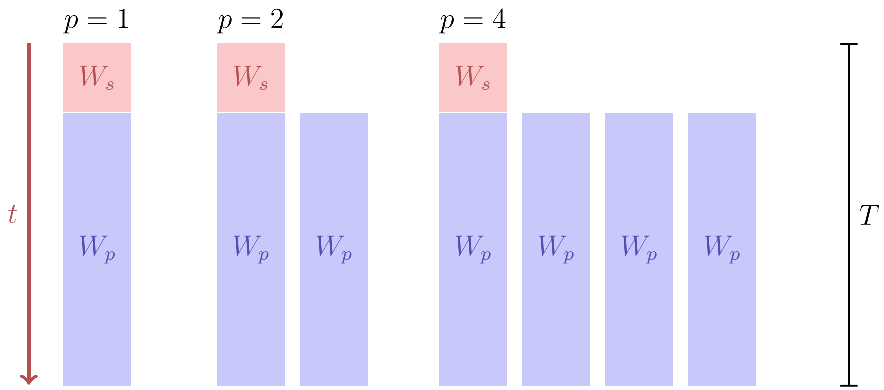
Amdahl's Law is bad news

All non-parallel parts of a program can cause problems

Gustafson's Law

- Fix the time of execution
- Vary the problem size.
- Assumption: the sequential part stays constant, the parallel part becomes larger

Illustration Gustafson's Law



Gustafson's Law

Work that can be executed by one processor in time T:

$$W_s + W_p = T$$

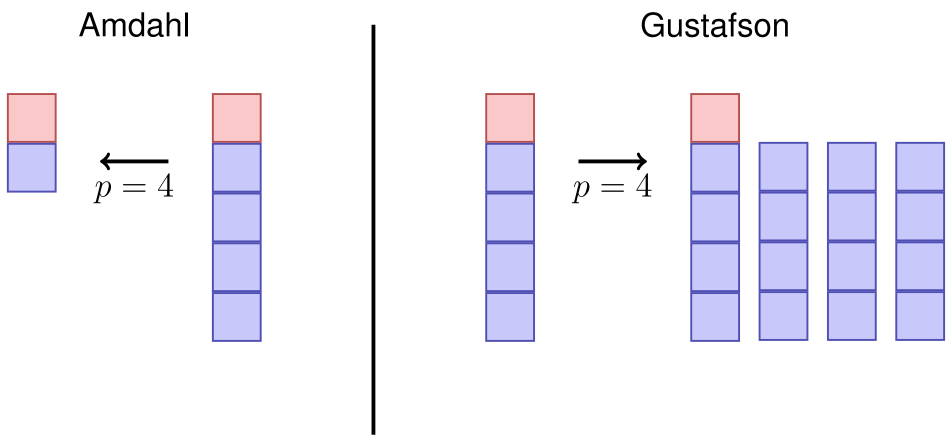
Work that can be executed by p processors in time T:

$$W_s + p \cdot W_p = \lambda \cdot T + p \cdot (1 - \lambda) \cdot T$$

Speedup:

$$\begin{aligned} S_p &= \frac{W_s + p \cdot W_p}{W_s + W_p} = p \cdot (1 - \lambda) + \lambda \\ &= p - \lambda(p - 1) \end{aligned}$$

Amdahl vs. Gustafson



Amdahl vs. Gustafson

The laws of Amdahl and Gustafson are models of speedup for parallelization.

Amdahl assumes a fixed *relative* sequential portion, Gustafson assumes a fixed *absolute* sequential part (that is expressed as portion of the work W_1 and that does not increase with increasing work).

The two models do not contradict each other but describe the runtime speedup of different problems and algorithms.

27.5 Task- and Data-Parallelism

Parallel Programming Paradigms

- *Task Parallel:* Programmer explicitly defines parallel tasks.
- *Data Parallel:* Operations applied simulatenously to an aggregate of individual items.

Example Data Parallel (OMP)

```
double sum = 0, A[MAX];
#pragma omp parallel for reduction (+:ave)
for (int i = 0; i< MAX; ++i)
  sum += A[i];
return sum;
```


Example Task Parallel (C++11 Threads/Futures)

```
double sum(Iterator from, Iterator to)
{
  auto len = from − to;
  if (len > threshold){
    auto future = std::async(sum, from, from + len / 2);
    return sumS(from + len / 2, to) + future.get();
  }
  else
    return sumS(from, to);
}
```

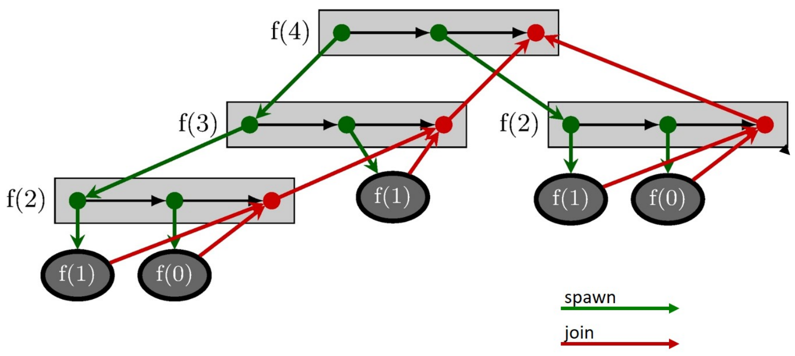

Work Partitioning and Scheduling

- Partitioning of the work into parallel task (programmer or system)
  - One task provides a unit of work
  - Granularity?
- Scheduling (Runtime System)
  - Assignment of tasks to processors
  - Goal: full resource usage with little overhead

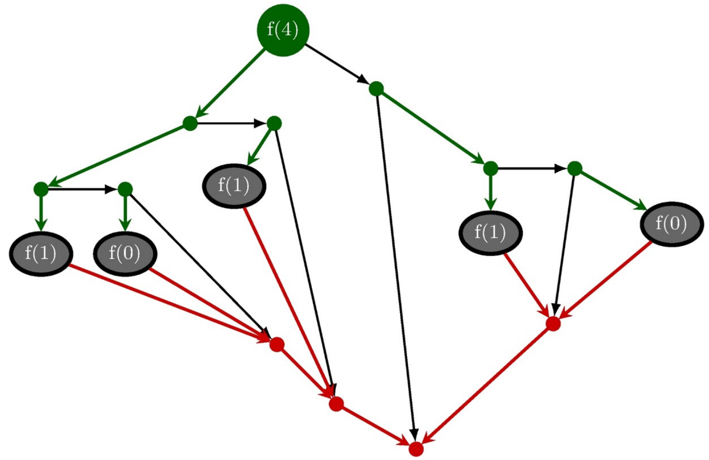
Example: Fibonacci P-Fib

if $n \leq 1$ **then**
  return n
else
  $x \leftarrow$ **spawn** P-Fib$(n-1)$
  $y \leftarrow$ **spawn** P-Fib$(n-2)$
  sync
  return $x+y$;

P-Fib Task Graph

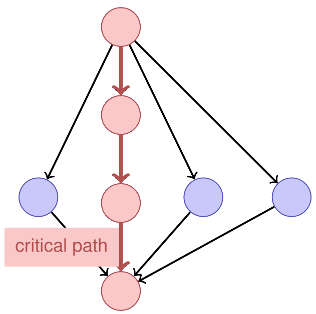


P-Fib Task Graph



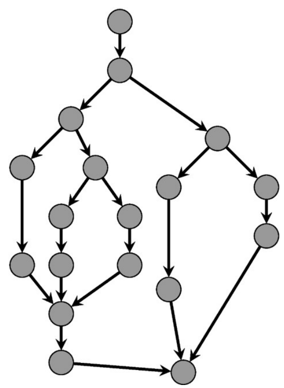
Question

- Each Node (task) takes 1 time unit.
- Arrows depict dependencies.
- Minimal execution time when number of processors = ∞?



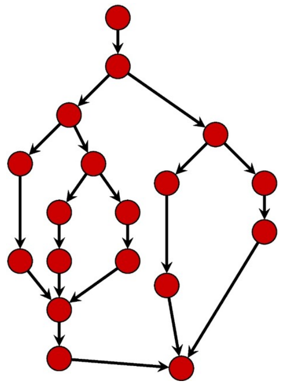
Performance Model

- p processors
- Dynamic scheduling
- T_p: Execution time on p processors



Performance Model

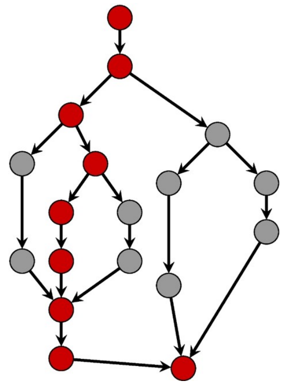
- T_p: Execution time on p processors
- T_1: *work*: time for executing total work on one processor
- T_1/T_p: Speedup



Performance Model

- T_∞: *span*: critical path, execution time on ∞ processors. Longest path from root to sink.
- T_1/T_∞: *Parallelism:* wider is better
- Lower bounds:

$$T_p \geq T_1/p \quad \text{Work law}$$
$$T_p \geq T_\infty \quad \text{Span law}$$



Greedy Scheduler

Greedy scheduler: at each time it schedules as many as availbale tasks.

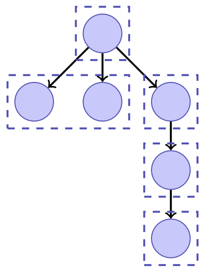
Theorem

On an ideal parallel computer with p processors, a greedy scheduler executes a multi-threaded computation with work T_1 and span T_∞ in time

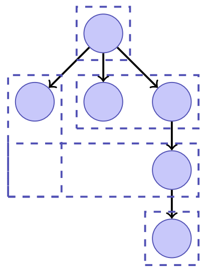
$$T_p \leq T_1/p + T_\infty$$

Beispiel

Assume $p = 2$.



$T_p = 5$



$T_p = 4$

Proof of the Theorem

Assume that all tasks provide the same amount of work.

- Complete step: p tasks are available.
- incomplete step: less than p steps available.

Assume that number of complete steps larger than $\lfloor T_1/p \rfloor$. Executed work $\geq \lfloor T_1/p \rfloor \cdot p + p = T_1 - T_1 \mod p + p > T_1$. Contradiction. Therefore maximally $\lfloor T_1/p \rfloor$ complete steps.

We now consider the graph of tasks to be done. Any maximal (critical) path starts with a node t with $\deg^-(t) = 0$. An incomplete step executes all available tasks t with $\deg^-(t) = 0$ and thus decreases the length of the span. Number incomplete steps thus limited by T_∞.

Consequence

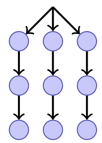
if $p \ll T_1/T_\infty$, i.e. $T_\infty \ll T_1/p$, then $T_p \approx T_1/p$.

Example Fibonacci

$T_1(n)/T_\infty(n) = \Theta(\phi^n/n)$. For moderate sizes of n we can use a lot of processors yielding linear speedup.

Granularity: how many tasks?

- #Tasks = #Cores?
- Problem if a core cannot be fully used
- Example: 9 units of work. 3 core. Scheduling of 3 sequential tasks.



Exclusive utilization:

P1	s1
P2	s2
P3	s3

Execution Time: 3 Units

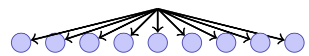
Foreign thread disturbing:

P1	s1	
P2	s2	s1
P3	s3	

Execution Time: 5 Units

Granularity: how many tasks?

- #Tasks = Maximum?
- Example: 9 units of work. 3 cores. Scheduling of 9 sequential tasks.



Exclusive utilization:

P1	s1	s4	s7
P2	s2	s5	s8
P3	s3	s6	s9

Execution Time: $3 + \varepsilon$ Units

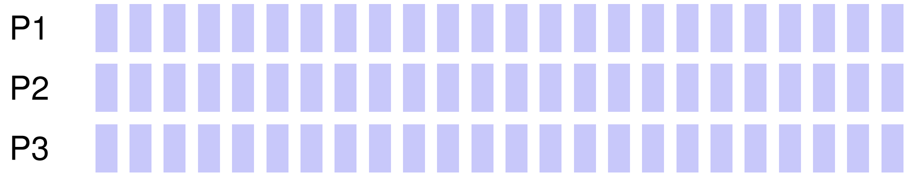
Foreign thread disturbing:

P1	s1			
P2	s2	s4	s5	s8
P3	s3	s6	s7	s9

Execution Time: 4 Units. Full utilization.

Granularity: how many tasks?

- #Tasks = Maximum?
- Example: 10^6 tiny units of work.



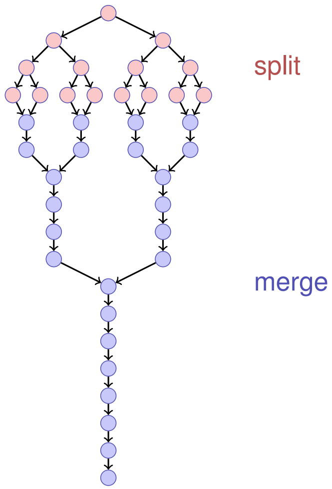
Execution time: dominiert vom Overhead.

Granularity: how many tasks?

Answer: as many tasks as possible with a sequential cutoff such that the overhead can be neglected.

Example: Parallelism of Mergesort

- Work (sequential runtime) of Mergesort $T_1(n) = \Theta(n \log n)$.
- Span $T_\infty(n) = \Theta(n)$
- Parallelism $T_1(n)/T_\infty(n) = \Theta(\log n)$ (Maximally achievable speedup with $p = \infty$ processors)

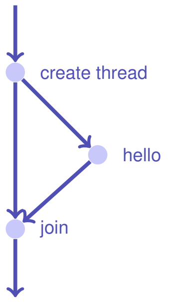


28. Parallel Programming II

C++ Threads, Shared Memory, Concurrency, Excursion: lock algorithm (Peterson), Mutual Exclusion Race Conditions [C++ Threads: Williams, Kap. 2.1-2.2], [C++ Race Conditions: Williams, Kap. 3.1] [C++ Mutexes: Williams, Kap. 3.2.1, 3.3.3]

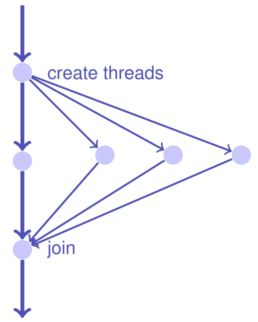
C++11 Threads

```
#include <iostream>
#include <thread>

void hello(){
  std::cout << "hello\n";
}

int main(){
  // create and launch thread t
  std::thread t(hello);
  // wait for termination of t
  t.join();
  return 0;
}
```



C++11 Threads

```
void hello(int id){
  std::cout << "hello from " << id << "\n";
}

int main(){
  std::vector<std::thread> tv(3);
  int id = 0;
  for (auto & t:tv)
    t = std::thread(hello, ++id);
  std::cout << "hello from main \n";
  for (auto & t:tv)
        t.join();
  return 0;
}
```



Nondeterministic Execution!

One execution:

hello from main
hello from 2
hello from 1
hello from 0

Other execution:

hello from 1
hello from main
hello from 0
hello from 2

Other execution:

hello from main
hello from 0
hello from hello from 1
2

Technical Detail

To let a thread continue as background thread:

```
void background();

void someFunction(){
  ...
  std::thread t(background);
  t.detach();
  ...
} // no problem here, thread is detached
```

More Technical Details

- With allocating a thread, reference parameters are copied, except explicitly std::ref is provided at the construction.
- Can also run Functor or Lambda-Expression on a thread
- In exceptional circumstances, joining threads should be executed in a catch block

More background and details in chapter 2 of the book *C++ Concurrency in Action*, Anthony Williams, Manning 2012. also available online at the ETH library.

28.2 Shared Memory, Concurrency

Sharing Resources (Memory)

- Up to now: fork-join algorithms: data parallel or divide-and-conquer
- Simple structure (data independence of the threads) to avoid race conditions
- Does not work any more when threads access shared memory.

Managing state

Managing state: Main challenge of concurrent programming.

Approaches:

- Immutability, for example constants.
- Isolated Mutability, for example thread-local variables, stack.
- Shared mutable data, for example references to shared memory, global variables

Protect the shared state

- Method 1: locks, guarantee exclusive access to shared data.
- Method 2: lock-free data structures, exclusive access with a much finer granularity.
- Method 3: transactional memory (not treated in class)

Canonical Example

```
class BankAccount {
  int balance = 0;
public:
  int  getBalance(){ return balance; }
  void setBalance(int x) { balance = x; }
  void withdraw(int amount) {
    int b = getBalance();
    setBalance(b − amount);
  }
  // deposit etc.
};
```

(correct in a single-threaded world)

Bad Interleaving

Parallel call to `widthdraw(100)` on the same account

Thread 1	Thread 2
`int b = getBalance();`	
	`int b = getBalance();`
	`setBalance(b−amount);`
`setBalance(b−amount);`	

t

Tempting Traps

WRONG:

```
void withdraw(int amount) {
  int b = getBalance();
  if (b==getBalance())
        setBalance(b − amount);
}
```

Bad interleavings cannot be solved with a repeated reading

Tempting Traps

also WRONG:

```
void withdraw(int amount) {
        setBalance(getBalance() − amount);
}
```

Assumptions about atomicity of operations are almost always wrong

Mutual Exclusion

We need a concept for mutual exclusion

Only one thread may execute the operation withdraw *on the same account* at a time.

The programmer has to make sure that mutual exclusion is used.

More Tempting Traps

```
class BankAccount {
  int balance = 0;
  bool busy = false;
public:
  void withdraw(int amount) {
    while (busy); // spin wait
    busy = true;
    int b = getBalance();
    setBalance(b − amount);
    busy = false;
  }

  // deposit would spin on the same boolean
};
```

does not work!

Just moved the problem!

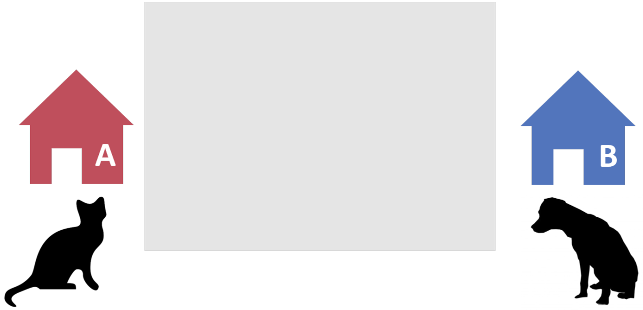
t	Thread 1	Thread 2
	`while (busy); //spin`	
		`while (busy); //spin`
	`busy = true;`	
		`busy = true;`
	`int b = getBalance();`	
		`int b = getBalance();`
		`setBalance(b − amount);`
↓	`setBalance(b − amount);`	

How ist this correctly implemented?

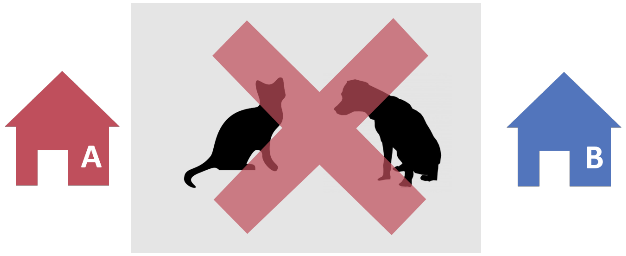
- We use *locks* (mutexes) from libraries
- They use hardware primitives, *Read-Modify-Write* (RMW) operations that can, in an atomic way, read and write depending on the read result.
- Without RMW Operations the algorithm is non-trivial and requires at least atomic access to variable of primitive type.

28.3 Excursion: lock algorithm

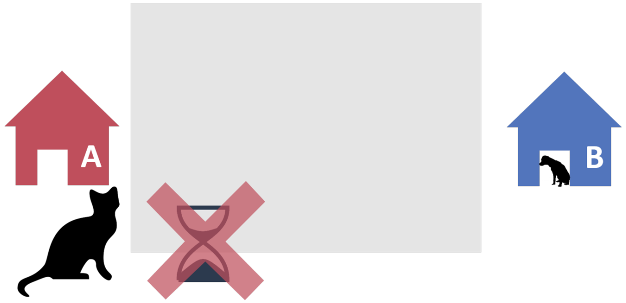
Alice's Cat vs. Bob's Dog



Required: Mutual Exclusion

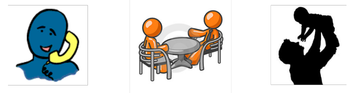


Required: No Lockout When Free



Communication Types

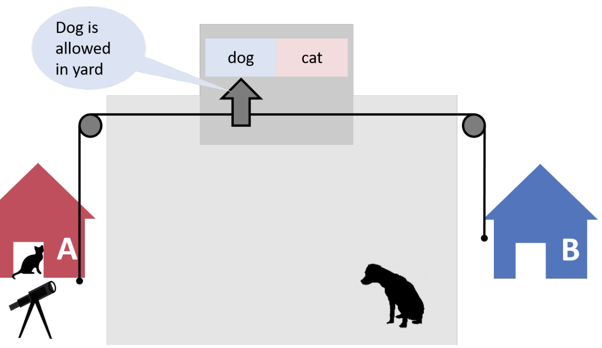
- Transient: Parties participate at the same time



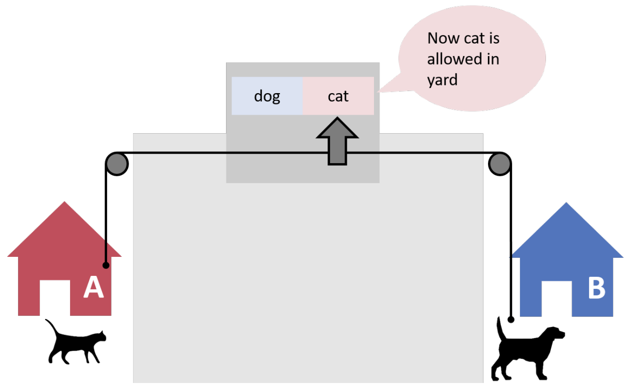
- Persistent: Parties participate at different times

Mutual exclusion: persistent communication

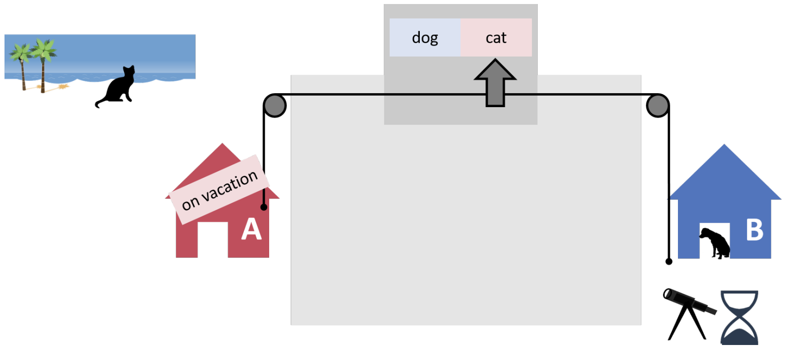
Communication Idea 1



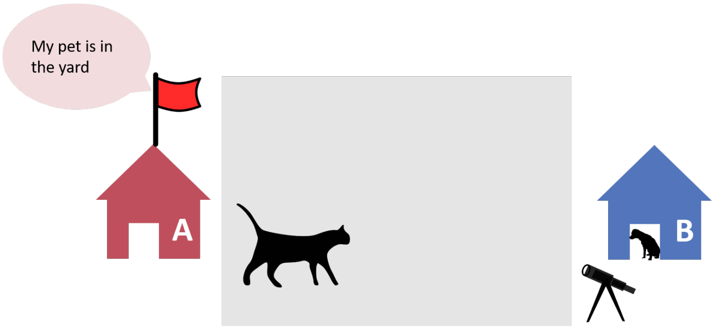
Access Protocol



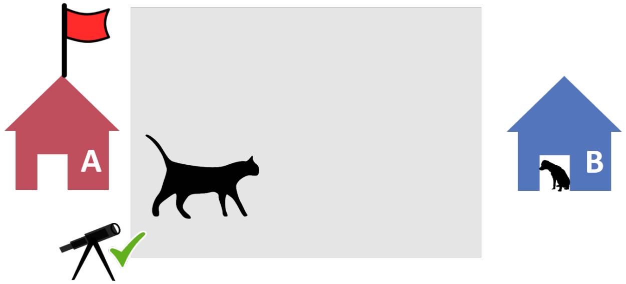
Problem!



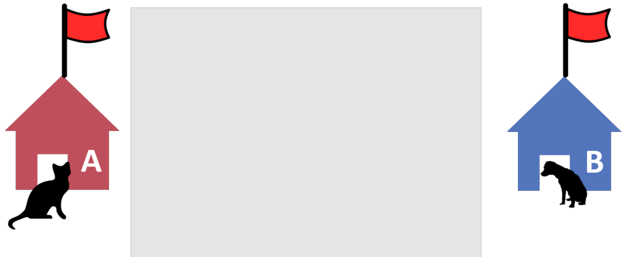
Communication Idea 2



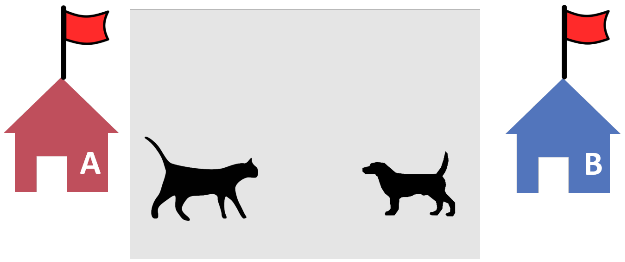
Access Protocol 2.1



Different Scenario

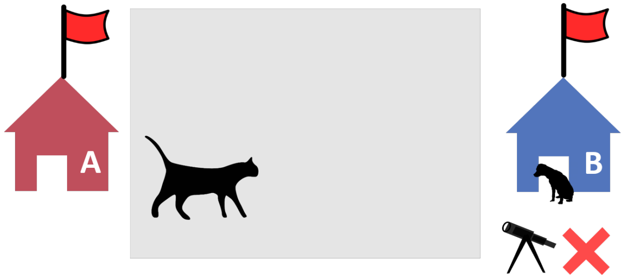


Problem: No Mutual Exclusion

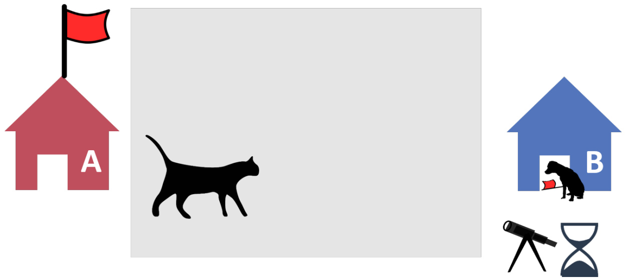


Checking Flags Twice: Deadlock

Access Protocol 2.2

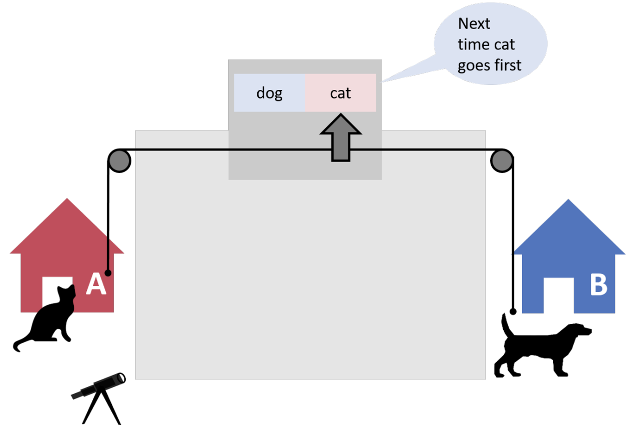


Access Protocol 2.2:provably correct

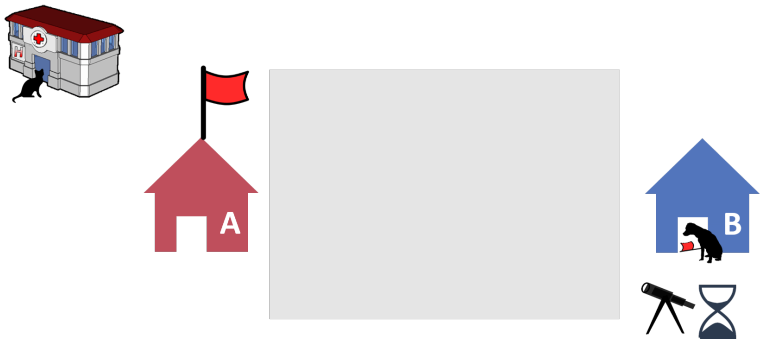


Weniger schwerwiegend: Starvation

Final Solution



General Problem of Locking remains



Peterson's Algorithm[41]

for two processes is provable correct and free from starvation

```
non—critical section

flag[me] = true // I am interested
victim = me // but you go first
// spin while we are both interested and you go first:
while (flag[you] && victim == me) {};

critical section

flag[me] = false
```

The code assumes that the access to flag / victim is atomic and particularly linearizable or sequential consistent. An assumption that – as we will see below – is not necessarily given for normal variables. The Peterson-lock is not used on modern hardware.

[41] not relevant for the exam

28.4 Mutual Exclusion

Critical Sections and Mutual Exclusion

Critical Section
Piece of code that may be executed by at most one process (thread) at a time.

Mutual Exclusion
Algorithm to implement a critical section

```
acquire_mutex();     // entry algorithm\\
...                  // critical section
release_mutex();     // exit algorithm
```

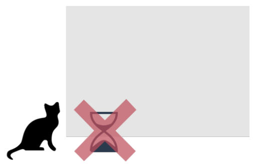

Required Properties of Mutual Exclusion

Correctness (Safety)

- At most one process executes the critical section code

Liveness

- Acquiring the mutex must terminate in finite time when no process executes in the critical section



Almost Correct

```
class BankAccount {
  int balance = 0;
  std::mutex m; // requires #include <mutex>
public:
  ...
  void withdraw(int amount) {
    m.lock();
    int b = getBalance();
    setBalance(b - amount);
    m.unlock();
  }
};
```

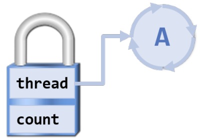
What if an exception occurs?

RAII Approach

```
class BankAccount {
  int balance = 0;
  std::mutex m;
public:
  ...
  void withdraw(int amount) {
    std::lock_guard<std::mutex> guard(m);
    int b = getBalance();
    setBalance(b - amount);
  } // Destruction of guard leads to unlocking m
};
```

What about getBalance / setBalance?

Reentrant Locks



Reentrant Lock (recursive lock)

- remembers the currently affected thread;
- provides a counter
  - Call of lock: counter incremented
  - Call of unlock: counter is decremented. If counter = 0 the lock is released.

Account with reentrant lock

```
class BankAccount {
  int balance = 0;
  std::recursive_mutex m;
  using guard = std::lock_guard<std::recursive_mutex>;
public:
  int  getBalance(){ guard g(m); return balance;
  }
  void setBalance(int x) { guard g(m); balance = x;
  }
  void withdraw(int amount) { guard g(m);
    int b = getBalance();
    setBalance(b - amount);
  }
};
```


28.5 Race Conditions

Race Condition

- A *race condition* occurs when the result of a computation depends on scheduling.
- We make a distinction between *bad interleavings* and *data races*
- *Bad interleavings* can occur even when a mutex is used.

Example: Stack

Stack with correctly synchronized access:

```
template <typename T>
class stack{
  ...
  std::recursive_mutex m;
  using guard = std::lock_guard<std::recursive_mutex>;
public:
  bool isEmpty(){ guard g(m); ...   }
  void push(T value){ guard g(m);   ... }
  T pop(){ guard g(m); ...}
};
```


Peek

Forgot to implement peek. Like this?

```
template <typename T>
T peek (stack<T> &s){
  T value = s.pop();
  s.push(value);
  return value;
}
```

not thread-safe!

Despite its questionable style the code is correct in a sequential world. Not so in concurrent programming.

Bad Interleaving!

Initially empty stack s, only shared between threads 1 and 2.

Thread 1 pushes a value and checks that the stack is then non-empty. Thread 2 reads the topmost value using peek().

t	Thread 1	Thread 2
↓	`s.push(5);`	
		`int value = s.pop();`
	`assert(!s.isEmpty());`	
		`s.push(value);`
		`return value;`

The fix

Peek must be protected with the same lock as the other access methods

Bad Interleavings

Race conditions as bad interleavings can happen on a high level of abstraction

In the following we consider a different form of race condition: data race.

How about this?

```
class counter{
  int count = 0;
  std::recursive_mutex m;
  using guard = std::lock_guard<std::recursive_mutex>;
public:
  int increase(){
    guard g(m); return ++count;
  }
  int get(){
    return count;
  }
}
```

not thread-safe!

Why wrong?

It looks like nothing can go wrong because the update of count happens in a “tiny step”.

But this code is still wrong and depends on language-implementation details you cannot assume.

This problem is called *Data-Race*

Moral: *Do not introduce a data race, even if every interleaving you can think of is correct. Don’t make assumptions on the memory order.*

A bit more formal

Data Race (low-level Race-Conditions) Erroneous program behavior caused by insufficiently synchronized accesses of a shared resource by multiple threads, e.g. Simultaneous read/write or write/write of the same memory location

Bad Interleaving (High Level Race Condition) Erroneous program behavior caused by an unfavorable execution order of a multithreaded algorithm, even if that makes use of otherwise well synchronized resources.

We look deeper

```
class C {
  int x = 0;
  int y = 0;
public:
  void f() {
Ⓐ   x = 1;
Ⓑ   y = 1;
  }
  void g() {
Ⓒ   int a = y;
Ⓓ   int b = x;
    assert(b >= a);
  }
}
```

Can this fail?

There is no interleaving of f and g that would cause the assertion to fail:

- A B C D ✓
- A C B D ✓
- A C D B ✓
- C A B D ✓
- C C D B ✓
- C D A B ✓

It can nevertheless fail!

One Resason: Memory Reordering

Rule of thumb: Compiler and hardware allowed to make changes that do not affect the *semantics of a sequentially* executed program

```
void f() {
        x = 1;
        y = x+1;
        z = x+1;
}
```

$\Longleftrightarrow$ sequentially equivalent

```
void f() {
        x = 1;
        z = x+1;
        y = x+1;
}
```


From a Software-Perspective

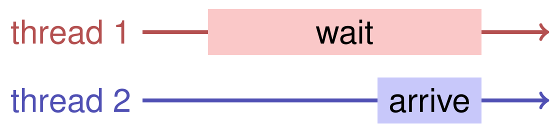
Modern compilers do not give guarantees that a global ordering of memory accesses is provided as in the sourcecode:

- Some memory accesses may be even optimized away completely!
- Huge potential for optimizations – and for errors, when you make the wrong assumptions

Example: Self-made Rendevouz

```
int x; // shared

void wait(){
  x = 1;
  while(x == 1);
}

void arrive(){
  x = 2;
}
```

Assume thread 1 calls wait, later thread 2 calls arrive. What happens?



Compilation

Source

```
int x; // shared

void wait(){
  x = 1;
  while(x == 1);
}

void arrive(){
  x = 2;
}
```

Without optimisation

```
wait:
movl $0x1, x
test:
mov x, %eax
cmp $0x1, %eax
je  test

arrive:
movl $0x2, x
```

if equal

With optimisation

```
wait:
movl $0x1, x
test:
jmp test

arrive
movl $0x2, x
```

always

Hardware Perspective

Modern multiprocessors do not enforce global ordering of all instructions for performance reasons:

- Most processors have a pipelined architecture and can execute (parts of) multiple instructions simultaneously. They can even reorder instructions internally.
- Each processor has a local cache, and thus loads/stores to shared memory can become visible to other processors at different times

Memory Hierarchy

Registers — fast,low latency, high cost, low capacity

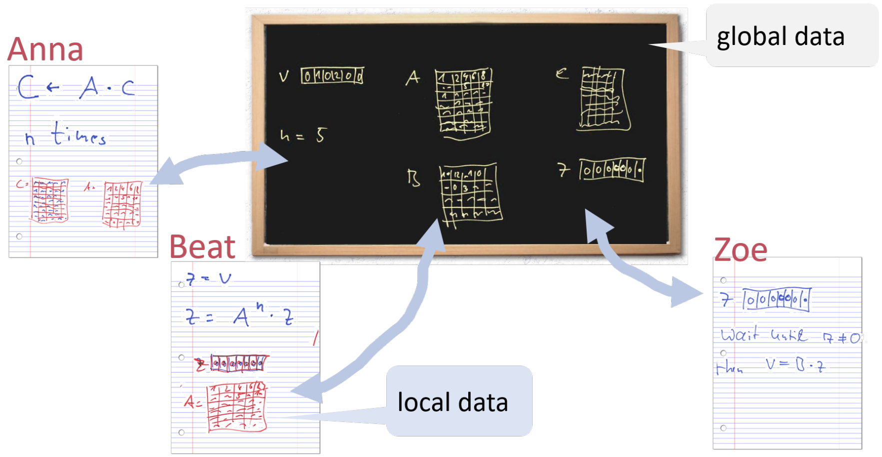
L1 Cache

L2 Cache

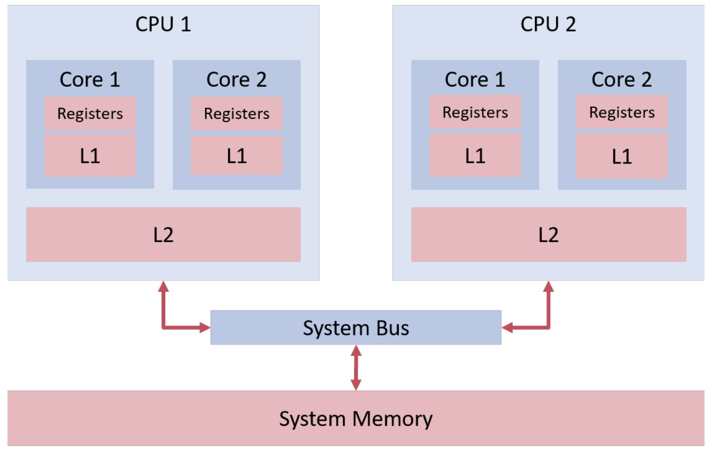
...

System Memory — slow,high latency,low cost,high capacity

An Analogy



Schematic



Memory Models

When and if effects of memory operations become visible for threads, depends on hardware, runtime system and programming language.

A *memory model* (e.g. that of C++) provides minimal guarantees for the effect of memory operations

- leaving open possibilities for optimisation
- containing guidelines for writing thread-safe programs

For instance, C++ provides *guarantees when synchronisation with a mutex* is used.

Fixed

```
class C {
  int x = 0;
  int y = 0;
  std::mutex m;
public:
  void f() {
    m.lock(); x = 1; m.unlock();
    m.lock(); y = 1; m.unlock();
  }
  void g() {
    m.lock(); int a = y; m.unlock();
    m.lock(); int b = x; m.unlock();
    assert(b >= a); // cannot fail
  }
};
```

Atomic

Here also possible:

```
class C {
  std::atomic_int x{0}; // requires #include <atomic>
  std::atomic_int y{0};
public:
  void f() {
    x = 1;
    y = 1;
  }
  void g() {
    int a = y;
    int b = x;
    assert(b >= a); // cannot fail
  }
};
```


29. Parallel Programming III

Deadlock and Starvation Producer-Consumer, The concept of the monitor, Condition Variables [Deadlocks : Williams, Kap. 3.2.4-3.2.5] [Condition Variables: Williams, Kap. 4.1]

Deadlock Motivation

```
class BankAccount {
  int balance = 0;
  std::recursive_mutex m;
  using guard = std::lock_guard<std::recursive_mutex>;
public:
  ...
  void withdraw(int amount) { guard g(m); ... }
  void deposit(int amount){ guard g(m); ... }

  void transfer(int amount, BankAccount& to){
      guard g(m);
      withdraw(amount);
      to.deposit(amount);
  }
};
```

Problem?

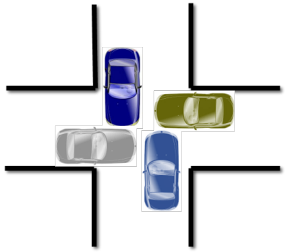
Deadlock Motivation

Suppose BankAccount instances `x` and `y`

`Thread 1: x.transfer(1,y);`	`Thread 2: y.transfer(1,x);`
acquire lock for x ← x	
withdraw from x	acquire lock for y ← y
acquire lock for y	withdraw from y
	acquire lock for x

Deadlock

Deadlock: two or more processes are mutually blocked because each process waits for another of these processes to proceed.

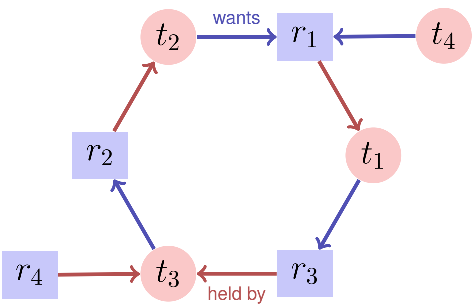


Threads and Resources

- Grafically t and Resources (Locks) r
- Thread t attempts to acquire resource a: $t \rightarrow a$
- Resource b is held by thread q: $s \leftarrow b$

Deadlock – Detection

A deadlock for threads $t_1, \ldots, t_n$ occurs when the graph describing the relation of the n threads and resources $r_1, \ldots, r_m$ contains a cycle.



Techniques

- *Deadlock detection* detects cycles in the dependency graph. Deadlocks can in general not be healed: releasing locks generally leads to inconsistent state
- *Deadlock avoidance* amounts to techniques to ensure a cycle can never arise
  - Coarser granularity "one lock for all"
  - Two-phase locking with retry mechanism
  - Lock Hierarchies
  - ...
  - Resource Ordering

Back to the Example

```
class BankAccount {
  int id; // account number, also used for locking order
  std::recursive_mutex m; ...
public:
  ...
   void transfer(int amount, BankAccount& to){
      if (id < to.id){
        guard g(m); guard h(to.m);
        withdraw(amount); to.deposit(amount);
      } else {
        guard g(to.m); guard h(m);
        withdraw(amount); to.deposit(amount);
      }
  }
};
```


C++11 Style

```
class BankAccount {
  ...
  std::recursive_mutex m;
  using guard = std::lock_guard<std::recursive_mutex>;
public:
  ...
   void transfer(int amount, BankAccount& to){
      std::lock(m,to.m); // lock order done by C++
      // tell the guards that the lock is already taken:
      guard g(m,std::adopt_lock); guard h(to.m,std::adopt_lock);
      withdraw(amount);
      to.deposit(amount);
  }
};
```

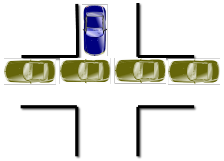

By the way...

```
class BankAccount {
  int balance = 0;
  std::recursive_mutex m;
  using guard = std::lock_guard<std::recursive_mutex>;
public:
  ...
  void withdraw(int amount) { guard g(m); ... }
  void deposit(int amount){ guard g(m); ... }

  void transfer(int amount, BankAccount& to){
      withdraw(amount);
      to.deposit(amount);
  }
};
```

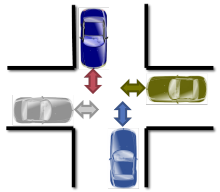
This would have worked here also. But then for a very short amount of time, money disappears, which does not seem acceptable (transient inconsistency!)

Starvation und Livelock

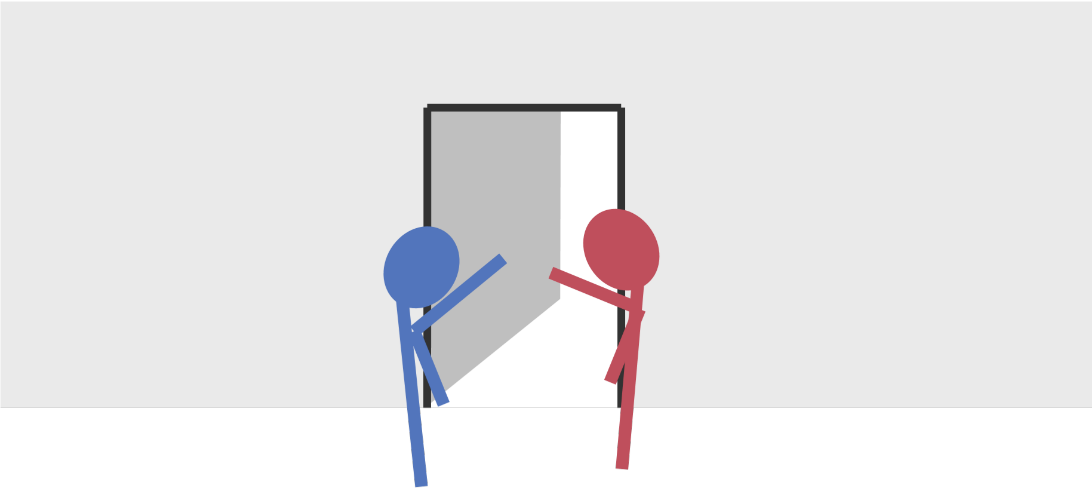
Starvation: the repeated but unsuccessful attempt to acquire a resource that was recently (transiently) free.



Livelock: competing processes are able to detect a potential deadlock but make no progress while trying to resolve it.



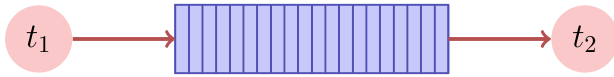
Politelock



Producer-Consumer Problem

Two (or more) processes, producers and consumers of data should become decoupled by some data structure.

Fundamental Data structure for building pipelines in software.



Sequential implementation (unbounded buffer)

```
class BufferS {
  std::queue<int> buf;
public:
    void put(int x){
        buf.push(x);
    }

    int get(){
        while (buf.empty()){} // wait until data arrive
        int x = buf.front();
        buf.pop();
        return x;
    }
};
```

not thread-safe

How about this?

```
class Buffer {
  std::recursive_mutex m;
  using guard = std::lock_guard<std::recursive_mutex>;
  std::queue<int> buf;
public:
    void put(int x){ guard g(m);
        buf.push(x);
    }
    int get(){ guard g(m);
        while (buf.empty()){}
        int x = buf.front();
        buf.pop();
        return x;
    }
};
```

Deadlock

Well, then this?

```
void put(int x){
    guard g(m);
    buf.push(x);
}
int get(){
    m.lock();
    while (buf.empty()){
        m.unlock();
        m.lock();
    }
    int x = buf.front();
    buf.pop();
    m.unlock();
    return x;
}
```

Ok this works, but it wastes CPU time.

Better?

```
void put(int x){
  guard g(m);
  buf.push(x);
}
int get(){
  m.lock();
  while (buf.empty()){
    m.unlock();
    std::this_thread::sleep_for(std::chrono::milliseconds(10));
    m.lock();
  }
  int x = buf.front(); buf.pop();
  m.unlock();
  return x;
}
```

Ok a little bit better, limits reactivity though.

Moral

We do not want to implement waiting on a condition ourselves.

There already is a mechanism for this: *condition variables*.

The underlying concept is called *Monitor*.

Monitor

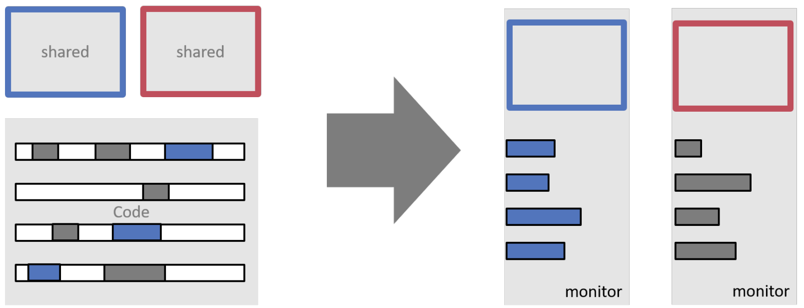
Monitor abstract data structure equipped with a set of operations that run in mutual exclusion and that can be synchronized.

Invented by C.A.R. Hoare and Per Brinch Hansen (cf. Monitors – An Operating System Structuring Concept, C.A.R. Hoare 1974)

C.A.R. Hoare, *1934

Per Brinch Hansen (1938-2007)

Monitors vs. Locks



Monitor and Conditions

Monitors provide, in addition to mutual exclusion, the following mechanism:

Waiting on conditions: If a condition does not hold, then

- Release the monitor lock
- Wait for the condition to become true
- Check the condition when a signal is raised

Signalling: Thread that might make the condition true:

- Send signal to potentially waiting threads

Condition Variables

```
#include <mutex>
#include <condition_variable>
...

class Buffer {
  std::queue<int> buf;

  std::mutex m;
  // need unique_lock guard for conditions
  using guard = std::unique_lock<std::mutex>;
  std::condition_variable cond;
public:
        ...
};
```


Condition Variables

```
class Buffer {
...
public:
    void put(int x){
        guard g(m);
        buf.push(x);
        cond.notify_one();
    }
    int get(){
        guard g(m);
        cond.wait(g, [&]{return !buf.empty();});
        int x = buf.front(); buf.pop();
        return x;
    }
};
```

Technical Details

- A thread that waits using `cond.wait` runs at most for a short time on a core. After that it does not utilize compute power and "sleeps".
- The notify (or signal-) mechanism wakes up sleeping threads that subsequently check their conditions.
  - `cond.notify_one` signals *one* waiting thread
  - `cond.notify_all` signals *all* waiting threads. Required when waiting thrads wait potentially on *different* conditions.

Technical Details

- Many other programming langauges offer the same kind of mechanism. The checking of conditions (in a loop!) has to be usually implemented by the programmer.

Java Example

```
synchronized long get() {
  long x;
  while (isEmpty())
    try {
      wait ();
    } catch (InterruptedException e) { }
  x = doGet();
  return x;
}

synchronized put(long x){
  doPut(x);
  notify ();
}
```


By the way, using a bounded buffer..

```
class Buffer {
  ...
  CircularBuffer<int,128> buf; // from lecture 6
public:
    void put(int x){ guard g(m);
        cond.wait(g, [&]{return !buf.full();});
        buf.put(x);
        cond.notify_all();
    }
    int get(){ guard g(m);
        cond.wait(g, [&]{return !buf.empty();});
        cond.notify_all();
        return buf.get();
    }
};
```


30. Parallel Programming IV

Futures, Read-Modify-Write Instructions, Atomic Variables, Idea of lock-free programming

[C++ Futures: Williams, Kap. 4.2.1-4.2.3] [C++ Atomic: Williams, Kap. 5.2.1-5.2.4, 5.2.7] [C++ Lockfree: Williams, Kap. 7.1.-7.2.1]

Futures: Motivation

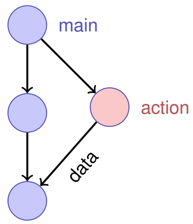
Up to this point, threads have been functions without a result:

```
void action(some parameters){
  ...
}

std::thread t(action, parameters);
...
t.join();
// potentially read result written via ref−parameters
```


Futures: Motivation

Now we would like to have the following

```
T action(some parameters){
  ...
  return value;
}

std::thread t(action, parameters);
...
value = get_value_from_thread();
```



We can do this already!

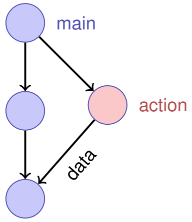
- We make use of the producer/consumer pattern, implemented with condition variables
- Start the thread with reference to a buffer
- We get the result from the buffer.
- Synchronisation is already implemented

Reminder

```
template <typename T>
class Buffer {
  std::queue<T> buf;
  std::mutex m;
  std::condition_variable cond;
public:
  void put(T x){ std::unique_lock<std::mutex> g(m);
    buf.push(x);
    cond.notify_one();
  }
  T get(){ std::unique_lock<std::mutex> g(m);
    cond.wait(g, [&]{return (!buf.empty());});
    T x = buf.front(); buf.pop(); return x;
  }
};
```

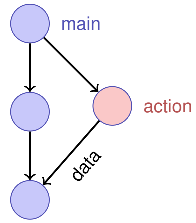
Application

```
void action(Buffer<int>& c){
  // some long lasting operation ...
  c.put(42);
}

int main(){
  Buffer<int> c;
  std::thread t(action, std::ref(c));
  t.detach(); // no join required for free running thread
  // can do some more work here in parallel
  int val = c.get();
  // use result
  return 0;
}
```



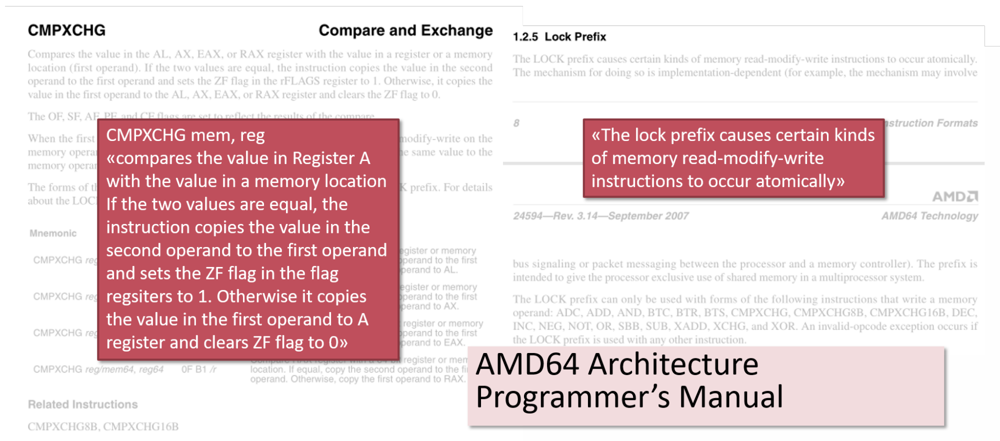
With features of C++11

```
int action(){
  // some long lasting operation
  return 42;
}

int main(){
  std::future<int> f = std::async(action);
  // can do some work here in parallel
  int val = f.get();
  // use result
  return 0;
}
```



30.2 Read-Modify-Write

Example: Atomic Operations in Hardware



Read-Modify-Write

Concept of Read-Modify-Write: Read, modify and write back at one point in time (atomic).

Example: Test-And-Set

Pseudo-code for TAS (C++ style):

```
bool TAS(bool& variable){
atomic  bool old = variable;
        variable = true;
        return old;
}
```


Application example TAS in C++11

We build our own lock:

```
class SpinLock{
std::atomic_flag taken {false};
public:

  void lock(){
    while (taken.test_and_set()); // TAS returns old value
  }

  void unlock(){
    taken.clear();
  }
};
```


30.3 Lock-Free Programming

Ideas

Compare-And-Swap

```
bool CAS(int& variable, int& expected, int desired){
  if (variable == expected){
    variable = desired;
    return true;
  }
  else{
    expected = variable;
    return false;
  }
}
```

atomic

Lock-free programming

Data structure is called

- *lock-free*: at least one thread always makes progress in bounded time even if other algorithms run concurrently. Implies system-wide progress but not freedom from starvation.
- *wait-free*: all threads eventually make progress in bounded time. Implies freedom from starvation.

Progress Conditions

	Non-Blocking	Blocking
Everyone makes progress	Wait-free	Starvation-free
Someone makes progress	Lock-free	Deadlock-free

Implication

- Programming with locks: each thread can block other threads indefinitely.
- Lock-free: failure or suspension of one thread cannot cause failure or suspension of another thread !

Lock-free programming: how?

Beobachtung:

- RMW-operations are implemented *wait-free* by hardware.
- Every thread sees his result of a CAS or TAS in bounded time.

Idea of lock-free programming: read the state of a data sructure and change the data structure *atomically* if and only if the previously read state remained unchanged meanwhile.

Example: lock-free stack

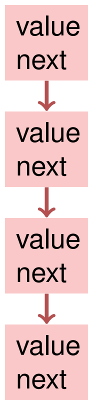
Simplified variant of a stack in the following

- pop prüft nicht, ob der Stack leer ist
- pop gibt nichts zurück

(Node)

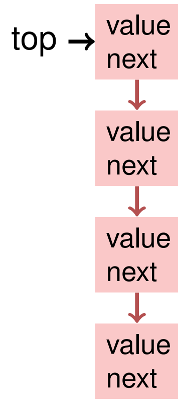
Nodes:

```
struct Node {
  T value;

  Node<T>* next;
  Node(T v, Node<T>* nxt): value(v), next(nxt) {}
};
```



(Blocking Version)

```
template <typename T>
class Stack {
    Node<T> *top=nullptr;
    std::mutex m;
public:
    void push(T val){ guard g(m);
        top = new Node<T>(val, top);
    }
    void pop(){ guard g(m);
        Node<T>* old_top = top;
        top = top->next;
        delete old_top;
    }
};
```



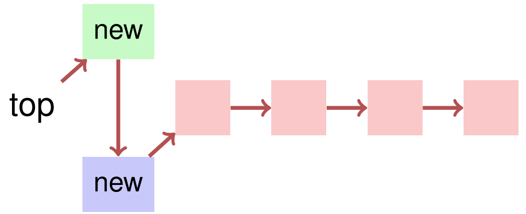
Lock-Free

```
template <typename T>
class Stack {
  std::atomic<Node<T>*> top {nullptr};
public:
  void push(T val){
    Node<T>* new_node = new Node<T> (val, top);
    while (!top.compare_exchange_weak(new_node->next, new_node));
  }
  void pop(){
    Node<T>* old_top = top;
    while (!top.compare_exchange_weak(old_top, old_top->next));
    delete old_top;
  }
};
```


Push

```
void push(T val){
  Node<T>* new_node = new Node<T> (val, top);
  while (!top.compare_exchange_weak(new_node->next, new_node));
}
```

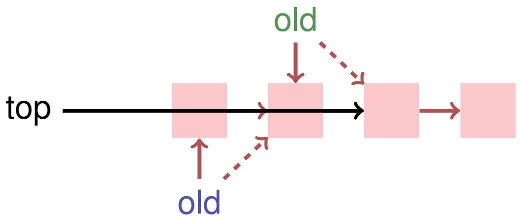
2 Threads:



Pop

```
void pop(){
  Node<T>* old_top = top;
  while (!top.compare_exchange_weak(old_top, old_top->next));
  delete old_top;
}
```

2 Threads:



Lock-Free Programming – Limits

- Lock-Free Programming is complicated.
- If more than one value has to be changed in an algorithm (example: queue), it is becoming even more complicated: threads have to “help each other” in order to make an algorithm lock-free.
- The *ABA problem* can occur if memory is reused in an algorithm.